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A GAME-THEORETIC PROOF OF ANALYTIC RAMSEY THEOREM

by KAZUYUKI TANAKA in Sendai (Japan)

Abstract

We give a simple game-theoretic proof of Silver's theorem that every analytic set is Ramsey. A set P of subsets of ω is called Ramsey if there exists an infinite set H such that either all infinite subsets of H are in P or all out of P. Our proof clarifies a strong connection between the Ramsey property of partitions and the determinacy of infinite games.
MSC: 03E15, 03E60, 05A17.
Key words: Analytic Ramsey theorem, determinacy of infinite games.

Let ω be the set of non-negative integers. For an infinite subset X of ω let $[X]^n$ denote the set of all subsets of X with exactly n elements $(n \in \omega)$. Suppose that $[X]^n$ is partitioned into P_1 and P_2. Then the classical version of RAMSEY's theorem asserts that there is an infinite subset H of ω such that either $[H]^n \subseteqq P_1$ or $[H]^n \subseteqq P_2$.

In this paper, we discuss the following natural generalization of RAMSEY's theorem. For any infinite subset X of ω, let $[X]^\omega$ denote the set of all infinite subsets of X. We say that a partition $P \subseteqq [\omega]^\omega$ is *Ramsey* if there exists an infinite subset H of ω such that either $[H]^\omega \subseteqq P$ or $[H]^\omega \subseteqq [\omega]^\omega - P$. By the axiom of choice, we can easily construct a partition which is not Ramsey. But it is natural to ask which sets, in terms of logical complexity, are Ramsey. This problem was first posed by DANA SCOTT (unpublished) in the mid-1960's.

After GALVIN-PRIKRY [3] proved that all Borel sets are Ramsey, SILVER [7] has given a complete answer to the problem, by showing in ZF + DC that (i) Σ_1^1 sets are Ramsey, (ii) the statement of Δ_2^1 Ramseyness contradicts with GÖDEL's axiom $V = L$, and (iii) if there is a measurable cardinal, then $\mathbf{\Sigma}_2^1$ sets are Ramsey. Recall some notation. A set is $\mathbf{\Sigma}_1^1$ (or *analytic*) iff it is a projection of a Borel ($\mathbf{\Delta}_1^1$) set iff it is defined by a Σ_1^1 formula with parameters. A set is $\mathbf{\Sigma}_2^1$ iff it is a projection of a co-analytic ($\mathbf{\Pi}_1^1$) set iff it is defined by a Σ_2^1 formula with parameters. A set is Δ_2^1 iff it can be defined both by a Σ_2^1 formula and a Π_2^1 formula without parameter. For more details, see MOSCHOVAKIS [6]. A nice exposition of SILVER's theorem can be found in BOLLOBAS [2].

In [4], KASTANAS shows that the Ramsey property of a partition P can be deduced from the determinacy of a certain infinite game of the same logical complexity as P. Although his proof has many interesting points, it does not provide an alternative proof of SILVER's theorem since analytic determinacy over the reals is not provable in ZF + DC. So we improve his game construction by an unfolding trick. Our game is similar to an asymmetric game of KECHRIS [5] which KASTANAS might try to use at the end of the paper, but it is actually more elementary. We will prove analytic Ramseyness from Σ_2^0 determinacy over the reals, and generally Σ_{n+1}^1 Ramseyness from Σ_n^1 determinacy over the reals. Since Σ_2^0 determinacy over the reals is provable in ZF, this gives another proof of SILVER's theorem.

We will treat only the lightface statements, since the boldface versions (including parameters) are straightforward from them by the usual relativization argument. Let P be a Σ_1^1 subset of $[\omega]^\omega$. Then there exists a Σ_2^0 formula $\varphi(f, X)$ such that $P(X) \leftrightarrow \exists f \in 2^\omega \varphi(f, X)$. We define the game G_P as follows:

I	II
d_0, A_0	
	n_0, B_0
d_1, A_1	
	n_1, B_1
$\vdots$	$\vdots$

The rules of G_P are

(i) $A_0 \in [\omega]^\omega$, $A_{i+1} \in [B_i]^\omega$, and $d_i = 0$ or 1,

(ii) $B_i \in [A_i]^\omega$, $n_i \in A_i$, and $n_i < b$ for all $b \in B_i$.

The first person who disobeys one of the above rules loses the game. When all the rules are obeyed, player I wins iff $\varphi(f, H)$ holds, where $f(i) = d_i$ for all $i \in \omega$ and $H = \{n_0, n_1, \ldots\}$. Thus G_P is a Σ_2^0 game and I is a Π_2^0 player.

For a Σ_{n+1}^1 partition P, we also define the game G_P in the same way. Supposing $P(X) \leftrightarrow \exists f \in 2^\omega \varphi(f, X)$ with $\varphi \in \Pi_n^1$, player I wins iff $\varphi(f, H)$ holds, and so G_P is a Σ_n^1 game and I is a Π_n^1 player.

Regarding a Σ_n^1 partition P $(n \geqq 1)$ and its associated game G_P, we have

Theorem.

(a) I *has a winning strategy in* $G_P \Rightarrow \exists H \in [\omega]^\omega \, \forall X \in [H]^\omega P(X)$,

(b) II *has a winning strategy in* $G_P \Rightarrow \forall A \in [\omega]^\omega \, \exists H \in [A]^\omega \, \forall X \in [H]^\omega \neg P(X)$.

As a corollary to the above theorem, we have

Corollary. (a) *Every analytic set is Ramsey.* (b) *Σ_n^1-determinacy over the reals implies Σ_{n+1}^1-Ramseyness. In particular, if there is a measurable cardinal* $(\geqq 2^{\aleph_0})$, *every Σ_2^1 set is Ramsey.*

Note that WOLFE's proof of Σ_2^0-determinacy and MARTIN's proof of analytic determinacy (based on a measurable cardinal) both can be carried out for the games over the reals as well as the natural numbers (see MOSCHOVAKIS [6]). Now we prove the theorem.

Proof. (a) Let σ be a winning strategy for I. We will construct an infinite set H such that for each $X = \{x_0, x_1, \ldots\} \in [H]^\omega$, there is a play $(d_0, A_0)^\frown(x_0, B_0)^\frown(d_1, A_1)^\frown(x_1, B_1)^\frown\ldots$ which is consistent with σ, i.e.

$$(d_i, A_i) = \sigma((d_0, A_0)^\frown(x_0, B_0)^\frown\ldots^\frown(x_{i-1}, B_{i-1})) \quad \text{for all } i \in \omega.$$

Since I wins at this play, we have $\varphi(f, X)$, where $f(i) = d_i$ for all $i \in \omega$, and so $P(X)$ holds.

If a partial play $(d_0, A_0)^\frown(n_0, B_0)^\frown\ldots^\frown(n_{i-1}, B_{i-1})^\frown(d_i, A_i)$ is consistent with σ, we say that it *realizes* a sequence $(n_0, n_1, \ldots, n_{i-1})$. To construct a set $H = \{n_0, n_1, \ldots\}$ of the above property, we simultaneously build an ω-sequence of finite trees $T_0 \subseteqq T_1 \subseteqq \ldots$ such that for each $i \in \omega$, T_i consists of certain σ-consistent partial plays extending plays in T_{i-1} and every subset of $n_0, n_1, \ldots, n_{i-1}$ is realized in a partial play in T_i. Once such T_i's are built, it is clear that for each $X \in [H]^\omega$ there is a path through $\bigcup T_i$ generates (realizes) X.

We now show the inductive c…
I's first move given by σ. Put T_0…
the induction step, we assume…
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in T_i end with (d, A) such that…
ments of T_i. Let n_i be the least…

$Y_0 = X_i - \{n_i\}, \quad (d_j,$…

Then we define T_{i+1} as follows:

$T_{i+1} = T_i \cup \{p_j{}^\frown(n_i, Y_j)$…

It is obvious that any subset of…
that all the partial plays in T_{i+}…
of (a).

(b) The basic idea of the follo…
need some extra treatment for th…
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finite set H such that for each…
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We here say that a τ-consist…
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$H = \{n_0, n_1, \ldots\}$ together with a…
$i \in \omega$, T_i consists of some τ-cons…
each subset s of $n_0, n_1, \ldots, n_{i-1}$…
s there is a partial play in T_i whi…
each $X \in [H]^\omega$ and for each f…
pair (X, f).

Before the construction of su…

Lemma. (cf. KASTANAS' σ_∞ L…
p with the last move (n, B) such…
and d (= 0 or 1) there exist X a…

Proof of the Lemma. We…

$(m_0, Y_0) = \tau(p^\frown(0, B)$…

Then put $Y_\infty = \{m_0, m_1, \ldots\}$. Ne…

$(m'_0, Y'_0) = \tau(p^\frown(1, Y_\infty$…

Then put $Y'_\infty = \{m'_0, m'_1, \ldots\}$. C…

We are now back to the con…
realizes the pair of the empty…
$\{n_0, n_1, \ldots, n_{i-1}\}$ and $T_0 \subseteqq T_1 \subseteqq$…
there is an infinite set C_i such t…
Let $\{p_0, p_1, \ldots, p_{k-1}\}$ be an enu…

We now show the inductive construction of $H = \{n_0, n_1, \dots\}$ and T_i's. Let (d_0, A_0) be player I's first move given by σ. Put $T_0 = \{(d_0, A_0)\}$. The empty sequence is realized by (d_0, A_0). For the induction step, we assume that $\{n_0, n_1, \dots, n_{i-1}\}$ and $T_0 \subseteqq T_1 \subseteqq \dots \subseteqq T_i$ have been constructed, and additionally assume that there is an infinite set X_i such that all the partial plays in T_i end with (d, A) such that $X_i \subseteqq A$. Let $\{p_0, p_1, \dots, p_{k-1}\}$ be an enumeration of the elements of T_i. Let n_i be the least element of X_i. We define d_j $(j < k)$ and Y_j $(j \leqq k)$ by

$$Y_0 = X_i - \{n_i\}, \qquad (d_j, Y_{j+1}) = \sigma(p_j{}^\frown(n_i, Y_j)).$$

Then we define T_{i+1} as follows:

$$T_{i+1} = T_i \cup \{p_j{}^\frown(n_i, Y_j)^\frown(d_j, Y_{j+1}) : j < k\}.$$

It is obvious that any subset of $\{n_0, n_1, \dots, n_i\}$ is realized by a play in T_{i+1}. We also notice that all the partial plays in T_{i+1} end with supersets of $X_{i+1} = Y_k$. This completes the proof of (a).

(b) The basic idea of the following proof is the same as that of part (a). However, we here need some extra treatment for the sequence $f = \{d_0, d_1, \dots\}$, which was automatically decided by I's winning strategy in part (a). Let τ be a winning strategy for II. We will construct an infinite set H such that for each $X \in [H]^\omega$ and for each $f \in 2^\omega$ there is a τ-consistent play which generates X and f. Clearly such an H is homogeneous for $\neg P(X)$.

We here say that a τ-consistent partial play $(d_0, A_0)^\frown(n_0, B_0)^\frown \dots {}^\frown(d_{i-1}, A_{i-1})^\frown(n_{i-1}, B_{i-1})$ *realizes* the pair of sequences $(n_0, n_1, \dots, n_{i-1})$ and $(d_0, d_1, \dots, d_{i-1})$. We construct $H = \{n_0, n_1, \dots\}$ together with an ω-sequence of finite trees $T_0 \subseteqq T_1 \subseteqq \dots$ such that for each $i \in \omega$, T_i consists of some τ-consistent partial plays extending plays in T_{i-1}, and such that for each subset s of $n_0, n_1, \dots, n_{i-1}$ and for each sequence d of 0's and 1's with the same length as s there is a partial play in T_i which realizes the pair (s, d). If we have such H and T_i's, then for each $X \in [H]^\omega$ and for each $f \in 2^\omega$ there is a path through $\bigcup T_i$ generating (realizing) the pair (X, f).

Before the construction of such H and T_i's, we prove the following lemma:

Lemma. (cf. KASTANAS' σ_∞ Lemma [4]). *Let C be an infinite subset of ω. For every partial play p with the last move (n, B) such that $B \supseteqq C$ there is an infinite set $A \subseteqq C$ such that for every $m \in A$ and d $(= 0$ or $1)$ there exist X and Y such that $\tau(p^\frown(d, X)) = (m, Y)$ and $Y \supseteqq A - \{m\}$.*

Proof of the Lemma. We first define the sequence of pairs (m_i, Y_i) as follows:

$$(m_0, Y_0) = \tau(p^\frown(0, B)), \qquad (m_{i+1}, Y_{i+1}) = \tau(p^\frown(0, Y_i)), \quad \text{for } i \in \omega.$$

Then put $Y_\infty = \{m_0, m_1, \dots\}$. Next define the sequence of pairs (m'_i, Y'_i) as follows:

$$(m'_0, Y'_0) = \tau(p^\frown(1, Y_\infty)), \qquad (m'_{i+1}, Y'_{i+1}) = \tau(p^\frown(1, Y'_i)), \quad \text{for } i \in \omega.$$

Then put $Y'_\infty = \{m'_0, m'_1, \dots\}$. Clearly, $A = Y'_\infty$ satisfies the lemma.

We are now back to the construction of H and T_i's. Let $T_0 = \{\varnothing\}$. The empty sequence $\varnothing$ realizes the pair of the empty sequences $(\varnothing, \varnothing)$. For the induction step, we assume that $\{n_0, n_1, \dots, n_{i-1}\}$ and $T_0 \subseteqq T_1 \subseteqq \dots \subseteqq T_i$ have been constructed, and additionally assume that there is an infinite set C_i such that all the partial plays in T_i end with (n, Y) such that $C_i \subseteqq Y$. Let $\{p_0, p_1, \dots, p_{k-1}\}$ be an enumeration of the elements of T_i.

We then apply the above lemma repeatedly as follows: let A_0 be a set obtained from C_i and p_0 in the lemma, and A_1 a set obtained from A_0 and p_1 in the lemma, ..., and A_{k-1} a set obtained from A_{k-2} and p_{k-1} in the lemma. Then let n_i be the least element of A_{k-1}, and $C_{i+1} = A_{k-1} - \{n_i\}$. By the lemma, there exist X_j, Y_j, X'_j, and Y'_j $(j < k)$ such that all of them are supersets of C_{i+1} and

$$\tau(p_j{}^\frown(0, X_j)) = (n_i, Y_j), \qquad \tau(p_j{}^\frown(1, X'_j)) = (n_i, Y'_j), \text{ for } j < k.$$

Finally, we define T_{i+1} as follows:

$$T_{i+1} = T_i \cup \{p_j{}^\frown(0, X_j)^\frown(n_i, Y_j) \;:\; j < k\} \cup \{p_j{}^\frown(1, X'_j)^\frown(n_i, Y'_j) \;:\; j < k\}$$

Obviously T_{i+1} satisfies all the required conditions. This completes the proof. □

Recently, BLASS [1] also stresses a connection between partitions and games, though he does not establish such an effective relation as in this paper.

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AN ELEMENTARY SYSTEM
AND ITS SEMI-COMPLETEN

by QIN JUN in Tempe, Arizona (

Abstract

The author establishes an e
constant 0 and then proves
theory of systems of inequal
MSC: 03F30, 03B25.
Key words: recursive arithm
tem of equations.

§ 0. Introduction

The principal purpose of thi
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tem. Since our proof is effecti

An *elementary system* is an a
elementary rules about equati
tions over the natural numbers
to be *semi-complete* if any true
elementary system is *complete*
MOH SHAWKWEI and SEN BAIY
NA, NM, AM, NAM and a co
decision procedures for thos
N. GEORGIEVA proved the deci
metic, but no elementary syste

In section 1, we first develo
to be used in section 6 to pro
blish our system AS and list s
theorems and rules will be p
theorems about system AS.

In this paper "$\alpha = \langle(a)\rangle\beta$" is
or replacement, and "$\alpha = \langle(a);$
is obtained by applying rule (a
refer readers to [4], [5]. But ba

[1]) The author is very grateful to P