Algorithms for Maximal Ind. Set

Exposition by William Gasarch
Credit Where Credit is Due

This talk is based on parts of the **AWESOME** book

**Exact Exponential Algorithms**

by

**Fedor Formin and Dieter Kratsch**
What is Maximum Ind Set?

**Definition:** If $G = (V, E)$ is a graph then $I \subseteq V$ is an Ind. Set if $(\forall x, y \in V)[(x, y) \notin E]$. The set $I$ is a MAXIMUM IND SET if it is an Ind Set and there is NO ind set that is bigger.

**Goal:** Given a graph $G$ we want the SIZE of the Maximum Ind. Set. Obtaining the set itself will be an easy modification of the algorithms which we will omit.

**Abbreviation:** MIS is the Maximum Ind Set problem.

**BILL** - Do examples and counterexamples on the board.
Why Do We Care About MIS?

1. MIS is NP-complete.
2. MIS comes up in applications (so my friends in systems tell me).
OUR GOAL

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If we had $1,000,000 then we wouldn't have to worry about whether the REU grant gets renewed.

2. We will show algorithms for MIS that
2.1 Run in time $O(\alpha n)$ for various $\alpha < 1$. NOTE: By $O(\alpha n)$ we really mean $O(p(n)\alpha n)$ where $p$ is a poly. We ignore such factors.
2.2 Quite likely run even better in practice.
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If all of the degrees are $\leq 2$ then the problem is EASY. BILL- HAVE THEM DO THIS.
If $G = (V, E)$ is a graph and $v \in V$ then

$$N[v] = \{v\} \cup \{u \mid (v, u) \in E\}.$$

The NEIGHBORS of $v$ AND $v$ itself.
MIN DEG ALGORITHM

ALG($G = (V, E)$: A Graph)

$v = \text{vertex of min degree}
\text{for } u \in N[v]
\quad m_u = ALG(G - N[m_u])
\quad m = \min\{m_u \mid u \in N[v]\}.
\text{RETURN}(1 + m)

BILL: TELL CLASS TO FIGURE OUT WHY WORKS.
Let $N[v] = \{v, x_1, \ldots, x_{d(v)}\}$.

\[
T(n) \leq 1 + T(n - d(v) - 1) + \sum_{i=1}^{d(v)} T(n - d(x_i) - 1)
\leq 1 + T(n - d(v) - 1) + \sum_{i=1}^{d(v)} T(n - d(v) - 1)
\leq 1 + (d(v) + 1) T(n - (d(v) + 1))
\]

BILL: HAVE CLASS ANALYSE $T(n) = 1 + sT(N - s)$. THEN DO ON BOARD.
HOW GOOD?

1. Runs in $T(n) = O((3^{1/3})^n) \leq O((1.42)^n)$.
2. Works well on high degree graphs until they become low degree graphs.
4. Makes more sense to take High degree nodes.
MAX DEG ALG

ALG(G)

1. If (∃v)[d(v) = 0] then RETURN(1 + ALG(G − v)).
2. If (∃v)[d(v) = 1] then RETURN(1 + ALG(G − N[v])).
3. If (∀v)[d(v) ≤ 2] then CALL 2-MIS ALG.
4. If (∃v)[d(v) ≥ 3] then
   4.1 Let v* be of max degree
   4.2 Return MAX of 1 + ALG(G − N[v*]), ALG(G − v*).

BILL- HAVE CLASS DISCUSS WHY WORKS.
ANALYSIS

\[ T(n) \leq T(n - d(v) - 1) + T(n - 1) \]
\[ T(n) \leq T(n - 4) + T(n - 1) \]

Guess \( T(n) = \alpha^n \)

\[ \alpha^n = \alpha^{n-4} + \alpha^{n-1} \]

\[ \alpha^4 = 1 + \alpha \]

\[ \alpha^4 - \alpha - 1 = 0 \]

\( \alpha \sim 1.38. \)
HOW GOOD?

1. Runs in $T(n) = O((1.38)^n)$.
2. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.
3. WORKS really well in practice.
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It works in practice— can we make it work in theory?
NEED TO MEASURE progress better.

1. We measure a node of degree \( \leq 1 \) as having weight ZERO.
2. We measure a node of degree 2 as having weight \( \frac{1}{2} \).
3. We measure a node of degree \( \geq 3 \) as having weight ONE.

SO we view \( |V| \) as

\[
\frac{1}{2} \left( \text{number of verts of degree 2} \right) + \left( \text{number of verts of degree 3} \right)
\]

We still refer to this as \( n \).
Have picked $v^*$.

1. Assume there are no vertices of degree $\leq 1$ (else would not be in $v^*$ case)
2. Assume $v^*$ has $d_2$ vertices of degree 2.
3. Assume $v^*$ has $d_3$ vertices of degree 3.
4. Assume $v^*$ has $d_{\geq 4}$ vertices of degree $\geq 4$. 
**BETTER ANALYSIS OF $G - N[v]$ CASE**

$G - N[v^*]$:  
1. Loss of $v^*$ is loss of 1.  
2. Loss of $d_2$ vertices of degree 2: Loss is $\frac{d_2}{2}$.  
3. Loss of $d_3$ vertices of degree 3: Loss is $d_3$.  
4. Loss of $d_{\geq 4}$ vertices of degree $\geq 4$: Loss is $d_{\geq 4}$.  

Total Loss: $1 + \frac{d_2}{2} + d_3 + d_{\geq 4}$.  

Work to do:  
\[
T(n - (1 + \frac{d_2}{2} + d_3 + d_{\geq 4}))
\]
**Better Analysis of** $G - v$ **Case**

$G - v^*$:

1. Loss of $v^*$ is loss of 1.
2. The $d_2$ verts of deg 2 become $d_2$ verts of deg $\leq 1$. Loss is $\frac{d_2}{2}$.
3. The $d_3$ verts of deg 3 become $d_3$ verts of deg $\leq 2$. Loss is $\frac{d_3}{2}$.
4. The $d_{\geq 4}$ verts of deg $\geq 4$. No Loss.

Total Loss: $1 + \frac{d_2}{2} + \frac{d_3}{2}$.

Work to do:

$$T(n - (1 + \frac{d_2}{2} + \frac{d_3}{2}))$$
TOTAL ANALYSIS

\[ T(n) \leq T(n - (1 + \frac{d_2}{2} + d_3 + d_{\geq 4})) + T(n - (1 + \frac{d_2}{2} + \frac{d_3}{2})) \]
\[ \leq T(n - 1) + T(n - (1 + d_2 + \frac{3d_3}{2} + d_{\geq 4})) \]
\[ \leq T(n - 1) + T(n - (d(v^*) + 1)) \]

1. If \( d(v^*) \geq 4 \) then get

\[ T(n) \leq T(n - 1) + T(n - 5) \]

BILL- HAVE STUDENTS DO.

2. If \( d(v^*) = 3 \) then BILL- HAVE STUDENTS DO.
HOW GOOD?

1. Runs in $T(n) \leq O((1.3248)^n)$.

2. Using cleverer choice of weights can get $O((1.2905)^n)$. (Deg2 nodes weigh 0.596601, Deg3 nodes weigh 0.928643, Deg4 nodes weigh 1.)

3. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.

4. WORKS really well in practice, and this analysis may say why.
Best known runs in time $O((1.2109)^n)$.

1. Order constant is REASONABLE.
2. LOTS of cases depending on degree.
3. Sophisticated analysis.
4. Good in practice? A project for NEXT YEARS REU!!!