Algorithms for Maximal Ind. Set

Exposition by William Gasarch

Credit Where Credit is Due

This talk is based on parts of the **AWESOME** book

Exact Exponential Algorithms
by
Fedor Formin and Dieter Kratsch

What is Maximum Ind Set?

Definition: If $G = (V, E)$ is a graph then $I \subseteq V$ is an *Ind. Set* if $(\forall x, y \in V)[(x, y) \notin E]$. The set I is a MAXIMUM IND SET if it is an Ind Set and there is NO ind set that is bigger.

Goal: Given a graph G we want the SIZE of the Maximum Ind. Set. Obtaining the set itself will be an easy modification of the algorithms which we will omit.
Abbreviation: MIS is the Maximum Ind Set problem.

BILL - Do examples and counterexamples on the board.

Why Do We Care About MIS?

1. MIS is NP-complete.
2. MIS comes up in applications (so my friends in systems tell me).

OUR GOAL

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2. We will show algorithms for MIS that
   2.1 Run in time $O(\alpha^n)$ for various $\alpha < 1$. NOTE: By $O(\alpha^n)$ we really mean $O(p(n)\alpha^n)$ where p is a poly. We ignore such factors.
   2.2 Quite likely run even better in practice.

2MIS

If all of the degrees are ≤ 2 then the problem is EASY.
BILL- HAVE THEM DO THIS.

IMPORTANT DEFINITION

If $G = (V, E)$ is a graph and $v \in V$ then

$$N[v] = \{v\} \cup \{u \mid (v, u) \in E\}.$$

The NEIGHBORS of v AND v itself.

MIN DEG ALGORITHM

ALG($G = (V, E)$: A Graph)

$v =$ `vertex of min degree`
`for` $u \in N[v]$
$m_u = ALG(G - N[m_u])$
$m = \min\{m_u \mid u \in N[v]\}$.
RETURN$(1 + m)$

BILL: TELL CLASS TO FIGURE OUT WHY WORKS.

Analysis

Let $N[v] = \{v, x_1, \ldots, x_{d(v)}\}$.

$$
\begin{aligned}
T(n) &\le 1 + T(n - d(v) - 1) + \sum_{i=1}^{d(v)} T(n - d(x_i) - 1) \\
&\le 1 + T(n - d(v) - 1) + \sum_{i=1}^{d(v)} T(n - d(v) - 1) \\
&\le 1 + (d(v)+1)T(n - (d(v)+1))
\end{aligned}
$$

BILL: HAVE CLASS ANALYSE $T(n) = 1 + sT(N - s)$. THEN DO ON BOARD.

HOW GOOD?

1. Runs in $T(n) = O((3^{1/3})^n) \leq O((1.42)^n)$.
2. Works well on high degree graphs until they become low degree graphs.
3. Upshot: Would not use in practice.
4. Makes more sense to take High degree nodes.

MAX DEG ALG

ALG(G)

1. If $(\exists v)[d(v) = 0]$ then RETURN($1 + ALG(G - v)$.
2. If $(\exists v)[d(v) = 1]$ then RETURN($1 + ALG(G - N[v])$).
3. If $(\forall v)[d(v) \le 2]$ then CALL 2-MIS ALG.
4. If $(\exists v)]d(v) \ge 3]$ then
   4.1 Let v^* be of max degree
   4.2 Return MAX of $1 + ALG(G - N[v^*])$, $ALG(G - v^*)$.

BILL- HAVE CLASS DISCUSS WHY WORKS.

ANALYSIS

$$\begin{aligned} T(n) &\leq T(n - d(v) - 1) + T(n - 1) \\ T(n) &\leq T(n - 4) + T(n - 1) \end{aligned}$$

Guess $T(n) = \alpha^n$

$$\alpha^n = \alpha^{n-4} + \alpha^{n-1}$$

$$\alpha^4 = 1 + \alpha$$

$$\alpha^4 - \alpha - 1 = 0$$

$\alpha \sim 1.38.$

HOW GOOD?

1. Runs in $T(n) = O((1.38)^n)$.
2. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.
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It works in practice— can we make it work in theory?

BETTER ANALYSIS

Need to MEASURE progress better.

1. We measure a node of degree ≤ 1 as having weight ZERO.
2. We measure a node of degree 2 as having weight $\frac{1}{2}$.
3. We measure a node of degree ≥ 3 as having weight ONE.

SO we view $|V|$ as

$$\frac{1}{2}(\text{number of verts of degree 2}) + (\text{number of verts of degree 3})$$

We still refer to this as n.

BETTER ANALYSIS

Have picked v^*.

1. Assume there are no vertices of degree ≤ 1 (else would not be in v^* case)
2. Assume v^* has d_2 vertices of degree 2.
3. Assume v^* has d_3 vertices of degree 3.
4. Assume v^* has $d_{\geq 4}$ vertices of degree ≥ 4.

BETTER ANALYSIS OF $G - N[v]$ CASE

$G - N[v^*]$:

1. Loss of v^* is loss of 1.
2. Loss of d_2 vertices of degree 2: Loss is $\frac{d_2}{2}$.
3. Loss of d_3 vertices of degree 3: Loss is d_3.
4. Loss of $d_{\geq 4}$ vertices of degree ≥ 4: Loss is $d_{\geq 4}$.

Total Loss: $1 + \frac{d_2}{2} + d_3 + d_{\geq 4}$.
Work to do:

$$T(n - (1 + \frac{d_2}{2} + d_3 + d_{\geq 4}))$$

BETTER ANALYSIS OF $G - v$ CASE

$G - v^*$:

1. Loss of v^* is loss of 1.
2. The d_2 verts of deg 2 become d_2 verts of deg ≤ 1. Loss is $\frac{d_2}{2}$.
3. The d_3 verts of deg 3 become d_3 verts of deg ≤ 2. Loss is $\frac{d_3}{2}$.
4. The $d_{\geq 4}$ verts of deg ≥ 4. No Loss.

Total Loss: $1 + \frac{d_2}{2} + \frac{d_3}{2}$.
Work to do:

$$T(n - (1 + \frac{d_2}{2} + \frac{d_3}{2}))$$

TOTAL ANALYSIS

$$\begin{aligned} T(n) &\le T(n-(1+\tfrac{d_2}{2}+d_3+d_{\ge 4}))+T(n-(1+\tfrac{d_2}{2}+\tfrac{d_3}{2})) \\ &\le T(n-1)+T(n-(1+d_2+\tfrac{3d_3}{2}+d_{\ge 4})) \\ &\le T(n-1)+T(n-(d(v^*)+1)) \end{aligned}$$

1. If $d(v^*) \ge 4$ then get

$$T(n) \le T(n-1)+T(n-5)$$

   BILL- HAVE STUDENTS DO.

2. If $d(v^*) = 3$ then BILL- HAVE STUDENTS DO.

HOW GOOD?

1. Runs in $T(n) \leq O((1.3248)^n)$.
2. Using cleverer choice of weights can get $O((1.2905)^n)$. (Deg2 nodes weigh 0.596601, Deg3 nodes weigh 0.928643, Deg4 nodes weigh 1.)
3. Works well on high degree graphs until they become low degree graphs. But better than Min-Degree alg.
4. WORKS really well in practice, and this analysis may say why.

BEST KNOWN

Best known runs in time

$$O((1.2109)^n).$$

1. Order constant is REASONABLE.
2. LOTS of cases depending on degree.
3. Sophisticated analysis.
4. Good in practice? A project for NEXT YEARS REU!!!!