Review of
Tales of Impossibility:
The 2000-Year quest to Solve the Mathematical Problems of Antiquity[1]
Author: David Richeson
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1 The Problems of Antiquity

This book is about the 4 Greek problems of Antiquity. I will state them in a way that is absolutely whiggish.

By *construction* we mean *construction with just a straightedge and compass.*

1. (Trisecting an angle) Prove or disprove the following: Given an angle θ it is possible to construct an angle that is $\frac{\theta}{3}$.
2. (Doubling the cube) Prove or disprove the following: Given a line segment of length x it is possible to construct a line segment of length $x2^{1/3}$.
3. (Squaring the circle) Prove or disprove the following: Given a circle C it is possible to construct a square S such that the area of C and S are the same.
4. Determine for which n it is possible to construct the regular n-gon.

The above description is whiggish for the following reasons:

1. I used algebraic notation. They would have stated it all in terms of geometry. For example, doubling the cube might would have been stated: *given an edge of a cube construct an edge of a second cube whose volume is twice that of the original.*
2. They would not have stated the question so precisely as *prove or disprove.*

The book under review is about the progress made on these four problems from 400BC until it was shown that the first three were impossible, and the fourth was solved, in the 1800's. There are many twists and turns and other issues that arise. What is of more interest than these four problems is how math itself has changed.

Paul Erdos said of the Collatz conjecture

Mathematics is Not Ready for Such Problems.

That may or may not be true (thats a tautology). Now that we know the solution to the four problems of antiquity, we can say with authority that, when they were posed,

Mathematics was Not Ready for Such Problems.

Realize how far math had to go: Algebra was in its infancy, the concept of a number was barely understood (e.g., negative numbers and irrationals were suspect), and the notation was awful. That last point is not trivial: good notation and good ideas go hand in hand, with one inspiring the other.

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2 Summary of Contents

This book has 21 chapters and 21 tangents (each chapter is followed by a tangent). To summarize all of them in a book review of this length would be as absurd as squaring the circle. So I will talk about types of chapters.

2.1 Variations

Some of the chapters go over, and vary, the ground rules for what it means to *construct*. For proving a construction impossible this is quite important. Its not good enough to say *I could not square the circle, hence it can't be done* (though if Gauss said this, it would be good evidence that one cannot square the circle).

The most important ground rule, and the one ignored by cranks, is that there *are* ground rules. If you trisect an angle with a straightedge, compass, and protractor *you have not solved a 2000 year old math problem.*

The following variations of the ground rules are discussed:

- What if you have a straightedge and compass, but the compass is fixed at one angle (a *Rusty Compass*)? What if the angle can be of your choice? What if its arbitrary?
- What if you allow a two mark on your straightedge?
- What if you allow other mechanical devices?
- What if you only use a straightedge?
- What if you only use a compass?

2.2 How Math has Changed

How mathematics changed over the centuries is not really the subject of any one chapter; however, it permeates the entire book. I give two examples.

I) Connection to Geometry.

I present this as a conversation between Darling and me.

BEGIN CONVERSATION

Bill: Do you have any objection to the equation: $x^2 + x = 10$?

Darling: No. Why should I? Is this a trick question?

Bill: You are a 21st century person who correctly has no problem with that expression. But Viete and others in the 1600's though of Algebra as being so tightly connected to Geometry that since x is a *length* and x^2 is an *area*, $x^2 + x$ makes no sense.

Darling: That reminds me of the controversy over the definition of functions. When Math and Physics were more tied together, functions that were not differential were shunned since it was thought they had no real world counterpart.

Bill: I am glad to be a 21st century person.

END CONVERSATION

When it became apparent that algebra would be needed for mathematics people accepted it but didn't like it. Here is a quote. I will tell you who said it at the end of this review, but see if

you can guess. That may be to hard for you to guess, so try to guess when it was said, and what kind of person said it.
BEGIN QUOTE

Algebra is to the geometer what you might call the "Faustian offer"... Algebra is the offer made by the devil to the mathematician. The devil says "I will give you this powerful machine, it will answer any question, you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."... Of course, we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about meaning.

END QUOTE

II) Rigor. Later in this review I will talk about what Descartes did mathematically. But a prelude to that in the book warns us about viewing it with modern eyes. Here is a quote:

Unfortunately, his (Descartes) Geometry isn't set up in the modern way, with clearly articulated definitions, carefully stated theorems, and rock-solid proofs.

2.3 History of Mathematics

Many mathematicians grace the pages of this book.

1. In the 1630's Descartes made massive progress on the problems by (1) realizing that the impossibility is something that might be provable, (2) translating the problem into algebra, and (3) in modern notation he showed that the set of numbers that are are constructible is the field formed by taking the rationals and closing under $+$, $-, \times$, $\div$ and taking square roots. Later mathematicians and historians claimed (incorrectly) that Descartes showed trisecting the angle and doubling the cube were not possible. Descartes own claims on this are muddled.

2. In 1796, when Gauss was 19, he constructed a 17-gon (I wish he had done it two years earlier!). He also showed that if $n = 2^a p_1 \cdots p_k$, where the p_i's are Fermat primes then the n-gone is constructible. He didn't give the construction. He showed, using Descartes work, that it was possible. (In 1894 Johann Gustav Hermes completed the construction of the 65537-gone. It was 200 pages and took 10 years. From a 21st century prospective I cannot understand why he did that.) Some mathematicians thought that Gauss proved the converse, which he did not.

3. In 1837 Wantzel published a 7 page article showing that (1) trisecting the angle is impossible, (2) doubling the cube is impossible, (3) the n-gon is constructible IFF $n = 2^a p_1 \cdots p_k$ where the p_i's are distinct Fermat primes. He solved three of the four problems of antiquity! Problems that had been open for around 2000 years! Yet his paper was met with a deafening silence. Why? (a) Some thought Descartes had already proved it. (b) Some thought Gauss had already proved it. (c) Some thought this was already proven and all Wantzel did was write up up formally. (d) Some thought it was one of those things that everyone kind of knows is true, but doesn't really need a proof. This last point may be the most important and was the theme of this article on Wantzel:

   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/wantzel.pdf`

To drive home this last point I will tell you a story that was also one of my motivations to read the book. At the 12th Gathering for Gardner the author gave a talk on 12 ways to trisect an angle using straightedge, compass, and just-one-more-thing. Many of them predate the proof that it is impossible to trisect an angle with just straightedge and compass. The talk is here:

`https://www.youtube.com/watch?v=5VxMUhxkBqA&t=420s`

I asked him afterwards *did the people who trisected the angle using straightedge, compass, and just-one-more-thing have the hope of replacing that one-more-thing with a straightedge and compass?*. He responded *No, people pretty much knew that it was impossible.*

This shows a different way of thinking about math. Viewing the problems of antiquity in the way I began this review, as a *prove or disprove* question, is a modern viewpoint. The notion that one can show a construction cannot be done is a modern viewpoint (even though Descartes had some idea of this). Hence, to celebrate Wantzel as having solved an open problem is a modern viewpoint.

4. In 1837 Hermite showed that e is transcendental. Lindemann talked to Hermite about his proof and nine years later Lindemann showed that π was transcendental, and hence the problem of squaring the circle was shown to be impossible. How did Hermite feel about this? IF he was angry or thought he was unfairly scooped then we would know about it. E.T. Bell would have had some absurd exaggeration of it in his book *Men of Mathematics*. But no. Hermite was quite happy with how things turned out.

The book closes with a chapter asking if the quest to show these problems impossible was a siren (dangerous, leading people astray) or a muse (guiding people to math of interest). Clearly the author comes down on the side of muse.

3 Opinion

Who should read this book? For the math-inclined there are plenty of constructions (many using more than a straightedge and compass) which are probably new to the reader and interesting. To the non-math inclined there is still some interesting points *about* math. I was especially intrigued by how math changed over the centuries.

4 Who Said the Quote?

Clearly the quote was said by someone at the time when math was transitioning from geometry to algebra. Also note that modern people do not invoke *the devil* in their discussions.

The above paragraph is incorrect. The quote is by Sir Michael Atiyah who won the Fields medal in 1966 and the Abel Prize in 2004. He said this in 2001. He passed away in 2019 and hence can be considered a 21st century person.