Review [1] of
The New Mathematical Coloring Book:
Mathematics of Coloring and the Colorful Life of its Creators
Second Edition
by Alexander Soifer
Springer, 2024
889 pages
EBook $169, Hardcover $220

Review by
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1 The NEW Mathematical Coloring Book

In 2009 Alexander Soifer published *The Mathematical Coloring Book*. I will refer to this book by TMCB. TMCB is around 600 pages. I reviewed it for SIGACT News here:

`https://www.cs.umd.edu/~gasarch/bookrev/40-3.pdf`.

In 2024 Alexander Soifer published *The New Mathematical Coloring Book*. I will refer to this book by TNMCB. TNMCB is around 900 oversized pages.

In this review I first republish the first two sections of my review of TMCB (with a few comments added), then summarize what's common to both books, and what's new in TNMCB. I will then render an opinion.

Begin Excerpt from Prior Review

2 Introduction

I first had the pleasure of meeting Alexander Soifer at one of the *Southeastern International Conferences on Combinatorics, Computing, and Graph Theory*. If I was as careful a historian as he is, I would know which one. Over lunch he told me about van der Waerden's behavior when he was living as a Dutch Citizen in Nazi Germany. Van der Waerden later claimed that he opposed the firing of Jewish professors. Soifer explained to me that in 1933 the German government passed a law requiring universities to fire all Jewish professors *unless they were veterans of WW I* (there were other exceptions also). Van der Waerden protested that veterans were being fired, in violation of the law. So he was objecting to *the law not being carried out properly* and not to *the law itself*. Alex told me that the full story would soon appear in a book he was writing on Coloring Theorems. I couldn't tell if the book would be a math book or a history book. It is both.

BEGIN Added Comment

Both TMCB and TNMCB have a lot about van der Waerden and his actions under the Nazi regime. Alexander Soifer has a separate book on that, *The Scholar and the State* [2]. I reviewed that book for SIGACT News here:

`https://www.cs.umd.edu/~gasarch/bookrev/FRED/vdwhistory.pdf`

END Added Comment

The second time I met Alex was at the next *Southeastern International Conference on Combinatorics, Computing, and Graph Theory.* Alex gave a talk on the following:

1. Prove that for any 2-coloring of the plane there are two points an inch apart that are the same color. (This is easy: I was able to do it in 2 minutes.)

2. Prove that for any 3-coloring of the the plane there are two points an inch apart that are the same color. (This is easy: I was able to do it in 3 minutes.)

   BEGIN Added Comment

   D.N.J. de Grey showed, in 2018, that for any 4-coloring of the plane there are two points an inch apart that are the same color. This proof requires the use of a computer. (This is hard: I was unable to do it in n minutes for all $n \in \mathsf{N}$.)

   END Added Comment

3. Prove that there is a 7-coloring of the plane such that for all points p, q that are an inch apart, p and q are different colors. (This is easy: I was able to do it in 7 minutes.)

4. Find the number χ such that (1) for any $(\chi - 1)$-coloring of the plane there will be two points an inch apart that are the same color, and (2) there exists a χ-coloring of the plane such that for all points p, q that are an inch apart, p and q are different colors. (This is open: I was unable to do this in χ minutes.)

Alex uses the symbol χ for this quantity throughout the book; hence I will use the symbol χ for this quantity throughout the review.

The problem of determining χ is called the *Chromatic Number of the Plane Problem* and is abbreviated *CNP*. Alex told me *CNP is his favorite problem in all of mathematics.* He especially likes that the problem can be explained to a layperson, yet leads to advanced mathematical concepts.

After seeing Alex's talk I asked my colleague Clyde Kruskal what happens if only a subset of the plane is colored. For example, what is the largest square that can be 2-colored? 3-colored? Clyde then obtained full characterizations of 2 and 3-colorings for rectangles and regular polygons [1]. The paper contains the following marvelous result: an $s \times s$ square is 3-colorable iff $s \leq 8/\sqrt{65}$.

After talking to Alex I very much looked forward to his book. I first got my hands on it at the SODA (Symposium on Discrete Algorithms) conference of 2009. The Springer-Verlag book vendor let me read parts of it during the coffee break. I later got a copy and read the whole thing.

3 What Kind of Book is this?

When I first read the book I noticed something odd. The first sentence is *I recall April of 1970.* **Most of the book is written in the first person, like a memoir or autobiography!** The only parts that are not written in first person are when someone else is doing the talking.

In Alex's honor my review is written in his style.

Ordinary math books are not written in the first person; however, this is no ordinary math book! I pity the Library of Congress person who has to classify it. This book contains much math of interest and pointers to more math of interest. All of it has to do with coloring: Coloring the plane (Alex's favorite problem), coloring a graph (e.g., the four color theorem), and of course Ramsey Theory. However, the book also has biographies of the people involved and scholarly discussions of who-conjectured-what-when and who-proved-what-when. When I took Calculus the textbook had a 120-*word* passage about the life of Newton. This book has a 120-*page* passage about the life of van der Waerden.

Is this a math book? YES. Is this a book on history of Math? YES. Is this a personal memoir? YES in that the book explicitly tells us of his interactions with other mathematicians, and implicitly tells us of his love for these type of problems.

Usually I save my opinion of the book for the end. For this book, I can't wait:

This is a Fantastic Book! Go buy it Now!

BEGIN Added Comment

This comment was made of TMCB but it also holds for TNMCB.

END Added Comment

End of Excerpt of Prior Review

4 Something Old, Something New, Something Borrowed, Something k-Colored

TMCB has 11 parts and 49 chapters. TNMCB has 13 parts and 68 chapters. For a detailed description of what is in TMCB, see my review. For now, I list topics that are in both books and how many chapters each book devotes to them.

Topic	TMCB	TNMCB
Chromatic Number of the Plane	15	30
Vertex and Edge Colorings of a Graph	11	13
Ramsey Theory	10	11
History	8	9
Logic	3	3
Miscellaneous	2	2

The above chart was about *chapters*. I now discuss *Parts*. There are two new parts:

1. **Ask what your computer can do for you.** This is approximately 50 pages of completely new material. It covers the proof that $\chi \geq 5$ by de Grey and much of the work that it inspired.

2. **What About Chromatic 6?** This is approximately 30 pages of completely new material. Now that $\chi \geq 5$ is known, what about $\chi \geq 6$? This chapter does not answer that question; however, it gives many related results.

The list above does not capture the breadth and depth of TNMCB because (1) some chapters are hard to classify, and (2) some chapters are in both books but there is a lot more in TNMCB.

I describe some of the topics that are in TNMCB but not in TMCB.

4.1 The Chromatic Number of the Plane

Let χ be the least number so that there is a χ-coloring of the real plane such that no two points of the same color are an inch apart. For 68 years it was known that $4 \le \chi \le 7$. While people (including Soifer) studied variants of the problem there was no progress on the original problem.

Until 2018.

In that year D.N.J. de Grey showed that $\chi \ge 5$. The proof used a computer program. Soifer gives the history and context of the result since he had the best seat in the house to the events.

Soifer shares his opinion of what χ is. Before de Grey's result Soifer thought $\chi = 7$. He still thinks so. In fact, he has a more general conjecture. Let χ_n be the chromatic number of E^n where we connect two points iff they are an inch apart. Soifer thinks $\chi_n = 2^{n+1} - 1$. Note that this conjecture implies $\chi_2 = 7$.

4.2 Finite Sets Have A Role. Or Do They?

De Bruijn and Erdös (1951) proved that, for any graph G, G is k-colorable iff every finite subset of G is k-colorable. (Today this would be called a standard compactness argument; however, like many things it was harder then but easy now.) Hence there is a finite number of points in the plane such that χ is the chromatic number of the unit distance graph they form. This proof uses the Axiom of Choice. This result takes the book in two directions.

Unit Distance Graphs of Girth $\ge X$

Consider the following finite sets of points.

1. Three points of an equilateral triangle of side 1. This finite set shows that $\chi \ge 3$.

2. The Mosers Spindler is a 7-point set that shows $\chi \ge 4$. (Note that its *Mosers* not *Moser*. That is because two brothers, Leo and William Moser, came up with it together.) These seven points have 4 triangles of side 1.

A *Unit Distance Graph* is a set of points in the plane where if two of them are 1-apart, we put an edge between them.

Noting that the Mosers Spindle is a 4-chromatic unit distance graph with triangles, Paul Erdös, in 1975, posed the following problem:

Is there a 4-chromatic unit-distance graph of girth 4,5 or higher?

In 1979 Nicholas Wormald constructed a 4-chromatic unit-distance graph of girth 5 on 6448 vertices. In 1990 Alexander Soifer, in his journal *Geombinatorics*, asked for the smallest such graph. This led to a flurry of results by different people culminating (for a time) in the following which are described in TMCB:

1. In 1996 Paul O'Donnell and Rob Hochberg obtained a 23-node 4-chromatic unit-distance graph with girth 4. This was a joint paper.

2. In 1996 Paul O'Donnell and Rob Hochberg obtained a 45-node 4-chromatic unit-distance graph with girth 5. This was a joint paper.

TNMCB describes the following new results:

1. In 2016 Exoo-Ismailescu obtained a 17-node 4-chromatic graph with girth 4, and proved that 17 is the best possible.
2. Because of de Grey's result it now made sense to look for 5-chromatic unit-distance graphs of small order. This challenge led to a flurry of results, including results by the following two people:
   (a) In 2018 Marijn Heule had a series of results culminating in a 5-chromatic unit-distance graph on 510 vertices.
   (b) In 2020 Jaan Parts obtained a 5-chromatic unit-distance graph on 509 vertices.

The two graphs have girth 3. Soifer asks in the book for the smallest 5-chromatic unit-distance graph of girth 4.

Full details are given plus lots of color pictures of graphs. And much like de Grey's result, Soifer offers an insider view of both the math and the history.

Different Models of Set Theory

The Continuum Hypothesis is independent of ZFC (Zermelo-Frankel Set Theory with the Axiom of Choice) a system where one can do virtually all of mathematics except from some questions in logic—though we will come back to that point. Is it possible that the value of χ is independent of ZFC?

There are a few natural (we'll come back to that point too) problems that are independent of set theory. The most notable one is CH (the Continuum Hypothesis: is there a cardinality between countable and the reals). The independence of CH was shown in two parts: (1) Gödel showed there is a model where CH is true, and (2) Cohen showed that there is a model where CH is false. AC held in both models.

In looking at models where the value of χ might change, it may be useful to drop AC and replace it with something else. Is dropping AC a good idea?

Consider the graph $G = (V, E)$ where

- $V = \mathsf{R}$ (the reals),
- $E = \{((s,t) \in \mathsf{R}^2 : s - t - \sqrt{2} \in \mathsf{Q}\}$.

What is $\chi(G)$? The answer is not so simple.

ZFS is the set of axioms of $\mathrm{ZF} + \mathrm{AC}_{\aleph_0} + \mathrm{LM}$ where $\mathrm{AC}_{\aleph_0}$ is AC for countable sets and LM is the statement that all sets are Lebesgue measurable, hence guaranteeing no Banach-Tarski Paradox. The S stands for Solovay who proved that if ZFC is consistent then ZFS is consistent. Let $\chi^{\mathrm{ZFC}}(G)$ be $\chi(G)$ in ZFC, and let $\chi^{\mathrm{ZFS}}(G)$ be $\chi(G)$ in ZFS. Soifer and Shelah showed that

1. $\chi^{\mathrm{ZFC}}(G) = 2$
2. $\chi^{\mathrm{ZFS}}(G) > \aleph_0$.

These results were in TMCB. Between TMCB and TNMCB Soifer spoke to many logicians (Paul Cohen, Robert Solovay, and Saharon Shelah) and other mathematicians, about these results and what they mean.

Soifer then has a chapter on what he thinks. Most Mathematicians are Platonists. They think that questions such as CH or the chromatic number of the G *have answers*. In short, Mathematicians *imagine the real*. This is also what scientists do. To quote the book *Science reflects what is outside of the Man, in Nature, whereas Art reflects what is within.*

Soifer thinks of Math as being an art. Hence, to quote the book, *Mathematics is an invention that makes us realize reality.* Soifer coined a term for this Philosophy: *Imaginism.* He gives strong evidence that Einstein, Picasso, Wittgenstein, Baudelaire, and Camus were imaginists.

4.3 How Can a Math Book be Controversial?

TMCB has a few chapters on history that examine van der Waerden's behavior during WW II. He stayed in Nazi Germany when he could have left. The treatment of him given here was nuanced and fair. When I read TMCB I thought this material was interesting but not controversial.

I should have been right, but I was wrong.

Gunter M. Ziegler wrote a negative book review of TMCB. Soifer carefully refutes everything that Ziegler wrote. But why was Ziegler so negative? I thought that Soifer was going to say that even bringing up Germany's Nazi past was considered controversial. While that is surely true, there is more going on here.

The criticism of TMCB came after Soifer went on a campaign to rename *The Nevanlinna Prize.* Why rename it? Because Nevanlinna was a Nazi collaborator. Soifer speculates that the campaign brings up Germany's Nazi past, and that is what Ziegler objects to.

TNMCB also gives a careful account of the campaign. Spoiler Alert: the name was changed to *The Abacus Medal.* That's of course good, but there were other issues involved. Soifer discusses all of this.

Nevanlinna was a fine analyst but had no theoretical computer science credentials. I wonder if the name change would have happened if Nevanlinna was a brilliant theoretical computer scientist.

4.4 Pictures of People, Graphs, and Letters

TNMCB has the following

1. Many color pictures of mathematicians and others.

2. Many color pictures of graphs.

3. Some copies of handwritten letters that are of interest for history.

4.5 Other

I have given an account of changes that lead to new chapters or half-chapters. There are many new sections as well that are harder to summarize.

5 Opinion

(This is the same Opinion I gave in my review of TMCB.)

Who *could* read this book? The upward closure of the union of the following people: (1) an excellent high school student, (2) a very good college math major, (3) a good grad student in math or math-related field, (4) a fair PhD in combinatorics, or (5) a bad math professor.

Who *should* read this book? Anyone who is interested in math or history of math. This book has plenty of both. If you are interested in math then this book will make you interested in history of math. If you are interested in history of math then this book will make you interested in math. Any researcher in either mathematics or the history of mathematics, no matter how sophisticated, will find many interesting things they did not know.

And now the elephant in the room: If you have TMCB should you buy TNMCB? Yes. There is so much more here that is worth knowing.

References

[1] C. Kruskal. The chromatic number of the plane: the bounded case. *Journal of Computer and System Sciences*, 74:598–627, 2008. `www.cs.umd.edu/~kruskal/papers/papers.html`.

[2] A. Soifer. *The scholar and the state: In search of Van der Waerden*. Birkhäuser Springer-Verlag, Basel, 2015.