Theory of Computation
Swarthmore Honors Exam, Spring 2023
Follow Up Questions

1. (20 points—4 points each) Give an example of each of the following. No proof required.

   (a) Two languages L_1 and L_2 such that the following are all true:
   - L_1 IS NOT a regular language.
   - L_2 IS NOT a regular language.
   - $L_1 \cup L_2$ IS a regular language.

   (b) Two languages L_1 and L_2 such that the following are all true:
   - L_1 IS a context-free language.
   - L_2 IS a context-free language.
   - $L_1 \cap L_2$ IS NOT a context-free language.

   FOLLOWUP:

   a) CFL's are not closed under intersection. Other class of sets that is not closed under intersection?

   (ANSWERS: infinite sets.)

   b) ARE CFL's closed under complement? If not give an example. Other class of sets that is not closed under complement?

   (c) A language that is in P but is not context-free.

   (d) Assume P $\neq$ NP. A language that is in NP but not in P that is NOT a set of boolean formulas (so you CANNOT use SAT or 3SAT or anything of that type).

   (e) A language that is not decidable.

   FOLLOWUP: A language that is not r.e. (acceptable).

2. (20 points) Recall that a DFA is a tuple $(Q, \Sigma, \delta, s, F)$ where

   - Q is a set of states.
   - $\delta : Q \times \Sigma \rightarrow Q$.
   - $s \in Q$ is the start state.
   - $F \subseteq Q$ are the final states.

   Let L_1 be regular with DFA $(Q_1, \Sigma, \delta_1, s_1, F_1)$.

   Let L_2 be regular with DFA $(Q_2, \Sigma, \delta_2, s_2, F_2)$.

   Give a DFA $(Q, \Sigma, \delta, s, F)$ that accepts $L_1 \cap \overline{L_2}$.

   ($\overline{L_2}$ is the complement of L_2.)

   FOLLOWUP:

   HOW many states in the DFA for $L_1 \cup L_2$?

   Are there cases where this number of states are NEEDED?

   Are there cases where this number of states is NOT NEEDED?

3. (20 points). Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

   Let f be a computable function from $\mathbb{N}$ to $\mathbb{N}$ that is onto. Let g be the function that, on input x returns the least y such that $f(y) = x$. (Such a y must exist since f is onto.) Show that g is computable.

   FOLLOWUP: Set of 1-1 functiosn? set of bijections?

4. (20 points) For each statement say if it is TRUE or FALSE and prove your assertion.

   (a) The language $\{w : \#_a(w) \neq \#_b(w)\}$ is regular.

   (b) The language $\{w : \#_a(w)$ is a square $\}$ is regular

   NO FOLLOWUP

5. (20 points) Let

$$\mathrm{SAT}_c = \{\phi : \phi \text{ has at least } c \text{ satisfying assignments}\}.$$

Note that SAT_1 is the usual problem SAT.

(a) (5 points) Give an example of a boolean formula on exactly 2 variables that is in SAT_2 but not in SAT_3.

(b) (15 points) Show that if SAT_3 is in polynomial time then SAT is in polynomial time.

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FOLLOWUP: What if c is a function of n? For example, $\geq n^2$ satisfying assignments? What if $c = 2^{n/2}$?