1 Limits On How Well We Can Approximate
\(a + \sqrt{b}\) With Rationals

Let \(\text{SQ}\) be the set of squares of naturals. Let \(\text{SQQ}\) be the set of squares of rationals.

We want to find see how well we can approximation \(a + \sqrt{b}\). Since shifting by a natural number does not change how well the \(a + \sqrt{b}\) can be approximated, and we want \(\sqrt{b}\) to be real and irrational, we only look for \(a \in \{0\} \cup \mathbb{Q} - \mathbb{N}\) and \(b \in \mathbb{Q}^+ - \text{SQQ}\).

We get conditions on \(a, b\) and the approximation bounds at the end.

We will determine \(a, b, \Delta\) to satisfy the following:

\[
(\exists \infty p, q \in \mathbb{N}) \left[ \left| \frac{p}{q} - (a + \sqrt{b}) \right| < \frac{\Delta}{q^2} \right] \implies \text{A CONTRADICTION.}
\]

Assume \(p, q, \Delta\) are such that \(\left| \frac{p}{q} - (a + \sqrt{b}) \right| < \frac{\Delta}{q^2}\).

We will find \((a, b, \Delta)\) such that if \(q\) is large we get a contradiction.

There exists \(\delta < \Delta\) such that

\[
\left| \frac{p}{q} - (a + \sqrt{b}) \right| = \frac{\delta}{q^2},
\]

\[
p - q(a + \sqrt{b}) = \frac{\delta}{q}
\]

\[
\frac{\delta}{q} = p - aq - \sqrt{b}q
\]

\[
\frac{\delta}{q} + \sqrt{b}q = p - aq
\]

\[
\left( \frac{\delta}{q} + \sqrt{b}q \right)^2 = (p - aq)^2
\]

\[
\frac{\delta^2}{q^2} + 2\frac{\delta}{q}\sqrt{b}q + q^2b = p^2 - 2apq + q^2a^2
\]

\[
\frac{\delta^2}{q^2} + 2\delta\sqrt{b} = p^2 - 2apq + q^2a^2 - q^2b
\]
\[
\frac{\delta^2}{q^2} + 2\delta \sqrt{b} = p^2 - 2apq + q^2a^2 - q^2b = p^2 - 2apq + q^2(a^2 - b)
\]

Want that as \(q \to \infty\) LHS \(\notin \mathbb{Z}\) and RHS \(\in \mathbb{Z}\).

LHS \(\notin \mathbb{Z}\): \(2\delta \sqrt{b} < 1\), so \(\delta < \frac{1}{2\sqrt{b}}\).

SO we can take \(\Delta = \frac{1}{2\sqrt{b}}\).

RHS \(\in \mathbb{Z}\).

Note that \(p, q\) could be ANYTHING IN \(\mathbb{Z}\). Hence we need to make have \(2apq \in \mathbb{Z}\) and \(a^2 - b \in \mathbb{Z}\). Recall that \(a \in \{0\} \cup \mathbb{Q} - \mathbb{N}\) and \(b \in \mathbb{Q}^+ - \mathbb{SQ}\).

To make \(2apq \in \mathbb{Z}\) need \(a \in \{-\frac{1}{2}, 0, \frac{1}{2}\}\).

1. \(a = 0\): To make \(q^2(a^2 - b) \in \mathbb{Z}\) we need \(b \in \mathbb{N} - \mathbb{SQ}\).

   Upshot It works to take \(a = 0, b \in \mathbb{N} - \mathbb{SQ}\), and \(\Delta = \frac{1}{2\sqrt{b}}\).

2. \(a = \frac{1}{2}\): To make \(q^2(a^2 - b) \in \mathbb{Z}\) we need \(q^2(\frac{1}{4} - b) \in \mathbb{Z}\). Hence

   \[
   b \in X = \left\{ \frac{c}{4} : c \equiv 1 \pmod{4}, c \notin \mathbb{SQ} \right\}.
   \]

   Upshot It works to take \(a = \frac{1}{2}, b = \frac{c}{4} \) where \(c \equiv 1 \pmod{4}, c \notin \mathbb{SQ}\), and \(\Delta = \frac{1}{2\sqrt{b}} = \frac{1}{\sqrt{c}}\).

3. \(a = -\frac{1}{2}\): To make \(q^2(a^2 - b) \in \mathbb{Z}\) we need \(q^2(\frac{1}{4} - b) \in \mathbb{Z}\). Hence

   \[
   b \in Y = \left\{ \frac{c}{4} : c \equiv 1 \pmod{4}, c \notin \mathbb{SQ} \right\}.
   \]

   Upshot It works to take \(a = -\frac{1}{2}, b = \frac{c}{4}, c \equiv 1 \pmod{4}, c \notin \mathbb{SQ}\), and \(\Delta = \frac{1}{2\sqrt{b}} = \frac{1}{\sqrt{c}}\).
How big does $q$ have to be?

Need

$$\frac{\delta^2}{q^2} + 2\delta \sqrt{b} < 1$$

$\delta$ is at most $\Delta$, so we need

$$\Delta^2 + 2\Delta \sqrt{b}q^2 < q^2$$

$$\Delta^2 + 2\Delta \sqrt{b}q^2 < q^2$$

$$\Delta^2 < q^2(1 - 2\Delta \sqrt{b})$$

$$q^2 > \frac{\Delta^2}{1 - 2\Delta \sqrt{b}}$$
<table>
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<th>$a$</th>
<th>$b$</th>
<th>$\Delta = \frac{1}{2\sqrt{b}}$</th>
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