Open Problems Column
Edited by William Gasarch

This Issue's Column! This issue's Open Problem Column is by William Gasarch and is *Relationships Between the Busy Beaver Function and Mathematics I: What is the Smallest n such that* $\mathrm{BB}(n) > \cdots$

Request for Columns! I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere in between, and (2) really important or really unimportant or anywhere in between.

Relationships Between the Busy Beaver Function and Mathematics I:
What is the Smallest n such that $\mathrm{BB}(n) > \cdots$?
by William Gasarch

1 This is the first of (hopefully only) two papers

In this paper we ask the following question: for a variety of computable functions f, what is the smallest n such that $\mathrm{BB}(n) > f(n)$. In the sequel, we will ask questions such as: *What is the smallest Turing machine M such that Goldbach's conjecture is true iff M never halts?*

In both papers all concepts and definitions are due to others unless otherwise noted. For example, if I wrote *We define a hierarchy similar to the Wainer hierarchy* the definition we give is already known.

2 Introduction

Notation 2.1 We will abbreviate *Turing machine* by *TM*.

Def 2.2 $\mathrm{BB}(n)$ is the largest number of steps that an n-state TM will take to halt (BB stands for Busy Beaver function). We only consider TMs that halt, so $\mathrm{BB}(n)$ is finite.

Note that to define $\mathrm{BB}(n)$ rigorously you need to specify the details of the TM model you are using. See Scott Aaronson [Aar20] for these details.

It is easy to see that $\mathrm{BB}(n)$ is not computable. Indeed, it grows faster than any computable function. Hence we are interested in how quickly it is greater than various computable functions. That is the question we will investigate in this open problems column.

The first five Busy Beaver numbers are known. Lower bounds for $\mathrm{BB}(6)$ and $\mathrm{BB}(7)$ are also known. We summarize this information below. (See the definition of the $\uparrow$-notation in Section 4.)

n	BB(n)	Reference
1	1	Trivial
2	6	Shen Lin & Tibor Radó 1963 [LR65]
3	21	Shen Lin & Tibor Radó 1963 [LR65]
4	107	Allen Brady 1983 [Bra83]
5	$47,176,870$	The bbchallenge Collaboration [bC26]
6	$\geq 2 \uparrow^3 5$	Reported by Pascal Michel
7	$\geq 2 \uparrow^{11} (2 \uparrow^{11} 3)$	Reported by Pascal Michel

(We will discuss BB(6) and BB(7) more in Section 5.)

The lower bound on BB(6) is *ginormous*. We believe BB(6) will never be found. There is also a philosophical issue that was pointed out to me by Shawn Ligocki. We paraphrase his email:

Can we truly understand the number BB(6)*? I think this is a philosophically deep question that I cannot hope to answer. My current feeling is that even comparatively small numbers (like a billion) are large enough that I cannot completely understand them in many ways, but for these truly huge numbers my current feeling is that I understand them when I can comfortably answer arithmetic questions about them. For example, given two numbers in up-arrow notation, can I figure out which one is bigger? This is not trivial, but I think I have the algorithm for tetration at least.*

For more background on the Busy Beaver function see a nonempty subset of the following:

- The open problems column by Scott Aaronson [Aar20] which inspired the work that led to BB(5) being discovered.
- Many blog posts by Scott Aaronson on the topic (go to his blog, Shtetl-optimized, and search for *Busy Beaver*).
- Ben Brubaker's superb article in Quanta Magazine [Bru24].
- bbchallenge's own announcement [CW24] of the BB(5) result.

3 Our Goals and Motivation

In this open problems column we will, for a variety of computable fast growing functions f, ask

what is the smallest n *such that* $\mathrm{BB}(n) > f(n)$.

Each section will discuss faster and faster growing functions. So Section x will discuss functions that grow much faster than those in Section $x - 1$.

We have several motivations.

1. It is easy to see that BB(n) is not computable. Indeed, it grows faster than any computable function. So it's of interest to see which computable functions f grow so fast that, at least for small n, $BB(n) \le f(n)$.

2. We believe that BB(6) will never be known. One reason is how large it is, and hence, how many TMs would have to be checked (and checking them is difficult also). There are other reasons, discussed in the sequel, to believe that BB(n), for some small values of n, will likely never be known.

   So rather than compute (say) BB(6) we try to find lower bounds on it. But even this may not be satisfying. Relating these lower bounds to other functions in math may Alleviate some of Shawn Ligocki's concerns.

3. The work on BB has mostly been hacking at TMs. That last statement is perhaps the perception but **its not true!** Some of the work on BB relates to *other parts of math.* This paper, and the sequel, will bring these connections to a wider audience.

4. One of my proofreaders complained that this paper is just a survey of computable fast growing functions, some of which arise naturally in mathematics. While I disagree with the word *just* I will note that if what you, the reader, learn from the paper is about some fast growing functions, well that's still a win.

4 Knuth's Arrow Notation

Def 4.1 Let $a \ge 1$, $b \ge 0$, and $k \ge 1$.

1. $a \uparrow b = a^b$. This is a base case. Only one arrow ($k = 1$).

2. $a \uparrow^k 0 = 1$ if $k > 1$. This is a base case since $b = 0$.

3. $a \uparrow^k b = a \uparrow^{k-1} (a \uparrow^k (b-1))$ if $k \ge 2$ and $b \ge 1$.

4. The notation associates right to left. For example
   $3 \uparrow^2 4 \uparrow^3 8 = 3 \uparrow^2 (4 \uparrow^3 8)$.

Note the following:

1. $a \uparrow^2 b$ is $a^{a^{\cdots}}$ where there are b copies of a. This is often called *The Tower Function* or *Tetration.*

2. $a \uparrow^3 b$ is often called *The Wower Function* (that is not a typo) if you define *often* to mean *once.* That terminology was used in the Graham-Rothschild-Spencer book on Ramsey Theory [GRS90].

3. For every k, the function $f(a,b) = a \uparrow^k b$ is primitive recursive.

Open Problem 4.2 For triples (a,b,k), what is the smallest n such that $\mathrm{BB}(n) > a \uparrow^k b$? Some small cases of (a,b,k) can likely be worked out explicitly.

5 Ackermann's Function

Def 5.1 Ackermann's Function is defined as follows:

$\mathrm{ACK}(0,n) = n+1$
$\mathrm{ACK}(m+1,0) = \mathrm{ACK}(m,1)$
$\mathrm{ACK}(m+1,n+1) = \mathrm{ACK}(m,\mathrm{ACK}(m+1,n))$

The following are known:

1. $\mathrm{ACK}(1,n) = n+2$.
2. $\mathrm{ACK}(2,n) = 2n+3$.
3. $\mathrm{ACK}(3,n) = 2^{n+3} - 3$.
4. $\mathrm{ACK}(4,n) = 2 \uparrow^2 (n+3) - 3$.
5. $\mathrm{ACK}(5,n) = 2 \uparrow^3 (n+3) - 3$.
6. $\mathrm{ACK}(m,n)$ is very roughly $2 \uparrow^{m-2} (n+3) - 3$.
7. $\mathrm{ACK}(m,n)$ grows faster than any function involving a fixed number of $\uparrow$'s.

Open Problem 5.2

1. In Scott Aaronson's survey of the Busy Beaver function [Aar20] he states

   > *As I was writing this survey, my 7-year-old daughter Lily raised the following question:*
   >
   > *What's the first n such that $\mathrm{BB}(n) > \mathrm{ACK}(n,n)$?*

   Pavel Kropitz showed that $\mathrm{BB}(6) \geq 10 \uparrow^2 15$. This is a tower of 10s of height 15. See Scott Aaronson's blog post `https://scottaaronson.blog/?p=6673`.

   for the result and the reference.

   The lower bound on BB(6) has been improved since then. In addition, an enormous lower bound for BB(7) has been found. The new lower bounds from Pascal Michel's historical survey of champions are:

- $\mathrm{BB}(6) \ge 2 \uparrow^3 5$
- $\mathrm{BB}(7) \ge 2 \uparrow^{11} 2 \uparrow^{11} 3$

See his website for details and results about those and other BB numbers:

`https://bbchallenge.org/~pascal.michel/ha`

A calculation shows that we can conclude:

- $\mathrm{BB}(5) < \mathrm{ACK}(5,5)$.
- The status of BB(6) and $\mathrm{ACK}(6,6)$ is not known.
- $\mathrm{BB}(7) > \mathrm{ACK}(7,7)$.

Hence the answer to Lily's question is either 6 or 7. It is plausible that a 6 state TM will be constructed that runs for more than $\mathrm{ACK}(6,6)$ steps, hence showing that 6 is the answer. If in fact $\mathrm{BB}(6) \le \mathrm{ACK}(6,6)$, then proving this may be extremely difficult.

2. Determine triples (n,x,y) such that $\mathrm{BB}(n) > \mathrm{ACK}(x,y)$, and identify where the open problems are.

6 The Wainer Hierarchy: Functions Based on Ordinals $\le \epsilon_0$

Def 6.1 We give two equivalent definitions of the ordinal ϵ_0.

- ϵ_0 is $\omega^{\omega^{\cdots}}$ where the number of ω's is infinite.
- ϵ_0 is the least fixed point of the function that maps the ordinal α to the ordinal ω^α.

The following is well known so we omit the proof.

Lemma 6.2 *If $\lambda < \epsilon_0$ then there exist ordinals $\alpha_1 \ge \cdots \ge \alpha_k$ with $\alpha_1 < \lambda$ such that*

$$\lambda = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k}.$$

This representation is commonly called Cantor normal form

Def 6.3

1. If λ is a countable limit ordinal then **a** *fundamental sequence for* λ is an increasing sequence of ordinals, indexed by $n \in \mathbb{N}$, which are all $< \lambda$ and whose limit is λ. The nth element of the sequence is denoted $\lambda[n]$. Note that this was the definition of **a** fundamental sequence. We present, for many ordinals, what we will call **the** *fundamental sequence.*

2. If λ is a countable limit ordinal, $\lambda \le \epsilon_0$, then *the fundamental sequence for* λ is defined as follows (we give more cases than are needed for educational purposes).

   - If $\lambda = \omega$ then $\lambda[n] = n$.
   - If $\lambda = m\omega$ then $\lambda[n] = (m-1)\omega + n$.
   - If $\lambda = \omega^2$ then $\lambda[n] = \omega \cdot n$.
   - If $\lambda = \omega^{\alpha+1}$ then $\lambda[n] = \omega^{\alpha} \cdot n$.
   - If $\lambda = \omega^{\alpha}$ where α is a limit ordinal $< \lambda$ then $\lambda[n] = \omega^{\alpha[n]}$.
   - If $\lambda = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k}$ and $k \ge 2$ then $\lambda[n] = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k}[n]$.
   - If $\lambda = \epsilon_0$ then $\lambda[0] = 1$ and $\lambda[n+1] = \omega^{\lambda[n]}$.

Def 6.4 Let λ be a countable ordinal $\le \epsilon_0$. The function $f_\lambda\colon \mathbb{N} \to \mathbb{N}$ is defined as follows.

1. $f_0(n) = n + 1$.

2. If λ is a successor ordinal, $\lambda = \alpha + 1$, then

   $f_{\alpha+1} = f_{\alpha}^{n}(n)$. (The superscript means that you apply f n times.)

3. If λ is a limit ordinal then let $\lambda[n]$ be its fundamental sequence. Then

   $f_\lambda(n) = f_{\lambda[n]}(n)$.

   This set of functions is called *the Wainer hierarchy.*

The following are known.

1. For all $\lambda \le \epsilon_0$, f_λ is computable.

2. For all $\lambda < \epsilon_0$, f_λ can be proven total in Peano Arithmetic.

3. f_{ϵ_0} cannot be proven total in Peano Arithmetic.

4. For every primitive recursive function g there exists $i \in \mathbb{N}$ such that, for all but a finite number of n, $g(n) < f_i(n)$.

5. The Ackermann function has growth rate roughly f_ω.

6. The Goodstein function has growth rate roughly f_{ϵ_0}. This is why Goodstein's theorem is not provable in Peano Arithmetic. For information on Goodstein's theorem, see the Wikipedia entry on it.

7. The Paris-Harrington function, from Ramsey Theory, has growth rate roughly f_{ϵ_0}. This is why the Paris-Harrington theorem is not provable in Peano Arithmetic. For information on the Paris-Harrington theorem, see the Wikipedia Entry on it.

Open Problem 6.5 For each ordinal $\lambda \le \epsilon_0$, what is the smallest n such that $\mathrm{BB}(n) > f_\lambda(n)$?

7 The 0 Case of the Veblen Hierarchy: Functions Based on Ordinals from ϵ_0 to ϵ_ω

Def 7.1

1. Let $\varphi_0(\alpha) = \omega^\alpha$. (In the next section we will define φ_α for ordinals α. The set of functions φ_α will be called *the Veblen hierarchy.*)

2. Recall that ϵ_0 is the first fixed point of φ_0. Recall that the fundamental sequence for ϵ_0 is

   $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$.

3. For $i \in \mathbb{N}$ let ϵ_i be the ith fixed point of φ_0.

4. ϵ_ω is the limit of the ϵ_i's.

5. The fundamental sequence for ϵ_1 is

   $\epsilon_0 + 1, \omega^{\epsilon_0+1}, \omega^{\omega^{\epsilon_0+1}}, \ldots$.

6. For $i \ge 1$ The fundamental sequence for ϵ_i is

   $\epsilon_{i-1} + 1, \omega^{\epsilon_{i-1}+1}, \omega^{\omega^{\epsilon_{i-1}+1}}, \ldots$.

   We call this representation *Veblen-zero normal form.* This terminology is not standard since there does not seem to be any name for this representation in the literature.

7. The fundamental sequence for ϵ_ω is

   $\epsilon_\omega[n] = \epsilon_n$.

8. We define functions $f_{\epsilon_i}\colon \mathbb{N} \rightarrow \mathbb{N}$ similar to the Wainer hierarchy.

   $f_{\epsilon_i}(n) = f_{\epsilon_i[n]}(n)$.

   $f_{\epsilon_\omega}(n) = f_{\epsilon_\omega[n]}(n) = f_{\epsilon_n}(n)$.

1. For $i \in \mathbb{N}$, f_{ϵ_i} is computable.

2. The functions f_{ϵ_i} grow much faster than the functions in the Wainer hierarchy.

3. To the best of my knowledge, there are no natural functions whose growth rate is comparable to any f_{ϵ_i}, for $i \geq 1$, or to f_{ϵ_ω}.

Open Problem 7.2 For each ordinal ϵ_i, what is the smallest n such that $\mathrm{BB}(n) > f_{\epsilon_i}(n)$?

8 The Veblen Hierarchy: Functions Based on Ordinals from ϵ_ω to Γ_0

Def 8.1

1. $\varphi_0(\gamma) = \omega^\gamma$.

2. $\varphi_1(\gamma)$ is the $(\gamma + 1)$th fixed point of φ_0. (The $+1$ is used since if $\gamma = 0$ the γth fixed point makes no sense.)

3. $\varphi_2(\gamma)$ is the γth ordinal that is a fixed point of *both* φ_0 and φ_1.

4. For $\alpha > 0$, $\varphi_\alpha(\gamma)$ is the $(\gamma + 1)$st ordinal that is a fixed point for all φ_β for $\beta < \alpha$.

5. Γ_0 is the smallest ordinal α such that $\varphi_\alpha(0) = \alpha$.

Recall that Lemma 6.2 stated that all ordinals $\alpha < \epsilon_0$ could be put into Cantor normal form. We state a similar lemma for ordinals $\alpha < \Gamma_0$, without proof.

Lemma 8.2 *Let $\alpha < \Gamma_0$, $\alpha \neq 0$. There exist $\beta_1, \ldots, \beta_k$, $\gamma_1, \ldots, \gamma_k$ such that*

1. $\alpha = \varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_k}(\gamma_k)$.

2. $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$.

3. *For all $m \in \{1, \ldots, k\}$, $\gamma_m < \varphi_{\beta_m}(\gamma_m)$.*

This representation is commonly called Veblen normal form.

Def 8.3 Let $\alpha < \Gamma_0$. Let the Veblen normal form for α be

$$\alpha = \varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_k}(\gamma_k).$$

We use this form to define the fundamental sequence for $\alpha < \Gamma_0$.

1. $(\varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_k}(\gamma_k))[n] = (\varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_{k-1}}(\gamma_{k-1})) + \varphi_{\beta_k}(\gamma_k)[n]$.
   This definition is incomplete since we need to define $\varphi_{\beta_k}(\gamma_k)[n]$. The cases below take care of that.

2. Recall that $\varphi_0(\gamma) = \omega^\gamma$. We define $\varphi_0(\gamma+1)[n] = \omega^\gamma \cdot n$. (These definitions are standard.)

3. $\varphi_{\beta+1}(0)[n] = \varphi_\beta^n(0)$. (Recall that the superscript n means we apply the function n times.)

4. $\varphi_\beta(\gamma)[n] = \varphi_\beta(\gamma[n])$ for a limit ordinal $\gamma < \varphi_\beta(\gamma)$.

5. $\varphi_\beta(0)[n] = \varphi_{\beta[n]}(0)$ for a limit ordinal $\beta < \varphi_\beta(0)$.

6. $\varphi_\beta(\gamma+1)[n] = \varphi_{\beta[n]}(\varphi_\beta(\gamma)+1)$ for a limit ordinal β.

We can now define more fast growing functions from $\mathbb{N}$ to $\mathbb{N}$.

Def 8.4 Let λ be a countable ordinal, $\lambda \le \Gamma_0$. The function $f_\lambda \colon \mathbb{N} \to \mathbb{N}$ is defined as follows.

1. $f_0(n) = n+1$.

2. If λ is a successor ordinal, $\lambda = \alpha + 1$, then
   $f_{\alpha+1} = f_\alpha^n(n)$. (The superscript means that you apply f n times.)

3. If λ is a limit ordinal then let $\lambda[n]$ be its fundamental sequence. Then
   $f_\lambda(n) = f_{\lambda[n]}(n)$.

The following are known.

1. For all $\lambda \le \Gamma_0$, f_λ is computable.

2. For all $\lambda < \Gamma_0$, f_λ can be proven total in predicative analysis. This system is called ATR_0 in reverse mathematics.

3. f_{Γ_0} cannot be proven total in predicative analysis.

Open Problem 8.5 For each ordinal $\lambda \le \Gamma_0$, what is the smallest n such that $\mathrm{BB}(n) > f_\lambda(n)$?

9 Natural Examples of Fast Growing Functions Related to Well-Quasi-Orderings

9.1 Well-Quasi-Orders

Def 9.1 Let X be a set and $\preceq$ be a quasi-ordering (i.e., a reflexive, transitive relation) on X. We give two equivalent definitions of a *well-quasi-ordering*.

1. $(X, \preceq)$ is a *well-quasi-ordering* if, for all infinite sequences $x_1, x_2, \ldots$ of elements from X, there exists $i < j$ such that $x_i \preceq x_j$.

2. $(X, \preceq)$ is a *well-quasi-ordering* if there are no infinite decreasing sequences and there are no infinite antichains.

Def 9.2 Let $c \in \mathbb{N}$. A *c-colored graph* is a graph together with a function from the vertices to $\{1, \ldots, c\}$. There is no restriction on the coloring. For example, we do not require that two vertices connected by an edge have different colors.

The following theorem will lead to several fast growing functions. The proof is a standard compactness argument; however, we give more details than usual to underscore that the proof is nonconstructive.

Theorem 9.3 *Let X be the set of all graphs. Let $\preceq$ be a quasi-ordering on graphs. Assume $(X, \preceq)$ is a well-quasi-ordering.*

1. *For all k, there exists an n, such that the following holds:*

   For every infinite sequence $x_1, \ldots, x_n$ of elements from X, where x_m has $\leq m + k$ vertices, there is an $i < j$ with $x_i \preceq x_j$.

2. *(This is commentary, not a statement we will prove.) Part 1 implies that, for all k, there is an n. We will later refer to* **the function that exists by Theorem 9.3.** *That function will map k to* **the least** *n. In all cases we consider, f will be obviously computable. The proof of Part 1 is nonconstructive and hence is of no help in trying to compute f.*

Proof:

Assume, by way of contradiction, that the theorem is false. Then there exists k such that, for all n, there is an n-sequence

$$x_1, \ldots, x_n,$$

where x_m has $\leq m + k$ vertices, and there is no $i < j$ with $x_i \preceq x_j$.

Then, for all $n \in \mathbb{N}$ there is a sequence

$$x_{n1}, x_{n2}, \ldots, x_{nn}$$

where x_{nm} has $\le m+k$ vertices, such that there is no $i < j$ with $x_i \preceq x_j$.

We construct an infinite sequence of graphs that contradict $(X, \preceq)$ being a well-quasi-ordering.

Look at x_{11}, x_{21}, ... (the set of all first elements of sequences).

For all $1 \le i \le n$, x_{i1} has $\le m+1$ vertices. Hence some graph appears in this list infinitely often. Let y_1 be that graph.

Look at all second elements in sequences that have y_1 as the first element. Each second element has $\le m+2$ vertices. Hence some graph appears in this list infinitely often. Let y_2 be that graph.

Continue in this manner to create

$$y_1, y_2, \ldots$$

It is easy to show that there is no $i < j$ with $y_i \preceq y_j$. This contradicts $(X, \preceq)$ being a well-quasi-ordering.

Hence there is an n such that, for every sequence $x_1, \ldots, x_n$, of elements from X, where x_m has $\le m+k$ vertices, there is an $i < j$ with $x_i \preceq x_j$. ∎

Theorem 9.4 *Let $c \in \mathbb{N}$. Let X be the set of all c-colored graphs. Let $\preceq$ be a quasi-ordering on X. Assume $(X, \preceq)$ is a well-quasi-ordering.*

1. *For all c there exists an n such that the following holds:*

   For every sequence of elements from X,

   $$x_1, \ldots, x_n,$$

   where x_m has $\le m$ vertices and uses $\le c$ colors there is an $i < j$ with $x_i \preceq x_j$.

   We call this value of n $f_{X,\preceq}(c)$.

2. *(This is commentary, not a statement we will prove.) Part 1 implies that, for all c, there is an n. We will later refer to* **the function that exists by Theorem 9.4.** *That function will map c to* **the least** *n. In all cases we consider $f_{X,\preceq}$ will be obviously computable. The proof of Part 1 is nonconstructive and hence is of no help in trying to compute $f_{X,\preceq}$.*

The proof of Theorem 9.4 is similar to the proof of Theorem 9.3 and hence we omit it.

9.2 Four Theorems That Lead to Four Fast Growing Functions

Def 9.5 Let G be a graph or colored graph.

1. The operation RM-V (Remove Vertex) means that you remove a vertex v and all edges incident on v.
2. The operation RM-E (Remove Edge) means that you remove an edge (a, b).
3. The operation SU-V (Suppress Vertex) means that you take a vertex v of degree 2, with neighbors a, b, remove v, and add the edge (a, b), unless (a, b) is already and edge.
4. The operation CON-E (Contract Edge) means that you remove an edge (a, b), and replace $\{a, b\}$ with one vertex. Picture it as moving a towards b until the merge and the edge is gone.
5. H is a *homeomorphically embedded in* G if there is a sequence of operations $O_1, \ldots, O_m$ such that (1) each O_i is one of RM-V, RM-E, SU-V, and (2) when you apply those operations to G you obtain H. (There is an alternative definition that involves embedding H into G.)
6. H is a *minor of* G if there is a sequence of operations $O_1, \ldots, O_m$ such that (1) each O_i is one of RM-V, RM-E, CON-E, and (2) when you apply those operations to G you obtain H.

Notation 9.6 Let H, G be a graph or colored graphs.

1. $H \preceq_h G$ means that H is homeomophically embedded in G.
2. $H \preceq_m G$ means that H is a minor of G.

Theorem 9.7

1. *The set of uncolored trees under $\preceq_h$ is a well-quasi-ordering. (This is a corollary of Kruskal's tree theorem.) Let $F_{T,h}(c)$ be the function that exists via Theorem 9.3. $F_{T,h}(c)$ has growth rate approximately f_{Γ_0}. (We use $F_{T,h}$ instead of the more cumbersome $F_{T,\preceq_h}$.)*
2. *For all c, the set of c-colored trees under $\preceq_h$ is a well-quasi-ordering. (This is a corollary of the Kruskal's tree theorem.) Let $F_{T,h}(c)$ be the function that exists via Theorem 9.4. $F_{T,h}(c)$ has growth rate far greater than f_{Γ_0}.*

   This function is somewhat known to the general public and is called TREE. *There is a numberphile YouTube video on it:*

https://www.youtube.com/watch?v=3P6DWAwwViU.

TREE(1) = 1, TREE(2) = 3, TREE(3) *can be expressed using arrow notation; however, the number of arrows must itself be expressed using arrow notation.*

3. *The set of uncolored graphs under the $\preceq_m$ is a well-quasi-ordering. (This is the graph minor theorem.) Let $F_{G,m}$ be the function that exists via Theorem 9.3. $F_{G,m}$ has growth rate much bigger than $F_{T,h,c}$.*

4. *Fix c. The set of c-colored graphs under $\preceq_m$ is a well-quasi-ordering. (This is a slight variant of the graph minor theorem.) Let $F_{G,m,c}$ be the function that exists via Theorem 9.4. $F_{G,m,c}$ has growth rate only slightly faster than $F_{G,m}$.*

Open Problem 9.8

1. What is the smallest n such that $\mathrm{BB}(n) > F_{T,h}(n)$?
2. What is the smallest n such that $\mathrm{BB}(n) > F_{T,h,c}(n)$?
3. What is the smallest n such that $\mathrm{BB}(n) > F_{G,m}(n)$?
4. What is the smallest n such that $\mathrm{BB}(n) > F_{G,m,c}(n)$?

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