

Rectangle Free Coloring of Grids

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Abstract

A two-dimensional *grid* is a set $G_{n,m} = [n] \times [m]$. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi_{n,m} : G_{n,m} \rightarrow [c]$ such that there are no rectangles with all four corners the same color. We address the following question: for which values of n and m is $G_{n,m}$ *c-colorable*? This problem can be viewed as a bipartite Ramsey problem and is related to the Gallai-Witt theorem (also called the multidimensional Van Der Waerden Theorem). We determine (1) *exactly* which grids are 2-colorable, (2) *exactly* which grids are 3-colorable, and (3) (assuming a conjecture) *exactly* which grids are 4-colorable. We use combinatorics and finite fields.

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1 Introduction

Let $n, m \in \mathbb{N}$. A two-dimensional *grid* is a set $G_{n,m} = [n] \times [m]$ where $[t] = \{1, \dots, t\}$ for $t \in \mathbb{N}$. A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a, b), (a + c_1, b), (a + c_1, b + c_2), (a, b + c_2)\}$ for some $a, b, c_1, c_2 \in \mathbb{N}$. A grid $G_{n,m}$ is *c -colorable* if there is a function $\chi_{n,m} : G_{n,m} \rightarrow [c]$ such that there are no rectangles with all four corners the same color. Not all grids have c -colorings. As an example, for any c clearly $G_{c+1, c^{c+1}+1}$ does not have a c -coloring by two applications of the pigeonhole principle. If a grid has a c -coloring, we say it is *c -colorable*.

In this paper, we ask the following question: what are the exact values of m and n for which $G_{n,m}$ is c -colorable?

Def 1.1 Let $n, m, n', m' \in \mathbb{N}$. $G_{m,n}$ contains $G_{n',m'}$ if $n' \leq n$ and $m' \leq m$. $G_{m,n}$ is contained in $G_{n',m'}$ if $n \leq n'$ and $m \leq m'$. Proper containment means that at least one of the inequalities is strict.

Clearly, if $G_{n,m}$ is c -colorable, then all grids that it contains are c -colorable. Likewise, if $G_{n,m}$ is not c -colorable then all grids that contain it are not c -colorable.

Def 1.2 Fix $c \in \mathbb{N}$, then OBS_c is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not c -colorable but all grids properly contained in $G_{n,m}$ are c -colorable. OBS_c stands for *Obstruction Sets*. We also call such grids c -minimal.

We leave the proof of the following theorem to the reader.

Theorem 1.3 Fix $c \in \mathbb{N}$. A grid $G_{a,b}$ is c -colorable iff it does not contain any element of OBS_c .

By Theorem 1.3 we can rephrase the question of finding which grids are c -colorable: find OBS_c . Note that if $G_{n,m} \in \text{OBS}_c$, then $G_{m,n} \in \text{OBS}_c$.

This problem arises as follows. The Gallai-Witt theorem¹ (also called the multi-dimensional Van Der Waerden theorem) has the following corollary: *For all c , there exists $W = W(c)$ such that, for all c -colorings of $[W] \times [W]$ there exists a monochromatic square.* The classical proof of the theorem gives very large upper bounds on $W(c)$. Despite some improvements² the known bounds on $W(c)$ are still quite large; however, there has been some work on this by [1]. If we relax the problem to seeking a *monochromatic rectangle* then we can obtain far smaller bounds. In fact, in some cases we will obtain exact characterizations of when a grid is c -colorable.

Another motivation is the bipartite Ramsey problem: Given a, c , what is the least n such that for any c -coloring of the edges of $K_{n,n}$ there is a monochromatic $K_{a,a}$? A coloring of $G_{n,n}$ can be viewed as an edge coloring of $K_{n,n}$. A monochromatic rectangle corresponds to a monochromatic $K_{2,2}$. Beineke and Schwenk [3] studied a closely related problem: What is the minimum value of b such that any 2-coloring of $K_{b,b}$ results in a monochromatic $K_{n,m}$? In their work, this minimal value is denoted $R(n, m)$. Later, Hattingh and Henning [8] defined $b(n, m)$ as the minimum b for which any 2-coloring of $K_{b,b}$ contains a monochromatic $K_{m,m}$ or a monochromatic $K_{n,n}$.

In a related paper, Cooper, Fenner, and Purewal [4] generalize the problem to multiple dimensions and obtain upper and lower bounds on the sizes of the obstruction sets.

The remainder of this paper is organized as follows. In Sections 2 and 3 we develop tools to show grids *are not* c -colorable. In Section 4 we develop tools to show grids *are*

¹It was attributed to Gallai in [12] and [13]; Witt proved the theorem in [16].

²Both [7] and [5] can be used to obtain better bounds on $W(c)$.

c -colorable. In Section 5 we obtain upper and lower bounds on $|\text{OBS}_c|$. In Section 6 and 7 we find OBS_2 and OBS_3 respectively. In Section 8 we obtain a most elements of OBS_4 , as well as possible elements to complete the set. We also propose a conjecture which, if true, would yield the exact elements of OBS_4 . In Section 9 we apply the results to find some new bipartite Ramsey numbers. We conclude with some open questions. The appendix contains some sizes of maximum rectangle free sets (to be defined later).

2 Lower Bounds on Uncolorability

A *rectangle-free subset* $A \subseteq G_{n,m}$ is a subset that does not contain a rectangle as defined above. A problem that is closely related to grid-colorability is that of finding a rectangle-free subset of maximum cardinality. This relationship is illustrated by the following lemma.

Theorem 2.1 *If $G_{n,m}$ is c -colorable, then it contains a rectangle-free subset of size $\lceil \frac{nm}{c} \rceil$.*

Proof: A c -coloring partitions the elements of $G_{n,m}$ into c rectangle-free subsets. By the pigeon-hole principle, one of these sets must be of size at least $\lceil \frac{nm}{c} \rceil$. ■

Def 2.2 Let $n, m \in \mathbb{N}$. $\text{maxrf}(n, m)$ is the size of the maximum rectangle-free $A \subseteq G_{n,m}$.

Finding the maximum cardinality of a rectangle-free subset is equivalent to a special case of a well-known problem of Zarankiewicz [17] (see [6] or [15] for more information). The Zarankiewicz function, denoted $Z_{r,s}(n, m)$, counts the minimum number of edges in a bipartite graph with vertex sets of size n and m that guarantees a subgraph isomorphic to $K_{r,s}$. Zarankiewicz's problem was to determine $Z_{r,s}(n, m)$.

If $r = s$, the function is denoted $Z_r(n, m)$. If one views a grid as an incidence matrix for a bipartite graph with vertex sets of cardinality n and m , then a rectangle is equivalent to a subgraph isomorphic to $K_{2,2}$. Therefore the maximum cardinality of a rectangle-free set in $G_{n,m}$ is $Z_2(n, m) - 1$. We will use this lemma in its contrapositive form, i.e., we will often show that $G_{n,m}$ is not c -colorable by showing that $Z_2(n, m) \leq \lceil \frac{nm}{c} \rceil$.

Reiman [14] proved the following lemma. Roman [15] later generalized it.

Lemma 2.3 *Let $m \leq n \leq \binom{m}{2}$. Then $Z_2(n, m) \leq \left\lfloor \frac{n}{2} \left(1 + \sqrt{1 + 4m(m-1)/n} \right) \right\rfloor + 1$.*

Corollary 2.4 *Let $m \leq n \leq \binom{m}{2}$. Let $z_{n,m} = \left\lfloor \frac{n}{2} \left(1 + \sqrt{1 + 4m(m-1)/n} \right) \right\rfloor + 1$ be the upper-bound on $Z_2(n, m)$ in Lemma 2.3. If $z_{n,m} \leq \lceil \frac{nm}{c} \rceil$ then $G_{n,m}$ is not c -colorable.*

Corollary 2.4, and some 2-colorings of grids, are sufficient to find OBS_2 . To find OBS_3 and OBS_4 , we need more powerful tools to show grids are not colorable (along with some 3-colorings and 4-colorings of grids). This next lemma, which has a proof that is very similar to that of the previous lemma gives us two more uncolorability corollaries.

Def 2.5 Let $n, m, x_1, \dots, x_m \in \mathbf{N}$. (x_1, \dots, x_m) is (n, m) -placeable if there exists a rectangle-free $A \subseteq G_{n,m}$ such that, for $1 \leq j \leq m$, there are x_j elements of A in the j^{th} column.

Lemma 2.6 Let $n, m, x_1, \dots, x_m \in \mathbf{N}$ be such that (x_1, \dots, x_m) is (n, m) -placeable. Then $\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}$.

Proof: Let $A \subseteq G_{n,m}$ be a set that shows that (x_1, \dots, x_m) is (n, m) -placeable. Let $\binom{A}{2}$ be the set of pairs of elements of A . Let $2^{\binom{A}{2}}$ be the powerset of $\binom{A}{2}$.

Define the function $f : [m] \rightarrow 2^{\binom{A}{2}}$ as follows. For $1 \leq j \leq m$,

$$f(j) = \{\{a, b\} : (a, j), (b, j) \in A\}.$$

If $\sum_{j=1}^m |f(j)| > \binom{n}{2}$ then there exists $j_1 \neq j_2$ such that $f(j_1) \cap f(j_2) \neq \emptyset$. Let $\{a, b\} \in f(j_1) \cap f(j_2)$. Then

$$\{(a, j_1), (a, j_2), (b, j_1), (b, j_2)\} \subseteq A.$$

Hence A contains a rectangle. Since this cannot happen, $\sum_{j=1}^m |f(j)| \leq \binom{n}{2}$. Note that $|f(j)| = \binom{x_j}{2}$. Hence $\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}$. ■

Lemma 2.7 Let $a, n, m \in \mathbf{N}$. Let $q, r \in \mathbf{N}$ be such that $a = qn + r$ with $0 \leq r \leq n$. Assume that there exists $A \subseteq G_{n,m}$ such that $|A| = a$ and A is rectangle-free.

1. If $q \geq 2$ then

$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. If $q = 1$ then

$$r \leq \frac{m(m-1)}{2}.$$

Proof: The proof for the $q \geq 2$ and the $q = 1$ case begins the same; hence we will not split into cases yet.

Assume that, for $1 \leq j \leq m$, the number of elements of A in the j^{th} column is x_j . Note that $\sum_{j=1}^m x_j = a$. By Lemma 2.6 $\sum_{j=1}^m \binom{x_j}{2} \leq \binom{n}{2}$. We look at the least value that $\sum_{j=1}^m \binom{x_j}{2}$ can have.

Consider the following question:

Minimize $\sum_{j=1}^n \binom{x_j}{2}$

Constraints:

- $\sum_{j=1}^n x_j = a$.
- x_1, \dots, x_n are natural numbers.

One can easily show that this is minimized when, for all $1 \leq j \leq n$,

$$x_j \in \{\lfloor a/n \rfloor, \lceil a/n \rceil\} = \{q, q+1\}.$$

In order for $\sum_{j=1}^n x_j = a$ we need to have $n-r$ many q 's and r many $q+1$'s. Hence we obtain

$\sum_{j=1}^n \binom{x_j}{2}$ is at least

$$(n-r) \binom{q}{2} + r \binom{q+1}{2}.$$

Hence we have

$$\begin{aligned} (n-r) \binom{q}{2} + r \binom{q+1}{2} &\leq \sum_{j=1}^n \binom{x_j}{2} \leq \binom{m}{2} \\ nq(q-1) - rq(q-1) + r(q+1)q &\leq m(m-1) \\ nq(q-1) - rq^2 + rq + rq^2 + rq &\leq m(m-1) \\ nq(q-1) + 2rq &\leq m(m-1) \end{aligned}$$

Case 1: $q \geq 2$.

Subtract $2rq$ from both sides to obtain

$$nq(q-1) \leq m(m-1) - 2rq.$$

Since $q-1 \neq 0$ we can divide by $q(q-1)$ to obtain

$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

Case 2: $q = 1$.

Since $q-1 = 0$ we get

$$2r \leq m(m-1)$$

$$r \leq \frac{m(m-1)}{2}.$$

■

Corollary 2.8 *Let $m, n \in \mathbf{N}$. If there exists an r where $\frac{m(m-1)}{2} < r \leq n$ and $\lceil \frac{mn}{c} \rceil = n+r$, then $G_{m,n}$ is not c -colorable.*

Corollary 2.9 *Let $n, m \in \mathbf{N}$. Let $\lceil \frac{nm}{c} \rceil = qn+r$ for some $0 \leq r \leq n$ and $q \geq 2$. If $\frac{m(m-1)-2qr}{q(q-1)} < n$ then $G_{n,m}$ is not c -colorable.*

Note 2.10 In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

FENNER: THIS COROLLARY IS ONLY USED IN THE BOUNDING THE OBS SET PROOF. I CHANGED THE PROOF TO BE CONSISTENT WITH BOTH ITS USE THERE AND HERE. I ALSO ONLY MADE IT FOR $1 \leq c' < c$ - YOU HAD $1 \leq c' \leq c$. I DON'T THINK WE NEED THE $c = c'$ CASE ANYWHERE

PROOFREAD THE COR, ITS PROOF, AND ITS USE LATER (UNCHANGED ACTUALLY BUT MAKE SURE IT STILL WORKS).

Corollary 2.11 1. Let $c \geq 2$ and $1 \leq c' < c$. Let $n > \frac{c}{c'} \binom{c+c'}{2}$. Then $G_{n,c+c'}$ is not c -colorable.

2. Let $c \geq 2$ and $1 \leq c' < c$. Let $m > \frac{c}{c'} \binom{c+c'}{2}$. Then $G_{c+c',m}$ is not c -colorable. (This follows immediately from part a.)

Proof: Assume, by way of contradiction, that $G_{n,c+c'}$ is c -colorable. Then there is a rectangle free set of size

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = \left\lceil n + \frac{c'n}{c} \right\rceil = n + \left\lceil \frac{c'n}{c} \right\rceil.$$

Since $c' < c$ we have

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = n + \left\lceil \frac{c'n}{c} \right\rceil \leq n + \left\lceil \frac{c-1}{c} n \right\rceil = n + \left\lceil n - \frac{n}{c} \right\rceil.$$

The premise of this corollary implies $c < n$. Hence

$$\left\lceil \frac{n(c+c')}{c} \right\rceil \leq n + \left\lceil n - \frac{n}{c} \right\rceil \leq 2n - 1.$$

Therefore when we divide n into $r = \lceil \frac{c'n}{c} \rceil$.

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = n + \left\lceil \frac{c'n}{c} \right\rceil.$$

We want to apply Corollary 2.8 with $m = c + c'$ and $r = \lceil \frac{c'n}{c} \rceil$. We need

$$\frac{m(m-1)}{2} < r \leq n.$$

$$\frac{(c+c')(c+c'-1)}{2} < \left\lceil \frac{c'n}{c} \right\rceil \leq n.$$

The second inequality is obvious. The first inequality follows from $n > \frac{c}{c'} \binom{c+c'}{2}$.

■

Note 2.12 In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

3 Tools to Show Sets Contain Rectangles

3.1 Conventions

Throughout this section we will have the following notations and conventions.

Notation 3.1 If $n, m \in \mathbb{N}$ and $A \subseteq G_{n,m}$ then we assume the following.

1. The top row of a grid is row 1.
2. We will denote that $(a, b) \in A$ by putting an R in the (a, b) position.
3. For $1 \leq j \leq m$, x_j is the number of elements of A in column j .
4. The rows and columns are reordered so that the following holds (unless we explicitly say otherwise):
 - (a) $x_1 \geq x_2 \geq \dots \geq x_m$.
 - (b) The first column has x_1 contiguous elements of A starting at row 1.
 - (c) The second column has x_2 contiguous elements of A (unless we say otherwise).
5. For $1 \leq j \leq m$, C_j is the set of rows r such that A has an element in the r^{th} row of column j . Formally

$$C_j = \{r : (r, j) \in A\}.$$

6. For $1 \leq i \leq k$ let

$$I_i = \sum_{1 \leq j_1 < \dots < j_i \leq m} |C_{j_1} \cap \dots \cap C_{j_i}|.$$

Example 3.2

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17
1		R		R										R		R	R
2	R	R								R	R		R				
3	R								R							R	R
4						R			R			R	R	R			
5		R	R			R											

$$C_1 = \{2, 3\}, C_2 = \{1, 2, 5\}, C_3 = \{5\}, C_4 = \{1\}, C_5 = \emptyset,$$

$$C_6 = \{4, 5\}, C_7 = \{\}, C_8 = \{\}, C_9 = \{3, 4\}, C_{10} = \{2\},$$

$$C_{11} = \{2\}, C_{12} = \{4\}, C_{13} = \{2, 4\}, C_{14} = \{1, 4\}, C_{15} = \{3\},$$

$$C_{16} = \{1, 3\}, C_{17} = \{1\},$$

In this example A is rectangle free. Hence, for all $i < j$ $|C_i \cap C_j| \leq 1$. Hence we have the following observations.

1. I_1 is the number of R 's in the grid which is 22.
2. I_2 is the number of pairs of columns that intersect. We list all of the intersecting pairs that are nonempty by listing what $C_{j'}$ intersects with C_j where $j' > j$.

C_1 intersects $C_2, C_9, C_{10}, C_{11}, C_{13}, C_{15}, C_{16}$;

C_2 intersects $C_3, C_4, C_6, C_{10}, C_{11}, C_{13}, C_{14}, C_{16}, C_{17}$;

C_3 intersects C_6 ;

C_4 intersects C_{14}, C_{16}, C_{17} ;

C_6 intersects $C_9, C_{12}, C_{13}, C_{14}$;

C_9 intersects $C_{12}, C_{13}, C_{14}, C_{15}, C_{16}$;

C_{10} intersects C_{11}, C_{13} ,

C_{11} intersects C_{13} ;

C_{12} intersects C_{13}, C_{14} ;

C_{13} intersects C_{14}

C_{14} intersects C_{16}, C_{17} ;

C_{15} intersects C_{16} ;

C_{16} intersects C_{17} ;

Therefore $I_2 = 39$.

3. I_3 is the number of triples of columns that intersect. We list all of the intersecting triplets that are nonempty:

$(C_1, C_2, C_{10}), (C_1, C_2, C_{11}), (C_1, C_2, C_{13}), (C_1, C_9, C_{15}), (C_1, C_9, C_{16}), (C_1, C_{10}, C_{11}), (C_1, C_{10}, C_{13}), (C_1, C_{11}, C_{13}), (C_1, C_{15}, C_{16})$;

$(C_2, C_4, C_{14}), (C_2, C_4, C_{16}), (C_2, C_4, C_{17}), (C_2, C_{10}, C_{11}), (C_2, C_{10}, C_{13}), (C_2, C_{11}, C_{13}), (C_2, C_{14}, C_{16}), (C_2, C_{14}, C_{17}), (C_2, C_{16}, C_{17})$;

$(C_4, C_{14}, C_{16}), (C_4, C_{14}, C_{17}), (C_4, C_{16}, C_{17})$;

$(C_6, C_9, C_{12}), (C_6, C_9, C_{13}), (C_6, C_9, C_{14}), (C_6, C_{12}, C_{13}), (C_6, C_{12}, C_{14}), (C_6, C_{13}, C_{14})$;

$(C_9, C_{15}, C_{16}), (C_9, C_{12}, C_{13}), (C_9, C_{12}, C_{14}), (C_9, C_{13}, C_{14})$;

(C_{10}, C_{11}, C_{13}) ;

$(C_{12}, C_{13}, C_{14});$

$(C_{14}, C_{15}, C_{16});$

Hence $I_3 = 34$.

4. I_4 is the number of 4-tuples of columns that intersect. We list all of the intersecting 4-sets that are nonempty:

$(C_1, C_2, C_{10}, C_{11}), (C_1, C_2, C_{10}, C_{13}), (C_1, C_9, C_{15}, C_{16}), (C_1, C_{10}, C_{11}, C_{13});$

$(C_2, C_4, C_{14}, C_{16}), (C_2, C_4, C_{14}, C_{17}), (C_2, C_4, C_{16}, C_{17}), (C_2, C_{10}, C_{11}, C_{13}), (C_2, C_{14}, C_{16}, C_{17});$

$(C_4, C_{14}, C_{16}, C_{17});$

$(C_6, C_9, C_{12}, C_{13}), (C_6, C_9, C_{12}, C_{14}), (C_6, C_9, C_{13}, C_{14});$

$(C_6, C_{12}, C_{13}, C_{14});$

$(C_9, C_{12}, C_{13}, C_{14});$

Hence $I_4 = 15$.

5. I_5 is the number of 5-tuples of columns that intersect. We list all of the intersecting 5-sets that are nonempty:

$(C_1, C_2, C_{10}, C_{11}, C_{13});$

$(C_2, C_4, C_{14}, C_{16}, C_{17});$

$(C_6, C_9, C_{12}, C_{13}, C_{14});$

Hence $I_5 = 3$.

6. I_6 is the number of 6-tuples of columns that intersect. There are none of these, so $I_6 = 0$.

Def 3.3 Let $n, m \in \mathbb{N}$ and $A \subseteq G_{n,m}$. Let $1 \leq i_1 < i_2 \leq n$. C_{i_1} and C_{i_2} intersect if $C_{i_1} \cap C_{i_2} \neq \emptyset$. The following picture portrays this happening with C_1 and C_2 .

	1	2	...
1	R		...
2	R		...
\vdots	\vdots	\vdots	\vdots
$x_1 - 1$	R		...
x_1	R	R	...
$x_1 + 1$		R	...
$x_1 + 2$		R	...
\vdots	\vdots	\vdots	\vdots
$x_1 + x_2 - 1$		R	...
$x_1 + x_2$...
$x_1 + x_2 + 1$...
\vdots	\vdots	\vdots	\vdots
n			...

3.2 $\sum_{j=1}^k x_j$ and $|\bigcup_{j=1}^k C_j|$

Lemma 3.4 *Let $n, m \in \mathbf{N}$. Let A be a rectangle free subset of $G_{n,m}$. Let $1 \leq j_1 < j_2 \leq n$. Then $|C_{j_1} \cap C_{j_2}| \leq 1$.*

Proof:

The following picture shows what happens if $|C_1 \cap C_2| \geq 2$. Note that a rectangle is formed. We leave it to the reader to make this into a formal argument.

	1	2	...
1	R		...
2	R		...
\vdots	\vdots	\vdots	\vdots
$x_1 - 1$	R	R	...
x_1	R	R	...
$x_1 + 1$		R	...
$x_1 + 2$		R	...
\vdots	\vdots	\vdots	\vdots
$x_1 + x_2 - 2$		R	...
$x + x_2 - 1$...
\vdots	\vdots	\vdots	\vdots
n			...

■

Lemma 3.5 *Let $n, m \in \mathbf{N}$. Let $1 \leq k \leq m$. Let $x_1, \dots, x_M \in \mathbf{N}$. Assume (x_1, \dots, x_m) is (n, m) -placeable via A . (We need not assume that $x_1 \geq \dots \geq x_m$ and hence can use this for any set of columns.) Then the following two statements are true:*

1. $x_1 + \dots + x_k \leq n + \binom{k}{2}$.
2. If $\bigcap_{j=1}^k C_j \neq \emptyset$ then $\sum_{j=1}^k x_j \leq n + \sum_{j=2}^k (-1)^j \binom{k}{j}$.

Proof:

We begin with facts that are useful to prove both parts.

By the law of inclusion-exclusion

$$\left| \bigcup_{j=1}^k C_j \right| = \sum_{j=1}^k |C_j| - I_2 + I_3 - I_4 + \dots + (-1)^{k+1} I_k.$$

Since $|C_j| = x_j$ we have

$$\begin{aligned} \left| \bigcup_{j=1}^k C_j \right| &= \sum_{j=1}^k x_j - I_2 + I_3 - I_4 + \dots + (-1)^{k+1} I_k. \\ \sum_{j=1}^k x_j &= \left| \bigcup_{j=1}^k C_j \right| + I_2 - I_3 + I_4 + \dots + (-1)^k I_k. \end{aligned}$$

Since $\left| \bigcup_{j=1}^k C_j \right| \leq n$ we have

$$\sum_{j=1}^k x_j \leq n + I_2 - I_3 + I_4 + \dots + (-1)^k I_k.$$

1) Assume k is odd (the case of k even is similar).

$$\sum_{j=1}^k x_j \leq n + I_2 + (I_4 - I_3) + \dots + (I_{k-1} - I_{k-2}) - I_k.$$

Since $I_2 \leq \binom{k}{2}$ and, for $3 \leq j \leq k-2$, $(I_{j+1} - I_j) \leq 0$ and $-I_k \leq 0$ we have

$$\sum_{j=1}^k x_j \leq n + \binom{k}{2}.$$

2) Since $C_1 \cap \dots \cap C_k \neq \emptyset$, for all j , $I_j = \binom{k}{j}$.

Hence

$$\sum_{j=1}^k x_j \leq n + \sum_{j=2}^k (-1)^j I_j = n + \sum_{j=2}^k (-1)^j \binom{k}{j}.$$

■

It will be convenient to specify the $k = 2$ case of Lemma 3.5.1.

Lemma 3.6 *Let $n, m \in \mathbf{N}$. Assume (x_1, \dots, x_m) is (n, m) -placeable via A . Then $x_1 + x_2 \leq n + 1$.*

3.3 Using maxrf

Lemma 3.7 *Let $n, m \in \mathbf{N}$. Let $x \leq x_1 \leq n$. Assume (x_1, \dots, x_m) is (n, m) -placeable via A . Then*

$$|A| \leq x + m - 1 + \text{maxrf}(n - x, m - 1).$$

Proof: The following picture portrays what might happen in the case of $n = 12$, $x_1 = 8$. We use double lines to partition the grid in a way that will be helpful later.

	1	2	3	4	5	...	j	...	m
1	R	R				
2	R		R			
3	R			R		
4	R				R	
5	R					
6	R					
7	R					R
8	R					...	R	...	
9		R	R			
10		R		R		
11		R				
12		R				

We view this grid in three parts.

Part 1: The first column. This has x_1 elements of A in it.

Part 2: Consider the grid consisting of rows $1, \dots, x_1$ and columns $2, \dots, m$. Look at the j^{th} column, $2 \leq j \leq m$ in this grid. For each such j , this column has at most one element in A (else there would be a rectangle using the first column). Hence the total number of elements of A from this part of the grid is $m - 1$.

Part 3: The bottom most $n - x_1$ elements of the right most $m - 1$ columns. This clearly has $\leq \text{maxrf}(n - x_1, m - 1)$ elements in it.

Taking all the parts into account we obtain

$$|A| \leq x_1 + (m - 1) + \text{maxrf}(n - x_1, m - 1).$$

We leave it as an exercise to show that, if $x \leq x_1$, then

$$x_1 + (m - 1) + \text{maxrf}(n - x_1, m - 1) \leq x + (m - 1) + \text{maxrf}(n - x, m - 1).$$

■

4 Tools for Finding Proper c -colorings

4.1 Strong c -colorings and Strong (c, c') -colorings

Def 4.1 Let $c, c', n, m \in \mathbb{N}$ and let $\chi : G_{n,m} \rightarrow [c]$. Assume $c' \leq c$.

1. A *half-mono rectangle with respect to χ* is a rectangle where the left corners are the same color and the right corners are the same color.
2. χ is a *strong c -coloring* if there are no half-mono rectangles.
3. χ is a *strong (c, c') -coloring* if for any half-mono rectangle the color of the left corners and the right corners are (1) different, and (2) in $[c']$.

Example 4.2

1. The following is a strong 4-coloring of $G_{5,8}$.

1	1	1	4	1	1	4	4
2	2	4	1	2	4	1	4
3	4	2	2	4	2	4	1
4	3	3	3	4	4	2	2
4	4	4	4	3	3	3	3

2. The following is a strong 3-coloring of $G_{4,6}$.

1	1	3	1	3	3
2	3	1	3	1	3
3	2	2	3	3	1
3	3	3	2	2	2

3. The following is a strong $(4, 2)$ -coloring of $G_{6,15}$.

1	1	1	1	1	3	3	3	2	3	3	2	2	2	2
1	2	2	2	2	1	1	1	1	4	4	3	3	3	2
2	1	3	3	2	1	2	2	2	1	1	1	4	4	3
2	2	1	4	3	2	1	4	3	1	2	2	1	1	4
3	3	2	1	4	2	2	1	4	2	1	4	1	2	1
4	4	4	2	1	4	4	2	1	2	2	1	2	1	1

4. The following is a strong $(6, 2)$ -coloring of $G_{8,6}$.

1	1	2	2	3	6
1	2	1	2	4	5
2	1	2	1	5	4
2	2	1	1	6	3
3	4	5	6	1	2
4	5	6	4	1	1
5	6	3	3	1	2
6	3	4	5	1	2

5. The following is a strong $(5, 3)$ -coloring of $G_{8,28}$.

1	1	1	1	1	1	1	5	5	5	5	3	2	4	3	4	3	2	3	4	3	2	3	3	2	2	2	2
1	2	2	2	2	2	2	1	1	1	1	1	1	5	4	5	4	3	4	3	4	3	3	4	3	3	3	2
2	1	3	3	3	3	2	1	2	2	2	2	2	1	1	1	1	1	5	5	5	4	4	3	4	3	4	3
2	2	1	4	4	4	3	2	1	3	3	3	3	1	2	2	2	2	1	1	1	1	5	5	5	4	3	3
3	3	2	1	5	3	3	2	2	1	4	4	4	2	1	3	3	3	1	2	2	2	1	1	1	5	5	4
3	4	3	2	1	5	4	3	3	2	1	5	3	2	2	1	5	4	2	1	3	3	1	2	2	1	1	5
4	3	4	3	2	1	5	3	4	3	2	1	5	3	3	2	1	5	2	2	1	5	2	1	3	1	2	1
5	5	5	5	3	2	1	4	3	4	3	2	1	3	5	3	2	1	3	3	2	1	2	2	1	2	1	1

Lemma 4.3 *Let $c, c', n, m \in \mathbf{N}$. Let $x = \lfloor c/c' \rfloor$. If $G_{n,m}$ is strongly (c, c') -colorable then $G_{n, xm}$ is c -colorable.*

Proof:

Let χ be a strong (c, c') -coloring of $G_{n,m}$. Let the colors be $\{1, \dots, c\}$. Let χ^i be the coloring

$$\chi^i(a, b) = \chi(a, b) + i \pmod{c}.$$

(During calculations mod c we use $\{1, \dots, c\}$ instead of the more conventional $\{0, \dots, c-1\}$.)

Take $G_{n,m}$ with coloring χ . Place next to it $G_{n,m}$ with coloring $\chi^{c'}$. Then place next to that $G_{n,m}$ with coloring $\chi^{2c'}$. Keep doing this until you have $\chi^{(x-1)c'}$ placed. The following is an example using the strong $(6, 2)$ -coloring of $G_{8,6}$ in Example 4.2.4. Since $c' = 2$ and $x = 3$ we will be shifting the colors first by 2 then by 4.

1	1	2	2	3	6	3	3	4	4	5	2	5	5	6	6	1	4
1	2	1	2	4	5	3	4	3	4	6	2	5	6	5	5	2	4
2	1	2	1	5	4	4	3	4	3	1	6	6	5	6	5	3	2
2	2	1	1	6	3	4	4	3	3	2	5	6	6	5	5	4	2
3	4	5	6	1	2	5	6	2	2	3	4	1	2	4	4	5	6
4	5	6	4	1	1	6	1	2	6	3	3	2	3	4	2	5	5
5	6	3	3	1	2	1	2	5	5	3	4	3	4	1	1	5	6
6	3	4	5	1	2	2	5	6	1	3	4	4	1	2	3	5	6

We claim that the construction always creates a c -coloring of $G_{m,xn}$.

We show that there is no rectangle with the two leftmost points from the first $G_{n,m}$. From this, to show that there are no rectangles at all is just a matter of notation.

Assume that in column i_1 there are two points colored R (in this proof $1 \leq R, B, G \leq c$.) We call these *the i_1 -points*. The points cannot form a rectangle with any other points in $G_{n,m}$ since χ is a c -coloring of $G_{n,m}$. The i_1 -points cannot form a rectangle with points in columns $i_1 + m, i_1 + 2m, \dots, i_1 + (x-1)m$ since the colors of those points are $R + c' \pmod{c}, R + 2c' \pmod{c}, \dots, R + (x-1)c' \pmod{c}$, all of which are not equal to R . Is there a j , $1 \leq j \leq x-1$ and a i_2 , $1 \leq i_2 \leq m$ such that the i_1 -points form a rectangle with points in column $i_2 + jm$?

Since χ is a strong (c, c') -coloring, points in column i_2 and on the same row as the i_1 -points are either colored *differently*, or both colors are in $[c']$. We consider both of these cases.

Case 1: In column i_2 the colors are B and G where $B \neq G$ (it is possible that $B = R$ or $G = R$ but not both). By the construction, the points in column $i_2 + jm$ are colored $B + jc' \pmod{c}$ and $G + jc' \pmod{c}$. These points are colored differently, hence they cannot form a rectangle with the i_1 -points.

...	i_1	...	i_2	$i_1 + jm$...	$i_2 + jm$...
...	R	...	B	$R + jc'$...	$B + jc'$...
...	R	...	G	$R + jc'$...	$G + jc'$...

Case 2: In column i_2 the colors are both B .

...	i_1	...	i_2	$i_1 + jm$...	$i_2 + jm$...
...	R	...	B	$R + jc'$...	$B + jc'$...
...	R	...	B	$R + jc'$...	$B + jc'$...

We have $R, B \in [c']$. By the construction, the points in column $i_2 + jm$ are both colored $B + jc' \pmod{c}$. We show that $R \neq B + jc' \pmod{c}$. Since $1 \leq j \leq x-1$ we have

$$c' \leq jc' \leq (x-1)c'.$$

Hence

$$B + c' \leq B + jc' \leq B + (x-1)c'.$$

Since $B \in [c']$ we have $B + (x - 1)c' \leq xc'$. Hence

$$B + c' \leq B + jc' \leq xc'.$$

By the definition of x we have $xc' \leq c$. Since $B \in [c']$ we have $B + c' \geq c' + 1$. Hence

$$c' + 1 \leq B + jc' \leq c.$$

Since $R \in [c']$ we have that $R \neq B + jc'$. ■

4.2 Using Combinatorics and Strong (c, c') -colorings

Theorem 4.4 *Let $c \geq 2$.*

1. *There is a strong c -coloring of $G_{c+1, \binom{c+1}{2}}$.*
2. *There is a c -coloring of $G_{c+1, m}$ where $m = c \binom{c+1}{2}$.*

Proof:

1) We first do an example of our construction. In the $c = 5$ case we obtain the following coloring.

5	5	5	5	5	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	5	5	5	5	2	2	2	2	2	2
1	5	2	2	2	5	2	2	2	5	5	5	3	3	3
2	2	5	3	3	2	5	3	3	5	3	3	5	5	4
3	3	3	5	4	3	3	5	4	3	5	4	5	4	5
4	4	4	4	5	4	4	4	5	4	4	5	4	5	5

Here is our general construction. Index the columns by $\binom{[c+1]}{2}$. Color rows of column $\{x, y\}$, $x < y$, as follows.

1. Color rows x and y with color c .
2. On the other spots use the colors $\{1, 2, 3, \dots, c - 1\}$ in increasing order (the actual order does not matter).

CHARLES- this is new so please proofread

BILL- Instead of using $\{x, y\}$, why don't we use a function from $[c]$ to $\binom{[c+1]}{2}$? For instance, $f_c(x, y) = (x - 1)(c - 1) - (x^2 + x - 2)/2 + y$ should map the row-pair x, y to the right column.

CHARLES-Note that above I say that we index the columns by $\{x, y\}$. That is all we need to say. You are proposing a particular way to do it. We do not need this. If we propose

the function you mention we will need to show that it is a bijection. But all of that is unneeded.

BILL(3.7.11) - I see what you're saying. But the notation $\{x, y\}$ is just confusing on first read. Once I realized that what it meant, it followed, but it took some time to realize that. Initially I was sure that it was a mistake. Saying "column $\{x, y\}$ " just seems weird. I understand that you preface that by saying "index the columns by ...", but what I was saying is, instead of using this particular notation, say something like "let f be a function from $[c] \times [c]$ to the $\binom{c+1}{2}$ ". Then instead of using $\{x, y\}$ to refer to the column, we can say $f(x, y)$. I defined the function above to show (to myself mainly) that it exists. I agree that we shouldn't actually define an f because that would lead to more work and is unnecessary for this proof to work. But I do think its better notation to have $f(x, y)$ rather than $\{x, y\}$.

We call the coloring $\chi : G_{n,m} \rightarrow [c]$. We show that there are no half-mono rectangles. Let $RECT = \{p_1, p_2, q_1, q_2\}$ be a rectangle with p_1, p_2 in column $\{x, y\}$ and q_1, q_2 in column $\{x', y'\}$.

If any of p_1, p_2, q_1, q_2 have a color in $\{1, \dots, c-1\}$ then $RECT$ cannot be a half-mono rectangle since the colors $\{1, \dots, c-1\}$ only appear once in each column.

If $\chi(p_1) = \chi(p_2) = \chi(q_1) = \chi(q_2) = c$ then p_1 and p_2 are in rows x and y , and q_1 and q_2 are in rows x' and y' . Since $RECT$ is a rectangle $\{x, y\} = \{x', y'\}$. Hence p_1, p_2, q_1, q_2 are all in the same column. This contradicts $RECT$ being a rectangle.

2) This follows from Lemma 4.3 with $c = c$ and $c' = 1$, and Part (1) of this theorem. ■

The next theorem generalizes Theorem 4.4.

Theorem 4.5 *Let $c, c' \in \mathbb{N}$ with $c \geq 2$ and $1 \leq c' \leq c$.*

1. *There is a strong (c, c') -coloring of $G_{c+c', m}$ where $m = \binom{c+c'}{2}$.*
2. *There is a c -coloring of $G_{c+c', m'}$ where $m' = \lfloor c/c' \rfloor \binom{c+c'}{2}$.*

To prove Theorem 4.5, we will use a partition of $\binom{[2n]}{2}$ into perfect matchings of $[2n]$ for certain values of n . Each perfect matching thus has size n .

We first give some examples and then a general lemma.

Example 4.6

1. If $n = 3$, $2n = 6$, $2n - 1 = 5$. We show a partition of $\binom{[6]}{2}$ into 5 parts of size 3. We first pair up the elements as follows, each number in the top row being paired with the number below it:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 6 & 5 & 4 \\ \hline \end{array}$$

This corresponds to $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$. This is our first part of size 3.

We keep 1 fixed and keep rotating the other numbers clockwise to obtain the following parts.

$$\begin{array}{|c|c|c|} \hline 1 & 6 & 2 \\ \hline 5 & 4 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 5 & 6 \\ \hline 4 & 3 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 3 & 2 & 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & 5 \\ \hline \end{array}$$

Note that the first pair went $\{1,5\}$, $\{1,4\}$, $\{1,3\}$, $\{1,2\}$. That is, 1 was fixed but the other element decreased by 1. Also note that the second and third pair had both elements decrease by 1 except 2 goes to 6. This partition is a special case of a general construction we will have later. The same applies to the next example.

2. If $n = 4$, $2n = 8$, $2n - 1 = 7$. Here is a partition of $\binom{[8]}{2}$ into 7 parts of size 4.

We keep 1 fixed and keep rotating the other numbers clockwise to obtain the following parts.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 8 & 7 & 6 & 5 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 8 & 2 & 3 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 7 & 8 & 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 6 & 7 & 8 \\ \hline 5 & 4 & 3 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 5 & 6 & 7 \\ \hline 4 & 3 & 2 & 8 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 6 \\ \hline 3 & 2 & 8 & 7 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 8 & 7 & 6 \\ \hline \end{array}$$

The next lemma shows that such partitions always exist. The lemma (and the examples above) is based on the Wikipedia entry on Round Robin tournaments. We present a proof for completeness.

Lemma 4.7 *Let $n \in \mathbf{N}$. $\binom{[2n]}{2}$ can be partitioned into $2n - 1$ sets P_1, \dots, P_{2n-1} , each of size n , such that each P_i is itself a partition of $[2n]$ into pairs (i.e., a perfect matching).*

Proof: Following the examples above, we define the cyclic permutation ρ on $\{2, 3, \dots, 2n\}$ as follows:

$$\rho(x) = \begin{cases} x - 1 & \text{if } 2 < x \leq 2n, \\ 2n & \text{if } x = 2, \end{cases}$$

for all $x \in \{2, 3, \dots, 2n\}$. Then for each $1 \leq i \leq 2n - 1$, we define

$$P_i = \{ \{1, 2n - i + 1\} \} \cup \{ \{ \rho^{(i-1)}(j), \rho^{(i-1)}(2n - j + 1) \} \mid 2 \leq j \leq n \},$$

noting that $2n - i + 1 = \rho^{(i-1)}(2n)$. It is not too hard to see that each P_i contains exactly n pairwise disjoint pairs, so it suffices to show that no pair appears in two different P_i . Clearly, no pair of the form $\{1, x\}$ can appear in more than one P_i . Suppose $\{x, y\}$ appears in both P_i and P_j for some $i < j$, where $2 \leq x, y \leq 2n$. Then without loss of generality, x appears in the top row of P_i with y just below it. If x is still in the top row of P_j , then x has shifted to the right and y to the left, and so x and y are not vertically aligned in P_j , which means that $\{x, y\} \notin P_j$. So it must be that x is on the bottom row of P_j with y just above it. But for this to happen, x and y would have to rotate different amounts from P_i to P_j (one an even distance and the other an odd distance), but they rotate the same amount, namely, $j - i$ spaces—contradiction. Thus the P_i are as required. ■

Proof: [Proof of Theorem 4.5]

1) Here is our general construction. We split into two cases.

FENNER, WHEN YOU SAY, "WE BREAK UP THE COLUMNS", COLUMNS OF WHAT?

Case 1: $c + c'$ is even. Then $c + c' = 2n$ for some n . Since $c' \leq c$, we also have $c' \leq n$. Let P_1, \dots, P_{2n-1} be the partition of $[2n]$ of Lemma 4.7. Index the elements of each P_i as $p_{i,j}$ for $1 \leq j \leq n$, that is, $P_i = \{p_{i,1}, p_{i,2}, \dots, p_{i,n}\}$. We break up the columns into $2n - 1$ blocks of n columns each (note that $n(2n - 1) = \binom{[2n]}{2}$). We color the j^{th} column in the i^{th} block as follows:

- Assign color 1 to the two elements of $p_{i,(j+1) \bmod n}$,
- Assign color 2 to the two elements of $p_{i,(j+2) \bmod n}$,
- \vdots
- Assign color c' to the two elements of $p_{i,(j+c') \bmod n}$,

- Assign one each of the colors $c' + 1, \dots, c$ one each to the rest of the elements in the column in increasing order.

Suppose some pair $p_{i,k} = \{x, y\}$ is monochrome in two separate columns. Then both these columns must be in the i^{th} block, the j_1^{st} column (colored c_1) and j_2^{nd} column (colored c_2), say. Then we must have

$$k = (j_1 + c_1) \bmod n = (j_2 + c_2) \bmod n.$$

Since $j_1 \neq j_2$, we must have $c_1 \neq c_2$.

Case 2: $c + c'$ is odd. Then we choose a simpler partition. Let $c + c' = 2n + 1$ for some n . Since $c' < c$, we also have $c' \leq n$. For $1 \leq i \leq 2n + 1$ and $1 \leq j \leq n$, define

$$p_{i,j} = \{(i + j) \bmod (2n + 1), (i - j) \bmod (2n + 1)\} \in \binom{[2n + 1]}{2},$$

and let

$$P_i = \{p_{i,1}, \dots, p_{i,n}\}.$$

It is not too hard to see that all the pairs within the same P_i are pairwise disjoint and that no pair is contained in more than one P_i .

We now proceed with exactly the same recipe as in Case 1, except that $\binom{2n+1}{2} = n(2n+1)$, we group the columns into $2n + 1$ blocks of n columns each. We get a strong (c, c') -coloring just as in Case 1.

2) This follows from Lemma 4.3 and Part (1) of this theorem. ■

Corollary 4.8 *For all $c \geq 2$, there is a c -coloring of $G_{2c, 2c^2 - c}$.*

4.3 Using Finite Fields and Strong c -colorings

Def 4.9 Let X be a finite set and $q \in \mathbb{N}$, $q \geq 3$. Let $P \subseteq \binom{X}{q}$.

$$\text{pairs}(P) = \{\{a_1, a_2\} \in \binom{X}{2} : (\exists a_3, \dots, a_q)[\{a_1, \dots, a_q\} \in P]\}.$$

Example 4.10 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $q = 3$.

1. Let $P = \{\{1, 2, 6\}, \{1, 8, 9\}, \{2, 4, 6\}\}$. Then

$$\text{pairs}(P) = \{\{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 8\}, \{1, 9\}, \{8, 9\}, \{2, 4\}, \{4, 6\}\}$$

2. Let $P = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. Then

$$\text{pairs}(P) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{7, 8\}, \{7, 9\}, \{8, 9\}\}.$$

Lemma 4.11 *Let $c, m, r \in \mathbf{N}$. Assume there exist $P_1, \dots, P_m \subseteq \binom{[cr]}{r}$ such that the following hold.*

- *For all $1 \leq j \leq m$, P_j is a partition of $[cr]$ into c parts of size r .*
- *For all $1 \leq j_1 < j_2 \leq m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$.*

Then

1. $G_{cr,m}$ *is strongly c -colorable.*
2. $G_{cr,cm}$ *is c -colorable.*

Proof:

FENNER, IN THIS PROOF, WHERE DO THE 2K AND 2K-1 COME FROM?

1)

We define a strong c -coloring COL of $G_{cr,m}$ using P_1, \dots, P_m .

Let $1 \leq j \leq m$. Let

$$P_j = \{L_j^1, \dots, L_j^c\}$$

where each L_j^i is a subset of r elements from $[cr]$.

Let $1 \leq i \leq cr$ and $1 \leq j \leq m$. Since P_j is a partition of $[cr]$ there exists a unique u such that $i \in L_j^u$. Define

$$COL(i, j) = u.$$

We show that this is a strong c -coloring. Assume, by way of contradiction, that there exists $1 \leq i_1 < i_2 \leq 2k$ and $1 \leq j_1 < j_2 \leq 2k - 1$ such that $COL(i_1, j_1) = COL(i_1, j_2) = u$ and $COL(i_2, j_1) = COL(i_2, j_2) = v$. By definition of the coloring we have

$$i_1 \in L_{j_1}^u, i_1 \in L_{j_2}^u, i_2 \in L_{j_1}^v, i_2 \in L_{j_2}^v$$

Then

$$\{i_1, i_2\} \in \text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}),$$

contradicting the second premise on the P 's.

2) This follows from Part (1) and Lemma 4.3 with $c = c$ and $c' = 1$. ■

The Round Robin partition of Lemma 4.7 is an example of a partition satisfying the premises of Lemma 4.11, where $c = n$, $r = 2$, and $m = 2n - 1 = 2c - 1$. The next theorem yields partitions with bigger values of r .

CHARLES- proofread just the statement. BILL(3.7.11) - I Think you had a misplaced bracket. Shouldn't the p^{ds-s} be in the subscript?

Theorem 4.12 *Let p be a prime and $s, d \in \mathbf{N}$.*

1. $G_{p^{ds}, \frac{p^{ds-1}}{p-1}}$ *is strongly p^{ds-s} -colorable.*

2. $G_{p^{ds}, \frac{p^{ds}-1}{p-1} p^{ds-s}}$ is p^{ds-s} -colorable.

Proof: Let $c = p^{ds-s}$, $r = p^s$, and $m = \frac{p^{ds}-1}{p^s-1}$. We show that there exists P_1, \dots, P_m satisfying the premise of Lemma 4.11. The result follows immediately.

Let F be the finite field on p^s elements. We identify $[cr]$ with the set F^d .

Def 4.13

1. Let $\vec{x} \in F^d$, $\vec{y} \in F^d - \{0^d\}$. Then

$$L_{\vec{x}, \vec{y}} = \{\vec{x} + f\vec{y} \mid f \in F\}.$$

Sets of this form are called *lines*. Note that for all $\vec{x}, \vec{y}, a \in F$ with $a \neq 0$,

$$L_{\vec{x}, \vec{y}} = L_{\vec{x}, a\vec{y}}.$$

2. Two lines $L_{\vec{x}, \vec{y}}$, $L_{\vec{z}, \vec{w}}$ have the same slope if \vec{y} is a multiple of \vec{w} .

The following are easy to prove and well-known.

- If L and L' are two distinct lines that have the same slope, then $L \cap L' = \emptyset$.
- If L and L' are two distinct lines with different slopes, then $|L \cap L'| \leq 1$.
- If L is a line then there are exactly $r = p^s$ points on L .
- If L is a line then there are exactly $c = p^{ds-s}$ lines that have the same slope as L (this includes L itself).
- There are exactly $\frac{p^{ds}-1}{p^s-1}$ different slopes.

We define P_1, \dots, P_m as follows.

1. Pick a line L . Let P_1 be the set of lines that have the same slope as L .
2. Assume that P_1, \dots, P_{j-1} have been defined and that $j \leq m$. Let L be a line that is not in $P_1 \cup \dots \cup P_{j-1}$. Let P_j be the set of all lines that have the same slope as L .

We need to show that P_1, \dots, P_m satisfies the premises of Lemma 4.11

- a) For all $1 \leq j \leq m$, P_j is a partition of $[cr]$ into c parts of size r . Let $L \in P_j$. Note that P_j is the set of all lines with the same slope as L . Clearly this partitions F^d which is $[cr]$.
- b) For all $1 \leq j_1 < j_2 \leq m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$. Let L_1 be any line in P_{j_1} and L_2 be any line in P_{j_2} . Since $|L_1 \cap L_2| \leq 1 < 2$ we have the result.

Note that each P_j has $c = p^{ds-s}$ sets (lines) in it, each set (line) has $r = p^s$ numbers (points), and there are $m = \frac{p^{ds}-1}{p^s-1}$ many P 's. Hence the premises of Lemma 4.11 are satisfied.

■

It is convenient to state the $s = 1, d = 2$ case of Theorem 4.12.

BILL, ISN'T THIS A COMBINATION OF THEOREMS 4.14 AND THEOREM 4.13?

CHARLES- I HAVE CHANGED THEOREM 4.14 SO THAT NOW IT MAKES SENSE.

BILL, THIS COROLLARY CAN BE FOUND FROM EITHER THE THEOREM ABOVE IT (WITH $s = 1$ and $d = 2$ as stated) OR FROM THE LEMMA ABOVE IT (with $r = c = p$ and $m = p+1$).

CHARLES- THE COROLLARY DOES NOT FOLLOW FROM THE LEMMA SINCE THE LEMMA NEEDS THOSE SETS P_1, \dots, P_m . BILL(3.7.11) - OK.

Corollary 4.14 *Let p be a prime.*

1. *There is a strong p -coloring of $G_{p^2, p+1}$.*
2. *There is a p -coloring of G_{p^2, p^2+p} .*

Note 4.15 It would be of interest to obtain a lemma similar to Theorem 4.12 that does not need prime powers and possibly yields strong (c, c') -colorings.

5 Bounds on the Sizes of Obstruction Sets

5.1 An Upper Bound

Using the uncolorability bounds, we can obtain an upper-bound on the size of a c -colorable grid.

Theorem 5.1 *For all $c > 0$, G_{c^2+c, c^2+c} is not c -colorable.*

Proof: We apply Corollary 2.9 with $m = c^2 + c$ and $n = c^2 + c$. Note that

$$\begin{aligned} \left\lceil \frac{nm}{c} \right\rceil &= \left\lceil \frac{(c^2 + c)(c^2 + c)}{c} \right\rceil \\ &= (c + 1)(c^2 + c). \end{aligned}$$

Letting $q = c + 1$ and $r = 0$, we have

$$\begin{aligned} \frac{m(m-1) - 2qr}{q(q-1)} &= \frac{(c^2 + c)(c^2 + c - 1)}{(c + 1)c} \\ &= c^2 + c - 1 \\ &< c^2 + c \\ &= n. \end{aligned}$$

■

Using this, we can obtain an upper-bound on the size of an obstruction set.

Theorem 5.2 *If $c > 0$, then $|\text{OBS}_c| \leq 2c^2$.*

Proof: For each r , there can be at most one c -minimal grid of the form $G_{r,n}$. Likewise, there can be at most one c -minimal grid of the form $G_{n,r}$. If $r \leq c$ then for all n , $G_{r,n}$ and $G_{n,r}$ are trivially c -colorable and are, therefore, not c -minimal. Theorem 5.1 shows that for all $n, m > c^2 + c$, $G_{n,m}$ is not c -minimal. It follows that there can be at most two c -minimal grids for each integer r where $c < r \leq c^2 + c$. Therefore there are at most $2c^2$ c -minimal grids in OBS_c . ■

5.2 A Lower Bound

To get a lower bound on $|\text{OBS}_c|$, we will combine Corollary 2.11 and Theorem 4.5(2) with the following lemma:

Lemma 5.3 *Suppose that $G_{m_1,n}$ is c -colorable and $G_{m_2,n}$ is not c -colorable. Then there exists a grid $G_{x,y} \in \text{OBS}_c$ such that $m_1 < x \leq m_2$ (and in addition, $y \leq n$).*

Proof: Given n , let x be the least integer such that $G_{x,n}$ is not c -colorable. Clearly, $m_1 < x \leq m_2$. Now given x as above, let y be least such that $G_{x,y}$ is not c -colorable. Clearly, $y \leq n$ and $G_{x,y} \in \text{OBS}_c$. ■

Theorem 5.4 $|\text{OBS}_c| \geq 2\sqrt{c}(1 - o(1))$.

Proof: For any $c \geq 2$ and any $1 \leq c' \leq c$ we can summarize Corollary 2.11 and Theorem 4.5(2) as follows:

$$G_{c+c',n} \text{ is } \begin{cases} c\text{-colorable} & \text{if } n \leq \lfloor \frac{c}{c'} \rfloor \binom{c+c'}{2}, \\ \text{not } c\text{-colorable} & \text{if } n > \frac{c}{c'} \binom{c+c'}{2}. \end{cases}$$

(We won't use the fact here, but note that this is tight if c' divides c .)

Suppose $c' > 1$ and

$$\frac{c}{c'} \binom{c+c'}{2} < \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}. \quad (1)$$

Then letting $n := \lfloor \frac{c}{c'-1} \rfloor \binom{c+c'-1}{2}$, we see that $G_{c+c'-1,n}$ is c -colorable, but $G_{c+c',n}$ is not. Then by Lemma 5.3, there is a grid $G_{c+c',y} \in \text{OBS}_c$ for some y . So there are at least as many elements of OBS_c as there are values of c' satisfying Inequality (1)—actually twice as many, because $G_{n,m} \in \text{OBS}_c$ iff $G_{m,n} \in \text{OBS}_c$.

Fix any real $\varepsilon > 0$. Clearly, Inequality (1) holds provided

$$\frac{c}{c'} \binom{c+c'}{2} \leq \left(\frac{c}{c'-1} - 1 \right) \binom{c+c'-1}{2}.$$

A rather tedious calculation reveals that if $2 \leq c' \leq (1 - \varepsilon)\sqrt{c}$, then this latter inequality holds for all large enough c . Including the grid $G_{c+1,n} \in \text{OBS}_c$ where $n = c \binom{c+1}{2} + 1$, we then get $|\text{OBS}_c| \geq \lfloor (1 - \varepsilon)\sqrt{c} \rfloor$ for all large enough c , and since ε was arbitrary, we therefore have $|\text{OBS}_c| \geq \sqrt{c}(1 - o(1))$.

To double the count, we notice that $c + c' \leq \lfloor \frac{c}{c'} \rfloor \binom{c+c'}{2}$, hence $G_{c+c',c+c'}$ is c -colorable by Theorem 4.5(2). This means that $G_{c+c',y} \in \text{OBS}_c$ for some $y > c + c'$, and so we can count $G_{y,c+c'} \in \text{OBS}_c$ as well without counting any grids twice. ■

6 Which Grids Can be Properly 2-Colored?

Lemma 6.1

1. $G_{7,3}$ and $G_{3,7}$ are not 2-colorable
2. $G_{5,5}$ is not 2-colorable.
3. $G_{7,2}$ and $G_{2,7}$ are 2-colorable (this is trivial).
4. $G_{6,4}$ and $G_{4,6}$ are 2-colorable.

Proof:

We only consider grids of the form $G_{n,m}$ where $n \geq m$.
1,2)

CHARLES- PROOFREAD this proof, I have changed it a bit. BILL - Looks ok. In the following table we show that $G_{7,3}$ and $G_{5,5}$ are not 2-colorable. For each (n, m) we plan to use either Corollary 2.8 or 2.9. In the table we give, for each (n, m) , the value of $\lceil \frac{nm}{2} \rceil$, the q, r such that $\lceil \frac{nm}{2} \rceil = qn + r$ with $0 \leq r \leq n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (arith).

m	n	$\lceil \frac{nm}{2} \rceil$	q	r	Use	Prem	Arith
3	7	11	1	4	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n1$	$3 < 4 \leq 7$
5	5	13	2	3	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n1$	$4 < 5$

Table 1: Uncolorability values for $c = 2$

- 3) $G_{7,2}$ is clearly 2-colorable.
- 4) $G_{6,4}$ is 2-colorable by Corollary 4.14 with $p = 2$.

■

Theorem 6.2 $\text{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

Proof:

$G_{7,3}$ is not 2-colorable by Lemma 6.1. $G_{6,3}$ is 2-colorable by Lemma 6.1. $G_{7,2}$ is 2-colorable by Lemma 6.1. Hence $G_{7,3}$ is 2-minimal. The proof for $G_{3,7}$ is similar.

$G_{5,5}$ is not 2-colorable by Lemma 6.1. $G_{5,4}$ and $G_{4,5}$ are 2-colorable by Lemma 6.1. Hence $G_{5,5}$ is 2-minimal.

We need to show that $G_{7,3}$, $G_{5,5}$, and $G_{3,7}$ are the only 2-minimal grids. We consider the different possible values of n with $m \leq n$ and then use symmetry.

n	$m \leq n$	comment
1, 2, 3, 4	any $m \leq n$	$G_{n,m}$ is 2-colorable by Lemma 6.1
n	1, 2	$G_{n,m}$ is 2-colorable by Lemma 6.1
5	3, 4	$G_{n,m}$ is 2-colorable by Lemma 6.1
5	5	$G_{5,5} \in \text{OBS}_2$
6	3, 4	$G_{n,m}$ is 2-colorable by Lemma 6.1
6	5, 6	$G_{n,m}$ is not 2-minimal since $G_{5,5}$ not 2-colorable
7	3	$G_{7,3} \in \text{OBS}_2$
$n \geq 7$	$4, \dots, n$	$G_{n,m}$ is not 2-minimal since $G_{7,3}$ not 2-colorable

■

The following chart indicates exactly which grids are 2-colorable. The entry for (n, m) is C if $G_{n,m}$ is 2-colorable, and N if $G_{n,m}$ is not 2-colorable.

	2	3	4	5	6	7	8
2	C	C	C	C	C	C	C
3	C	C	C	C	C	N	N
4	C	C	C	C	C	N	N
5	C	C	C	N	N	N	N
6	C	C	C	N	N	N	N
7	C	N	N	N	N	N	N
8	C	N	N	N	N	N	N

Table 2: Table of 3-colorable grids (C) and non 3-colorable grids N

7 Which Grids Can be Properly 3-Colored?

Lemma 7.1

1. $G_{19,4}$ and $G_{4,19}$ are not 3-colorable.
2. $G_{16,5}$ and $G_{5,16}$ are not 3-colorable.
3. $G_{13,7}$ and $G_{7,13}$ are not 3-colorable.

4. $G_{12,10}$ and $G_{10,12}$ are not 3-colorable.
5. $G_{11,11}$ is not 3-colorable.
6. $G_{19,3}$ and $G_{3,19}$ are 3-colorable (this is trivial).
7. $G_{18,4}$ and $G_{4,18}$ are 3-colorable.
8. $G_{15,6}$ and $G_{6,15}$ are 3-colorable.
9. $G_{12,9}$ and $G_{9,12}$ are 3-colorable.

Proof: We just consider the grids $G_{n,m}$ where $n \geq m$.
1, 2, 3, 4, 5)

In the following table we show that several grids are not 3-colorable. For each (n, m) we plan to use either Corollary 2.8 or 2.9. In the table we give, for each (n, m) , the value of $\lceil \frac{nm}{3} \rceil$, the q, r such that $\lceil \frac{nm}{3} \rceil = qn + r$ with $0 \leq r \leq n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (arith).

m	n	$\lceil \frac{nm}{3} \rceil$	q	r	Use	Prem	Arith
4	19	26	1	7	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n$	$6 < 7 \leq 19$
5	16	27	1	11	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n$	$10 < 11 \leq 16$
7	13	31	2	5	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$11 < 13$
10	12	40	3	4	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$11 < 12$
11	11	41	3	8	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$10\frac{1}{3} < 11$

Table 3: Uncolorability values for $c = 3$

- 6) $G_{19,3}$ is clearly 3-colorable.
- 7) $G_{18,4}$ is 3-colorable by Theorem 4.4 with $c = 3$.
- 8) $G_{15,6}$ is 3-colorable by Corollary 4.8 with $c = 3$.
- 9) $G_{12,9}$ is 3-colorable by Corollary 4.14 with $p = 3$. ■

Lemma 7.2 $G_{10,10}$ is 3-colorable.

Proof: This is the 3-coloring:

■

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

Table 4: Proper 3-coloring of $G_{10,10}$.

Note 7.3 The coloring in Lemma 7.2 we found by first finding a size 34 rectangle free subset of $G_{10,10}$ (by hand). We used that for one of the colors. The final 3-coloring was then found with a simple computer program. It is an open problem to find a general theorem that has a corollary that $G_{10,10}$ is 3-colorable.

Lemma 7.4 *If $A \subseteq G_{11,10}$ and A is rectangle-free then $|A| \leq 36 = \lceil \frac{11 \cdot 10}{3} \rceil - 1$. Hence $G_{11,10}$ is not 3-colorable.*

Proof:

We divide the proof into cases. Every case will either conclude that $|A| \leq 36$ or A cannot exist.

For $1 \leq j \leq 10$ let x_j be the number of elements of A in column j . We assume

$$x_1 \geq \cdots \geq x_{10}.$$

1. $5 \leq x_1 \leq 11$.

By Lemma 3.7 with $x = 5$, $n = 11$, $m = 10$ we have

$$|A| \leq x + m - 1 + \text{maxrf}(n - x, m - 1) \leq 5 + 10 - 1 + \text{maxrf}(11 - 5, 10 - 1) \leq 14 + \text{maxrf}(6, 9).$$

By Lemma 12.1 we have $\text{maxrf}(6, 9) = 21$. Hence

$$|A| \leq 14 + 21 = 35 \leq 36.$$

2. $x_1 \leq 3$. Then $|A| \leq 3 \times 11 = 33 < 36$. Note that we now know that $x_1 = 4$.

3. There exists k , $1 \leq k \leq 6$, such that $x_1 = \cdots = x_k = 4$ and $x_{k+1} \leq 3$. Then

$$|A| = \sum_{j=1}^{10} x_j = \left(\sum_{j=1}^k x_j \right) + \left(\sum_{j=k+1}^{10} x_j \right) \leq 4k + 3(10 - k) = 30 + k$$

Since $k \leq 6$ this quantity is $\leq 30 + 6 = 36$. Hence $|A| \leq 36$.

4. $x_1 = \cdots = x_7 = 4$ and, for all, $1 \leq j_1 < j_2 < j_3 \leq 7$,

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 0.$$

Let G' be the grid restricted to the first 7 columns. Let B be A restricted to G' . Since every column of G' has 4 elements of B , $|B| = 7 \times 4 = 28$. Since every row of G' has ≤ 2 elements of B , $|B| \leq 2 \times 11 = 22$. Therefore A does not exist.

5. $x_1 = \cdots = x_7 = 4$ and there exists $1 \leq j_1 < j_2 < j_3 \leq 7$ such that

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 1,$$

but for all $1 \leq j_1 < j_2 < j_3 < j_4 \leq 7$

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 0.$$

By renumbering we can assume that

$$|C_1 \cap C_2 \cap C_3| = 1$$

and that the intersection is in row 11. Let G' be the grid restricted to the first 7 columns. Let B be A restricted to G' . The following picture portrays what is in the first 3 columns of G' .

	1	2	3	4	5	6	7
1	R						
2	R						
3	R						
4		R					
5		R					
6		R					
7			R				
8			R				
9			R				
10							
11	R	R	R				

Since there are no $1 \leq j_1 < j_2 < j_3 < j_4 \leq 7$ with

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 1,$$

there will be no other elements of A in row 11 of G' .

Consider C_4 . Since $|C_4| = 4$, $|C_1 \cap C_4| \geq 1$, $|C_2 \cap C_4| \geq 1$, $|C_3 \cap C_4| \geq 1$, and $11 \notin C_4$ we must have (1) $|C_1 \cap C_4| = 1$, $|C_2 \cap C_4| = 1$, $|C_3 \cap C_4| = 1$, and (2) $10 \in C_4$.

By the same reasoning on C_5, C_6, C_7 we have that, for all $5 \leq i \leq 7$, (1) $|C_1 \cap C_i| = 1$, $|C_2 \cap C_i| = 1$, $|C_4 \cap C_i| = 1$, and (2) $10 \in C_i$.

The following picture portrays all that we know so far.

	1	2	3	4	5	6	7
1	R						
2	R						
3	R						
4		R					
5		R					
6		R					
7			R				
8			R				
9			R				
10				R	R	R	R
11	R	R	R				

Note that $C_4 \cap C_5 \cap C_6 \cap C_7 \neq \emptyset$. This contradicts the premise of this case.

BILL- I CHANGED THE WRAPUP OF THE PROOF.

CHARLES-WHY? The lines above:

Note that $C_4 \cap C_5 \cap C_6 \cap C_7 \neq \emptyset$. This contradicts the premise of this case.

SEEM TO WRAP IT UP. I DO NOT KNOW WHY WE ARE SAYING ANY MORE. EITHER AGREE AND DELETE THE NEW WRAP UP AND LEAVE ME A NOTE SAYING YOU DID, OR TELL ME WHAT IS WRONG WITH WHAT I HAD.

BILL (3.7.11) - I see what you're saying now. Yeah, that makes sense and we can remove the rest of the proof after the intersection being nonempty.

Without loss of generality we can assume $C_4 = \{1, 4, 7, 10\}$. Consider C_5 . In addition to the above constraints on C_5 we also have that $1, 4, 7 \notin C_5$. Hence, without loss of generality, we can assume that $C_5 = \{2, 5, 8, 10\}$. In order to meet our requirements, we need $C_6 = \{3, 6, 9, 10\}$, but now there is no way to color C_7 without getting a rectangle. So A cannot exist.

6. $x_1 = \dots = x_7 = 4$ and there exists $1 \leq j_1 < j_2 < j_3 < j_4 \leq 7$ such that

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 1.$$

By renumbering we can assume that

$$|C_1 \cap C_2 \cap C_3 \cap C_4| = 1.$$

By Lemma 3.5.2 with $k = 4$ and $x_1 = x_2 = x_3 = x_4 = 4$:

$$16 = x_1 + x_2 + x_3 + x_4 \leq 11 + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 11 + 6 - 4 + 1 = 14.$$

Hence A does not exist.

■

Theorem 7.5

$$\text{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}, G_{10,11}, G_{7,13}, G_{5,16}, G_{4,19}\}.$$

Proof:

For each $G_{n,m}$ listed above, (1) $G_{n,m}$ is not 3-colorable by Lemma 7.1 or 7.4, (2) both $G_{n-1,m}$ and $G_{n,m-1}$ are 3-colorable by Lemma 7.1 or 7.2. Hence all of the grids listed are in OBS_3 . We need to show that no other grids are in OBS_3 . This is a straightforward use of Lemmas 7.1, 7.2, and 7.4. The proof is similar to how Theorem 6.2 was proven. We leave the details to the reader. ■

The following chart indicates exactly which grids are 3-colorable. The entry for (n, m) is C if $G_{n,m}$ is 3-colorable, and N if $G_{n,m}$ is not 3-colorable.

8 Which Grids Can be Properly 4-Colored?

In the first section we give absolute results about which grids are 4 colorable. In the second section we give results that assume a conjecture.

8.1 Absolute Results

Theorem 8.1

1. $G_{41,5}$ and $G_{5,41}$ are not 4-colorable.
2. $G_{31,6}$ and $G_{6,31}$ are not 4-colorable.
3. $G_{29,7}$ and $G_{7,29}$ are not 4-colorable.

	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
3	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
4	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>
5	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
6	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
7	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
8	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
9	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
10	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
11	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
12	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
13	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
14	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
15	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
16	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
17	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
18	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
19	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
20	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>

Table 5: Summary of information on which grids are 3-colorable.

4. $G_{25,9}$ and $G_{9,25}$ are not 4-colorable.
5. $G_{23,10}$ and $G_{10,23}$ are not 4-colorable.
6. $G_{22,11}$ and $G_{11,22}$ are not 4-colorable.
7. $G_{21,13}$ and $G_{13,21}$ are not 4-colorable.
8. $G_{20,17}$ and $G_{17,20}$ are not 4-colorable.
9. $G_{19,18}$ and $G_{18,19}$ are not 4-colorable.
10. $G_{41,4}$ and $G_{4,41}$ are 4-colorable (this is trivial).
11. $G_{40,5}$ and $G_{5,40}$ are 4-colorable.
12. $G_{30,6}$ and $G_{6,30}$ are 4-colorable.
13. $G_{28,8}$ and $G_{8,28}$ are 4-colorable.
14. $G_{20,16}$ and $G_{16,20}$ are 4-colorable.

Proof:

We only consider grids $G_{n,m}$ where $n \geq m$.

1,2,3,4,5,6,7,8,9)

In the following table we show that several grids are not 4-colorable. For each (n, m) we plan to use either Corollary 2.8 or 2.9. In the table we give, for each (n, m) , the value of $\lceil \frac{nm}{4} \rceil$, the q, r such that $\lceil \frac{nm}{4} \rceil = qn + r$ with $0 \leq r \leq n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (arith).

m	n	$\lceil \frac{nm}{4} \rceil$	q	r	Use	Prem	Arith
5	41	52	1	11	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n$	$10 < 11 \leq 41$
6	31	47	1	16	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n$	$15 < 16 \leq 31$
7	29	51	1	22	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n$	$21 < 22 \leq 29$
9	25	57	2	7	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$22 < 25$
10	23	58	2	12	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$21 < 23$
11	22	61	2	17	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$21 < 22$
13	21	69	3	6	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$20 < 21$
17	20	85	4	5	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$19\frac{1}{3} < 20$
18	19	86	4	10	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$18\frac{5}{6} < 19$

Table 6: Uncolorability values for $c = 4$

10) $G_{41,4}$ is clearly 4-colorable.

- 11) $G_{40,5}$ is 4-colorable by Theorem 4.4 with $c = 4$.
- 12) $G_{30,6}$ is 4-colorable by Theorem 4.5 with $c = 4$ and $c' = 2$.
- 13) $G_{28,8}$ is 4-colorable by Theorem 4.12 with $p = 2$, $d = 3$, and $s = 1$.
- 14) $G_{20,16}$ is 4-colorable by Theorem 4.12 with $p = 2$, $d = 2$, and $s = 2$

■

Lemma 8.2 *If $A \subseteq G_{19,17}$ and A is rectangle-free then $|A| \leq 80 = \lceil \frac{19 \cdot 17}{4} \rceil - 1$. Hence $G_{19,17}$ is not 4-colorable.*

Proof: We divide the proof into cases. Every case will either conclude that $|A| \leq 80$ or A cannot exist.

For $1 \leq j \leq 17$ let x_j be the number of elements of A in column j . We assume

$$x_1 \geq \cdots \geq x_{17}.$$

1. $6 \leq x_1 \leq 19$.

By Lemma 3.7 with $x = 6$, $n = 19$, $m = 17$,

$$|A| \leq x+m-1+\text{maxrf}(n-x, m-1) \leq 6+17-1+\text{maxrf}(19-6, 17-1) = 22+\text{maxrf}(13, 16).$$

Assume, by way of contradiction, that $|A| \geq 81$. Then $\text{maxrf}(13, 16) \geq 59$ By Lemma 2.7 with $n = 16$, $m = 13$, $a = 59$, $q = 3$, $r = 11$

$$16 \leq \left\lfloor \frac{13 \times 12 - 2 \times 3 \times 11}{3 \times 2} \right\rfloor = 15.$$

This is a contradiction.

2. There exists k , $1 \leq k \leq 12$, such that $x_1 = \cdots = x_k = 5$ and $x_{k+1} \leq 4$. Then

$$|A| = \sum_{j=1}^{17} x_j = \left(\sum_{j=1}^k x_j \right) + \left(\sum_{j=k+1}^{17} x_j \right) \leq 5k + 4(17 - k) = 68 + k.$$

Since $k \leq 12$ this quantity is $\leq 68 + 12 = 80$. Hence $|A| \leq 80$.

3. $x_1 = x_2 = \cdots = x_{13} = 5$. Look at the grid restricted to the first 13 columns. Let B be A restricted to that grid. Note that B is a rectangle-free subset of $G_{19,13}$ of size 65. By Lemma 2.7 with $n = 19$, $m = 13$, $a = 65$, $q = 3$, and $r = 8$ we have

$$19 \leq \left\lfloor \frac{13 \times 12 - 2 \times 8 \times 3}{3 \times 2} \right\rfloor = 18.$$

This is a contradiction, hence A cannot exist.

■

Lemma 8.3 $G_{24,9}$ is 4-colorable.

Proof: We show that $G_{9,6}$ is strongly $(4,1)$ -colorable and then apply Lemma 4.3 with $c = 4$ and $c' = 1$.

The following is a strong 4-coloring of $G_{9,6}$.

	1	2	3	4	5	6
1	Y	R	R	Y	R	R
2	Y	B	B	R	Y	B
3	Y	G	G	B	B	Y
4	R	Y	G	Y	G	B
5	B	Y	R	B	Y	G
6	G	Y	B	G	R	Y
7	G	B	Y	Y	B	G
8	R	G	Y	G	Y	R
9	B	R	Y	R	G	Y

Table 7: Strong 4-coloring of $G_{9,6}$.

■

Lemma 8.4 $G_{21,11}$ is 4-colorable

Proof: Brad Larsen has obtained the following 4-coloring of $G_{21,11}$.

■

Lemma 8.5 $G_{22,10}$ is 4-colorable

Proof: Brad Larsen has obtained the following 4-coloring of $G_{22,10}$.

■

Theorem 8.6

1. The following grids are in OBS_4 :

$G_{41,5}$, $G_{31,6}$, $G_{29,7}$, $G_{25,9}$, $G_{23,10}$, $G_{22,11}$, $G_{11,22}$, $G_{10,23}$, $G_{9,25}$, $G_{7,29}$, $G_{6,31}$, $G_{5,41}$.

2. For each of the following grids it is not known if it is 4-colorable. These are the only such. $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, $G_{18,18}$, $G_{21,12}$, $G_{12,21}$.

3. Exactly one of the following grids is in OBS_4 : $G_{21,12}$ $G_{21,13}$.

	1	2	3	4	5	6	7	8	9	10	11
1	<i>G</i>	<i>B</i>	<i>B</i>	<i>G</i>	<i>R</i>	<i>Y</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>Y</i>
2	<i>B</i>	<i>G</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>Y</i>	<i>R</i>	<i>R</i>	<i>B</i>	<i>R</i>
3	<i>R</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>B</i>	<i>Y</i>	<i>B</i>	<i>Y</i>	<i>G</i>	<i>G</i>	<i>R</i>
4	<i>Y</i>	<i>R</i>	<i>Y</i>	<i>G</i>	<i>B</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>R</i>	<i>G</i>	<i>B</i>
5	<i>R</i>	<i>Y</i>	<i>G</i>	<i>B</i>	<i>B</i>	<i>Y</i>	<i>Y</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>G</i>
6	<i>B</i>	<i>R</i>	<i>Y</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>B</i>
7	<i>Y</i>	<i>G</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>Y</i>	<i>R</i>
8	<i>Y</i>	<i>Y</i>	<i>G</i>	<i>B</i>	<i>R</i>	<i>B</i>	<i>G</i>	<i>R</i>	<i>G</i>	<i>B</i>	<i>Y</i>
9	<i>R</i>	<i>B</i>	<i>R</i>	<i>B</i>	<i>G</i>	<i>G</i>	<i>R</i>	<i>Y</i>	<i>Y</i>	<i>G</i>	<i>B</i>
10	<i>R</i>	<i>Y</i>	<i>Y</i>	<i>R</i>	<i>G</i>	<i>R</i>	<i>B</i>	<i>B</i>	<i>Y</i>	<i>B</i>	<i>G</i>
11	<i>B</i>	<i>Y</i>	<i>R</i>	<i>R</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>G</i>	<i>R</i>	<i>Y</i>	<i>G</i>
12	<i>R</i>	<i>B</i>	<i>Y</i>	<i>Y</i>	<i>Y</i>	<i>B</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>R</i>	<i>G</i>
13	<i>G</i>	<i>G</i>	<i>B</i>	<i>B</i>	<i>R</i>	<i>R</i>	<i>Y</i>	<i>Y</i>	<i>R</i>	<i>Y</i>	<i>G</i>
14	<i>G</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>G</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>Y</i>
15	<i>G</i>	<i>Y</i>	<i>G</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>B</i>	<i>B</i>	<i>B</i>
16	<i>B</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>Y</i>	<i>R</i>	<i>G</i>
17	<i>Y</i>	<i>G</i>	<i>B</i>	<i>G</i>	<i>Y</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>G</i>	<i>R</i>	<i>B</i>
18	<i>B</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>G</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>R</i>	<i>R</i>
19	<i>Y</i>	<i>G</i>	<i>R</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>G</i>	<i>B</i>	<i>B</i>	<i>G</i>	<i>R</i>
20	<i>B</i>	<i>R</i>	<i>Y</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>G</i>	<i>G</i>	<i>R</i>	<i>Y</i>
21	<i>G</i>	<i>R</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>Y</i>

Table 8: A 4-coloring of $G_{21,11}$ due to Brad Lorsen

	1	2	3	4	5	6	7	8	9	10
1	<i>Y</i>	<i>G</i>	<i>R</i>	<i>R</i>	<i>G</i>	<i>G</i>	<i>Y</i>	<i>Y</i>	<i>B</i>	<i>B</i>
2	<i>G</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>B</i>	<i>B</i>	<i>Y</i>	<i>R</i>	<i>Y</i>	<i>R</i>
3	<i>B</i>	<i>G</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>Y</i>	<i>B</i>
4	<i>Y</i>	<i>Y</i>	<i>G</i>	<i>G</i>	<i>R</i>	<i>R</i>	<i>B</i>	<i>B</i>	<i>G</i>	<i>Y</i>
5	<i>Y</i>	<i>B</i>	<i>Y</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>R</i>	<i>G</i>	<i>G</i>	<i>R</i>
6	<i>Y</i>	<i>B</i>	<i>R</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>G</i>	<i>G</i>
7	<i>G</i>	<i>Y</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>R</i>	<i>G</i>
8	<i>Y</i>	<i>R</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>G</i>	<i>B</i>	<i>R</i>
9	<i>Y</i>	<i>B</i>	<i>B</i>	<i>R</i>	<i>R</i>	<i>G</i>	<i>R</i>	<i>G</i>	<i>Y</i>	<i>G</i>
10	<i>R</i>	<i>R</i>	<i>B</i>	<i>B</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>B</i>	<i>G</i>	<i>Y</i>
11	<i>R</i>	<i>G</i>	<i>G</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>B</i>	<i>G</i>	<i>Y</i>	<i>R</i>
12	<i>R</i>	<i>B</i>	<i>R</i>	<i>G</i>	<i>G</i>	<i>Y</i>	<i>Y</i>	<i>B</i>	<i>B</i>	<i>G</i>
13	<i>B</i>	<i>R</i>	<i>G</i>	<i>B</i>	<i>G</i>	<i>R</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>Y</i>
14	<i>G</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>B</i>	<i>Y</i>	<i>R</i>	<i>R</i>	<i>G</i>	<i>B</i>
15	<i>R</i>	<i>G</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>Y</i>	<i>G</i>
16	<i>B</i>	<i>B</i>	<i>Y</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>R</i>
17	<i>G</i>	<i>Y</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>G</i>	<i>B</i>	<i>Y</i>	<i>B</i>	<i>R</i>
18	<i>R</i>	<i>B</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>G</i>	<i>Y</i>	<i>R</i>	<i>R</i>	<i>Y</i>
19	<i>G</i>	<i>B</i>	<i>R</i>	<i>Y</i>	<i>Y</i>	<i>R</i>	<i>B</i>	<i>G</i>	<i>R</i>	<i>B</i>
20	<i>B</i>	<i>R</i>	<i>Y</i>	<i>G</i>	<i>R</i>	<i>G</i>	<i>G</i>	<i>B</i>	<i>R</i>	<i>Y</i>
21	<i>B</i>	<i>R</i>	<i>G</i>	<i>R</i>	<i>B</i>	<i>Y</i>	<i>G</i>	<i>Y</i>	<i>B</i>	<i>Y</i>
22	<i>G</i>	<i>Y</i>	<i>Y</i>	<i>R</i>	<i>G</i>	<i>B</i>	<i>G</i>	<i>B</i>	<i>R</i>	<i>B</i>

Table 9: A 4-coloring of $G_{22,10}$ due to Brad Larsen

4. Exactly one of the following grids is in OBS_4 : $G_{19,17}$, $G_{18,17}$, $G_{17,17}$.

5. If $G_{19,17} \in \text{OBS}_4$ then it is possible that $G_{18,18} \in \text{OBS}_4$.

Proof: This is easily proven from Theorem 8.1, Lemmas 8.2, 8.3, 8.4, and 8.5. For a visual aid see the following chart where we put a C in the (n, m) spot if $G_{n,m}$ is Colorable, an N if it is not colorable, and a U if it is not known. ■

8.2 Assuming the Rectangle-Free Conjecture

We have the following conjecture which, if true, yields more 4-colorings and allows us to state exactly what OBS_4 is.

Rectangle-Free Conjecture (RFC): Let $n, m, c \geq 2$. If there exists a rectangle-free subset of $G_{n,m}$ of size $\lceil nm/c \rceil$ then $G_{n,m}$ is c -colorable.

Lemma 8.7 *There exists a rectangle-free subset of $G_{21,12}$ of size $63 = \lceil \frac{21 \cdot 12}{4} \rceil$. Hence, if RFC is true, there is a 4-coloring of $G_{21,12}$ and $G_{12,21}$.*

Proof:

We show there exists a rectangle free subset of $G_{21,12}$ of size 63 by showing it to you:

■

Lemma 8.8 *There exists a rectangle-free subset of $G_{18,18}$ of size $81 = \lceil \frac{18 \cdot 18}{4} \rceil$. Hence, if RFC is true, there is a 4-coloring of $G_{18,18}$.*

Proof:

Here is the rectangle-free set.

■

Note 8.9 If the 5th row and the 2nd column were removed then this would be a rectangle free set of $G_{17,17}$ of size 74. Note that $\lceil \frac{17 \times 17}{4} \rceil = 73$. Hence if we had a weaker version of RFC then we would have that $G_{17,17}$ is 4-colorable.

Theorem 8.10 *Assume RFC is true. Then*

$$\text{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \cup \\ \{G_{17,19}, G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}$$

	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21
8	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
9	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
10	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
11	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
12	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>
13	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>
14	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>
15	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>
16	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>
17	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>U</i>	<i>N</i>	<i>N</i>	<i>N</i>
18	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>U</i>	<i>N</i>	<i>N</i>	<i>N</i>
19	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
20	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
21	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
22	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
23	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
24	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
25	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
26	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
27	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
28	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
29	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
30	<i>C</i>	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
31	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
32	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
33	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
34	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
35	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
36	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
37	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
38	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
39	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
40	<i>C</i>	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>
41	<i>C</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>

Table 10: Summary of what is known about 4-coloring of grids.

	01	02	03	04	05	06	07	08	09	10	11	12
1	<i>R</i>	<i>R</i>										
2	<i>R</i>		<i>R</i>									
3		<i>R</i>	<i>R</i>									
4			<i>R</i>	<i>R</i>	<i>R</i>							
5		<i>R</i>		<i>R</i>		<i>R</i>						
6	<i>R</i>				<i>R</i>	<i>R</i>						
7						<i>R</i>	<i>R</i>	<i>R</i>				
8					<i>R</i>		<i>R</i>		<i>R</i>			
9				<i>R</i>				<i>R</i>	<i>R</i>			
10						<i>R</i>				<i>R</i>	<i>R</i>	
11					<i>R</i>					<i>R</i>		<i>R</i>
12				<i>R</i>							<i>R</i>	<i>R</i>
13			<i>R</i>			<i>R</i>			<i>R</i>			<i>R</i>
14		<i>R</i>			<i>R</i>			<i>R</i>			<i>R</i>	
15	<i>R</i>			<i>R</i>			<i>R</i>			<i>R</i>		
16			<i>R</i>				<i>R</i>				<i>R</i>	
17		<i>R</i>							<i>R</i>	<i>R</i>		
18	<i>R</i>							<i>R</i>				<i>R</i>
19			<i>R</i>					<i>R</i>		<i>R</i>		
20		<i>R</i>					<i>R</i>					<i>R</i>
21	<i>R</i>								<i>R</i>		<i>R</i>	

Table 11: A Rectangle Free Subset of $G_{21,12}$ of size 63.

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		<i>R</i>		<i>R</i>										<i>R</i>		<i>R</i>	<i>R</i>	
2	<i>R</i>	<i>R</i>								<i>R</i>	<i>R</i>		<i>R</i>					
3	<i>R</i>								<i>R</i>						<i>R</i>	<i>R</i>		<i>R</i>
4						<i>R</i>			<i>R</i>			<i>R</i>	<i>R</i>	<i>R</i>				
5		<i>R</i>	<i>R</i>			<i>R</i>												<i>R</i>
6	<i>R</i>			<i>R</i>		<i>R</i>	<i>R</i>											
7							<i>R</i>	<i>R</i>		<i>R</i>				<i>R</i>				<i>R</i>
8			<i>R</i>				<i>R</i>		<i>R</i>		<i>R</i>						<i>R</i>	
9		<i>R</i>			<i>R</i>		<i>R</i>					<i>R</i>			<i>R</i>			
10				<i>R</i>							<i>R</i>	<i>R</i>						<i>R</i>
11	<i>R</i>		<i>R</i>		<i>R</i>									<i>R</i>				
12			<i>R</i>	<i>R</i>				<i>R</i>					<i>R</i>		<i>R</i>			
13					<i>R</i>	<i>R</i>		<i>R</i>			<i>R</i>					<i>R</i>		
14	<i>R</i>							<i>R</i>				<i>R</i>					<i>R</i>	
15				<i>R</i>	<i>R</i>				<i>R</i>	<i>R</i>								
16						<i>R</i>				<i>R</i>					<i>R</i>		<i>R</i>	
17			<i>R</i>							<i>R</i>		<i>R</i>				<i>R</i>		
18					<i>R</i>								<i>R</i>				<i>R</i>	<i>R</i>

Table 12: A Rectangle Free Subset of $G_{18,18}$ of size 81.

Proof:

In this proof we only consider $G_{n,m}$ where $n \geq m$. The grids

$$G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}$$

are in the obstruction set by Theorem 8.6.

$G_{21,13}$ is not 4-colorable by Theorem 8.1. $G_{20,13}$ is 4-colorable by Theorem 8.1 (since $G_{20,16}$ is 4-colorable). Assuming RFC $G_{21,12}$ is 4-colorable by Lemma 8.7. Hence, assuming RFC, $G_{21,13}$ is a minimal grid that is not 4-colorable, hence it is in OBS_4 .

$G_{19,17}$ is not 4-colorable by Lemma 8.2. Assuming RFC $G_{18,17}$ is 4-colorable by Lemma 8.8 (since $G_{18,18}$ is 4-colorable). $G_{19,16}$ is 4-colorable by Theorem 8.1 (since $G_{20,16}$ is 4-colorable). Hence, assuming RFC, $G_{19,17}$ is a minimal grid that is not 4-colorable, hence it is in OBS_4 .

The proof that these are all of the elements in OBS_4 is similar to the proof of Theorem 6.2 and hence we omit it. ■

9 Application to Bipartite Ramsey Numbers

We state the Bipartite Ramsey Theorem. See [6] for history, details, and proof.

Def 9.1 A complete bipartite graph, $G = (V_1 + V_2, E)$, is a bipartite graph such that for any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, (v_1, v_2) is an edge in G . The complete bipartite graph with partitions of size $|V_1| = a$ and $|V_2| = b$, is denoted $K_{a,b}$.

Theorem 9.2 For all a, c there exists $n = BR(a, c)$ such that for all c -colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$.

The following theorem is easily seen to be equivalent to this.

Theorem 9.3 For all a, c there exists $n = BR(a, c)$ so that for all c -colorings of $G_{n,n}$ there will be a monochromatic $a \times a$ submatrix.

In this paper we are c -coloring $G_{n,m}$ and looking for a 2×2 monochromatic submatrix. We have the following theorems which, except where noted, seem to be new.

Theorem 9.4

1. $BR(2, 2) = 5$. (This was also shown in [11].)
2. $BR(2, 3) = 11$.
3. $17 \leq BR(2, 4) \leq 19$.
4. $BR(2, c) \leq c^2 + c$.

5. If p is a prime and $s \in \mathbb{N}$, then $BR(2, p^s) > p^{2s}$.

6. For almost all c , $BR(2, c) \geq c^2 - 2c^{1.525} + c^{1.05}$.

Proof:

1) By Lemma 6.1, $G_{5,5}$ is not 2-colorable and $G_{4,4}$ is 2-colorable.

2) By Lemma 7.4, $G_{11,11}$ is not 3-colorable. By Lemma 7.2 $G_{10,10}$ is 3-colorable.

3) By Lemma 8.2, $G_{19,19}$ is not 4-colorable. By Theorem 8.1 $G_{16,16}$ is 4-colorable.

4) By Theorem 5.1, G_{c^2+c, c^2+c} is not c -colorable.

5) By Theorem 4.12, $G_{cr, cm}$ is c -colorable where $c = p^s$, $r = p^s$, and $m = \frac{p^{2s}-1}{p^s-1}$. Note that $m \geq p^s$. Hence $G_{p^{2s}, p^{2s}}$ is p^s -colorable.

6) Baker, Harman, and Pintz [2] (see [9] for a survey) showed that for almost all c , there is a prime between c and $c - c^{0.525}$. Let p be that prime. By part 5 with $s = 1$, $BR(2, p) \geq p^2$. Hence

$$BR(2, c) \geq BR(2, p) \geq p^2 \geq (c - c^{0.525})^2 = c^2 - 2c^{1.525} + c^{1.05}.$$

■

10 Open Questions

1. Find OBS_4 . We feel this is possible since we are so close. A clever computer program may be needed. The second author has offered a prize of \$289.00 for a proper 4-coloring of $G_{17,17}$.
2. Refine our tools so that our ugly proofs can be corollaries of our tools.
3. Find an algorithm that will, given c , find OBS_c or $|OBS_c|$ quickly.
4. We know that $2\sqrt{c}(1 - o(1)) \leq |OBS_c| \leq 2c^2$. Bring these bounds closer together.
5. Is the Rectangle-Free Conjecture True? If so then this may help us find c -colorings. If not then this may open up new techniques for proving that a grid is not c -colorable.

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12 Appendix: Exact Values of $\text{maxrf}(n, m)$ for $0 \leq n \leq 6$, $m \leq n$

Lemma 12.1

0) For $m \geq 0$, $\text{maxrf}(0, m) = 0$.

1) For $m \geq 1$, $\text{maxrf}(1, m) = m$.

2) For $m \geq 2$, $\text{maxrf}(2, m) = m + 1$.

3) For $m \geq 3$, $\text{maxrf}(3, m) = m + 3$.

4)

$$\text{maxrf}(4, m) = \begin{cases} m + 5 & \text{if } 4 \leq m \leq 5 \\ m + 6 & \text{if } m \geq 6 \end{cases}$$

5)

$$\text{maxrf}(5, m) = \begin{cases} 12 & \text{if } m = 5 \\ m + 8 & \text{if } 6 \leq m \leq 7 \\ m + 9 & \text{if } 8 \leq m \leq 9 \\ m + 10 & \text{if } m \geq 10 \end{cases}$$

6)

$$\text{maxrf}(6, m) = \begin{cases} 2m + 4 & \text{if } 6 \leq m \leq 7 \\ 19 & \text{if } m = 8 \\ m + 12 & \text{if } 9 \leq m \leq 10 \\ m + 13 & \text{if } 11 \leq m \leq 12 \\ m + 14 & \text{if } 13 \leq m \leq 14 \\ m + 15 & \text{if } m \geq 15 \end{cases}$$

Proof:

Lemma 2.7 will provide all of the upper bounds. The lower bounds are obtained by actually exhibiting rectangle-free sets of the appropriate size. We do this for the case of $\text{maxrf}(6, m)$. Our technique applies to all of the other cases.

Case 1: $\text{maxrf}(6, m)$ where $6 \leq m \leq 7$ and $m = 8$: Fill the first four columns with 3 elements (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 12 are blocked. Hence we can fill the next $15 - 12 = 3$ columns with two elements each, and the remaining column (if $m = 8$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 8) = 19$; however, if you just look at the first 6 (7) columns you get the result for $\text{maxrf}(6, 6)$ ($\text{maxrf}(6, 7)$).

R		R		R			R
R			R		R		
R	R					R	
	R	R			R		
	R		R	R			
		R	R			R	

Case 2: $\text{maxrf}(6, m)$ where $9 \leq m \leq 10$: Fill the first three columns with 3 elements each (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 9 are blocked. Hence we can fill the next $15 - 9 = 6$ columns with two elements each and the remaining column (if $m = 10$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 10) = 22$; however, if you just look at the first 9 columns you get the result $\text{maxrf}(6, 9) = 21$.

R		R		R					
R					R		R	R	
R	R					R			
	R	R			R				
	R		R	R			R		
		R	R			R		R	R

Case 3: $\text{maxrf}(6, m)$ where $11 \leq m \leq 12$: Fill the first two columns with 3 elements each (they overlap). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 6 are blocked. Hence we can fill the next $15 - 6 = 9$ columns with two elements each and the remaining column (if $m = 12$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 12) = 25$; however, if you just look at the first 11 columns you get the result $\text{maxrf}(6, 11) = 24$.

R		R		R					R	
R					R		R	R		
R	R					R				
	R	R			R				R	
	R		R	R			R			
			R			R		R	R	R

Case 4: $\text{maxrf}(6, m)$ where $13 \leq m \leq 14$: Fill the first column with 3 elements. This column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs. Hence we can fill the next $15 - 3 = 12$ columns with two elements each and the remaining column (if $m = 14$) with 1 element. We omit the picture.

Case 5: $\text{maxrf}(6, m)$ where $m \geq 15$: Fill the first $\binom{6}{2} = 15$ columns with two elements each in a way so that each column has a distinct pair. Fill the remaining $m - 15$ columns with one element each. The result is a rectangle-free set of size $30 + m - 15 = m + 15$. ■

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