Andy Drucker's Derivation of Can Ramsey from Sz Theorem

Recall Sz's theorem:

Def 0.1 Let $A \subseteq \mathsf{N}$. The *densit of A* is

$$\limsup_{n\to\infty} \frac{|A \cup \{1,\ldots,n\}|}{n}.$$

Theorem 0.2 *Let $\epsilon \in \mathsf{R}^+$ and $k \in \mathsf{N}$. If $A \subseteq \mathsf{N}$ of density ϵ then A contains a k-AP.*

Here is a finite version you can get from compactness:

Theorem 0.3 *Let $\epsilon \in \mathsf{R}^+$ and $k \in \mathsf{N}$. There exists $N = N(i,\epsilon)$ such that the following holds: if $A \subseteq [N]$ of density ϵ then A contains a k-AP.*

We want to prove the Can Ramsey Theorem:

Theorem 0.4 *Let $k \in \mathsf{N}$. There exists $N = N(k)$ such that any coloring of $[N]$ has either a mono k-AP or a rainbow k-AP.*

Proof:

We will set $\epsilon = \frac{1}{2^k}$. Let $N = N(k,\epsilon)$ from Theorem 0.3.

Let COL: $[N] \rightarrow \omega$. If some color has density $\geq \frac{1}{2^k}$ then by Theorem 0.3 there is a mono k-AP. Hence we assume all colors have density $< \frac{1}{2^k}$.

Consider the following prob experiment: pick $(a,d) \in [N]\times[N]$ and then look at the set

$$a \pmod N, a+d \pmod N, a+2d \pmod N, \ldots, a+(k-1)d \pmod N$$

and return Y if these are all diff colors, and N if there are two that are the same color.

What is the prob that two of the $a+dj$ and $a+dj'$ have the same color?

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