The chromatic number of the plane: the bounded case

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ABSTRACT

The chromatic number of the plane is the smallest number of colors needed in order to paint each point of the plane so that no two points (exactly) unit distance apart are the same color. It is known that seven colors suffice and (at least) four colors are necessary. In order to understand two and three colorings better, it is interesting to see how large a region can be and still be two or three colorable, and how complicated the colorings need to be. This paper gives tight bounds for two and three coloring regions bounded by circles, rectangles, and regular polygons. In particular, a square is 2-colorable if and only the length of a side is $\leq 2/\sqrt{5}$, and is 3-colorable if and only the length of a side is $\leq 8/\sqrt{65}$.

1 INTRODUCTION

The chromatic number of the plane is the smallest number of colors needed in order to paint each point of the plane so that no two points (exactly) unit distance apart are the same color. Ed Nelson invented the problem in 1950 ([Soif1]). It is known that seven colors suffice and (at least) four colors are necessary. More about this problem can be found in Alexander Soifer's upcoming book ([Soif2]).

Simple proofs show that the chromatic number of the plane is not two or three. In order to understand two and three colorings better, it is interesting to see how large can a region be (here meaning a simple closed curve and its interior) and still be two or three colorable, and how complicated the colorings need to be.

This paper considers the coloring of regions bounded by circles, regular polygons, and rectangles. For comparison with later results, the following are obvious tight bounds for 1-coloring of regions (which are obtained by looking at the two furthest apart points in the region):

Boundary of region	*1-colorable if and only if*
circle	radius $< \frac{1}{2}$
equilateral triangle	side < 1
$a \times b$ rectangle	$b < \sqrt{1-a^2}$
square	side $< \frac{1}{\sqrt{2}}$
regular n-gon	
n is even	circumradius $< \frac{1}{2}$
n is odd	circumradius $< \frac{1}{\sqrt{2(1+\cos(\pi/n))}}$

We obtain the following tight bounds for 2-coloring of regions:

Boundary of region	*2-colorable if and only if*
circle	radius $\leq \frac{1}{2}$
equilateral triangle	side < 1
$a \times b$ rectangle, $a < b$	$b < 2\sqrt{1-a^2}$
square	side $\leq \frac{2}{\sqrt{5}}$
regular n-gon	
n is even	circumradius $\leq \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$
n is odd	circumradius $< \frac{1}{\sqrt{2(1+\cos(\pi/n))}}$

We obtain the following tight bounds for 3-coloring of regions:

Boundary of region	*3-colorable if and only if*
circle	radius $\leq \frac{1}{\sqrt{3}}$
equilateral triangle	side $\leq \sqrt{3}$
$a \times b$ rectangle, $a \leq b$	
$a \leq \frac{\sqrt{3}}{2}$	always
$\frac{\sqrt{3}}{2} < a \leq \frac{2}{\sqrt{5}}$	$b \leq 3\sqrt{1-a^2}$
$\frac{2}{\sqrt{5}} \leq a \leq \frac{8}{\sqrt{65}}$	$b \leq \sqrt{1-a^2} + \sqrt{1-a^2/4}$
square	side $\leq \frac{8}{\sqrt{65}}$
regular n-gon, $n \geq 5$	inradius $\frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$ where $\theta = \lfloor \frac{n+1}{3} \rfloor \frac{2\pi}{n}$ simplifies to inradius $\leq \frac{1}{\sqrt{3}}$, for $3 \vert n$

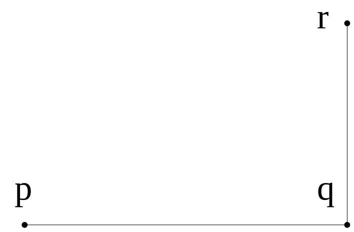


Figure 1: L-segment (p, q, r)

For comparison, here are (not necessarily tight) bounds for 4-coloring of regions.

Boundary of region	*4-colorable if*
circle	radius $\leq \frac{1}{\sqrt{2}}$
equilateral triangle	side ≤ 2
$a \times b$ rectangle, $a \leq b$	
$a \leq \frac{2\sqrt{2}}{3}$	always
$\frac{2\sqrt{2}}{3} < a \leq \sqrt{2}$	$b \leq \sqrt{4 - a^2}$
square	side $\leq \sqrt{2}$
regular n-gon, $4 \mid n$	inradius $\leq \frac{1}{\sqrt{2}}$

The equivalent problem was independently studied by Bohannon, et al. ([Boha]). For a given shape, they fix the size, let the forbidden distance vary, and see how the number of necessary colors changes. They only consider colorings of circle-regions and rectangle-regions. Their non-colorability arguments are different than the ones used in this paper. The appendix summarizes their results. Bauslaugh ([Baus]) studied 3-colorings of an infinite strip, and obtained tight bounds. The results are rederived here in the section on 3-colorings using our methods.

Section 2 gives the basic definitions. Sections 3, 4, and 5 present the coloring results for two, three, and four colors, respectively. Section 6 lists some open problems.

2 DEFINITIONS

Definition 2.1 A *circle-region* is the circle with its interior, and similarly with other geometric figures.

Definition 2.2 A region is *k-colored* if it is painted with k colors such that no two points (exactly) unit distance apart are the same color.

Definition 2.3 A region is *k-colorable* if it can be k-colored.

Definition 2.4 An *L-segment* (p, q, r) is a line segment (p, q) joined at a right angle with a line segment (q, r) at (their common endpoint) q. (Figure 1.)

NOTE ON FIGURES: The 2-colorable regions are colored using red and blue, the 3-colorable regions are colored using red, blue, and green. and the 4-colorable regions are colored

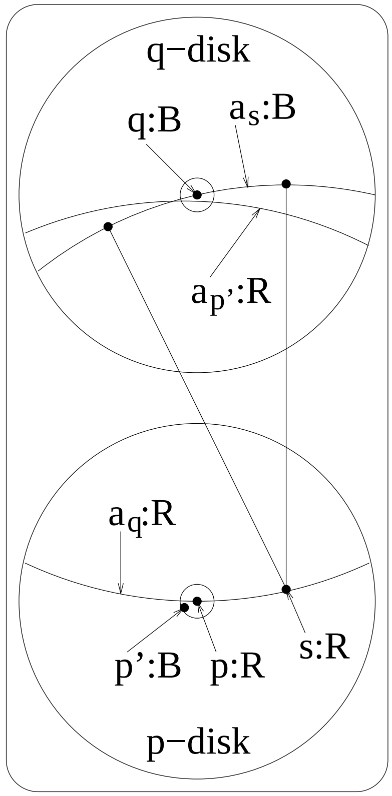


Figure 2:

using red, green, blue, and yellow. In the black and white version of this paper[1], red is represented by dark gray, green by medium gray, blue by light gray, and yellow by very light gray. When a closed subregion is monochromatic, the boundary is indicated by a black line, but the points on the boundary are colored the same as the interior. When two subregions share a boundary and/or a corner, the shared points can be either color, unless otherwise stated. Within each figure, all pictures are drawn to the same scale, but two different figures may have different scales.

3 2-COLORINGS

Definition 3.1 A *disk* is the interior of a circle (i.e., a circle-region without its boundary).

Definition 3.2 A *rod* (p, q) is a line segment of unit length with endpoints p, q.

The following lemma provides the workhorse for most of non-colorability arguments.

Lemma 3.3 *Let (p, q) be a rod, with the open disks of radius ϵ, $0 < \epsilon \leq 1/2$, centered at p and q (called the p-disk and q-disk) contained in a 2-colored region. Then the open disks of radius $\epsilon^2/8$ centered at p and q are monochromatic (with different colors).*

[1] A color verion of the paper will be available on my website at the University of Maryland.

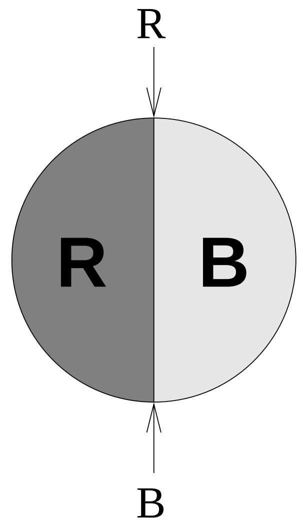


Figure 3: 2-coloring of circle-region for $r = \frac{1}{2}$.

Proof: Assume that the disk of radius $\epsilon^2/8$ centered at p is not monochromatic. (Figure 2.) Say p is red (so q is blue). Choose cartesian coordinates such that p is $(0,0)$ and q is $(0,1)$. Let p' be a blue point within $\epsilon^2/8$ of p (so p' is very close to p). The points unit distance from q lying in the p-disk form a red arc a_q, and the points unit distance from p' lying in the q-disk form a red arc $a_{p'}$. The point $s = (\epsilon/2, 1 - \sqrt{1-\epsilon^2/4})$ is on a_q so it must be red. So the unit-radius arc a_s of points centered at s in the q-disk must be blue.

The two arcs $a_{p'}$ and a_s intersect in the q-disk, since the point $s + (0,1)$ is above $a_{p'}$ and the point $s + (-\epsilon, \sqrt{1-\epsilon^2})$ is below $a_{p'}$. To see this, note that at $x = \epsilon/2$, the arc from s has y-coordinate $(1 - \sqrt{1-\epsilon^2/4}) + 1 = 2 - \sqrt{1-\epsilon^2/4}$, and the arc from p' has y-coordinate less than $\sqrt{1-\epsilon^2/4} + \epsilon^2/8$. So, we need, $2 - \sqrt{1-\epsilon^2/4} > \sqrt{1-\epsilon^2/4} + \epsilon^2/8$ or $1 - \epsilon^2/16 > \sqrt{1-\epsilon^2/4}$. At $x = -\epsilon/2$, the arc from s has y-coordinate $(1 - \sqrt{1-\epsilon^2/4}) + \sqrt{1-\epsilon^2}$ and the arc from p' has y-coordinate greater than $\sqrt{1-\epsilon^2/4} - \epsilon^2/8$. So, we need, $1 - \sqrt{1-\epsilon^2/4} + \sqrt{1-\epsilon^2} < \sqrt{1-\epsilon^2/4} - \epsilon^2/8$ or $1 + \sqrt{1-\epsilon^2} + \epsilon^2/8 < 2\sqrt{1-\epsilon^2/4}$. So the arcs intersect and have different colors, which is a contradiction. ∎

Lemma 3.4 *Assume a region is 2-colored. If we slide a rod continuously so that the endpoints stay strictly inside the region, the set of points passed over by a given endpoint are monochromatic.*

Proof: Follows from the previous lemma. ∎

Theorem 3.5 *A circle-region is 2-colorable if and only if its radius $r \leq \frac{1}{2}$.*

Proof: **COLORABILITY:** Let $r \leq \frac{1}{2}$. If $r < \frac{1}{2}$ then the circle-region is colorable with only one color (and a fortiori with two colors). Otherwise ($r = \frac{1}{2}$), bisect the circle-region along a diameter, and color one semicircle-region red and the other blue. The points on the diameter can be either color, except that the two end points of the diameter (which are on the circle) must be different colors. (Figure 3.)

NON-COLORABILITY: Let $r > \frac{1}{2}$. Suppose the region is 2-colored. Put a rod inside the circle-region so that the center of the rod is at the center of the circle, and rotate the rod 180 degrees around its center (halfway around). By Lemma 3.4 the points crossed by an end

Figure 4: Non-2-colorability of circle-region for $r > \frac{1}{2}$

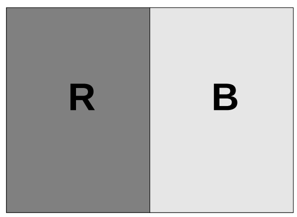


Figure 5: 2-coloring of $a \times b$ rectangle-region, where $a \leq b$ and $a^2 + b^2/4 < 1$

of the rod must all have same color (Figure 4a), but the start and finish points of an end are one unit apart (Figure 4b). Contradiction. ∎

Theorem 3.6 *A rectangle-region of dimensions $a \times b$, with $a \leq b$, is 2-colorable if and only if either it is not a square ($a < b$) and $a^2 + b^2/4 < 1$, or it is a square ($a = b$) and $a \leq \frac{2}{\sqrt{5}}$.*

Proof: **COLORABILITY:** If ($a \leq b$ and) $a^2 + b^2/4 < 1$ (the rectangle case and the square not of maximum allowed size), bisect the rectangle-region into two $a \times \frac{b}{2}$ sub-rectangle-regions, and color the one half red and the other half blue. The points on the boundary between the two subrectangles can be either color. (Figure 5.)

If $a = b$ and $a^2 + b^2/4 = 1$ (which implies $a = 2/\sqrt{5}$ the maximum allowed size square), then color the left half red and the right half blue, as before with the following modifications. Color

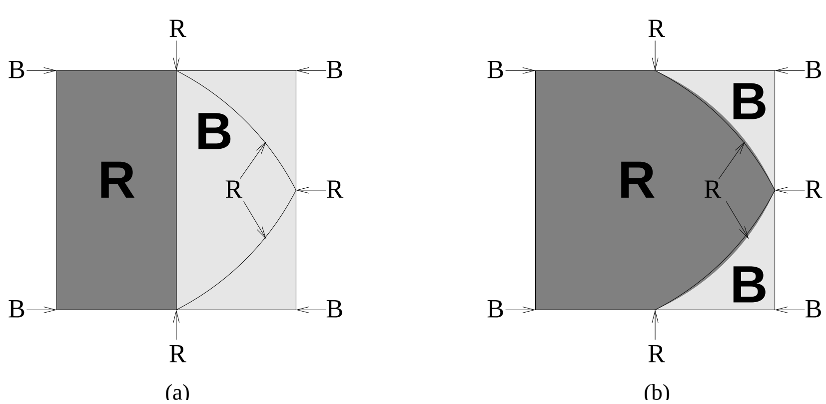


Figure 6: 2-coloring of square-region with side length $s = 2/\sqrt{5}$

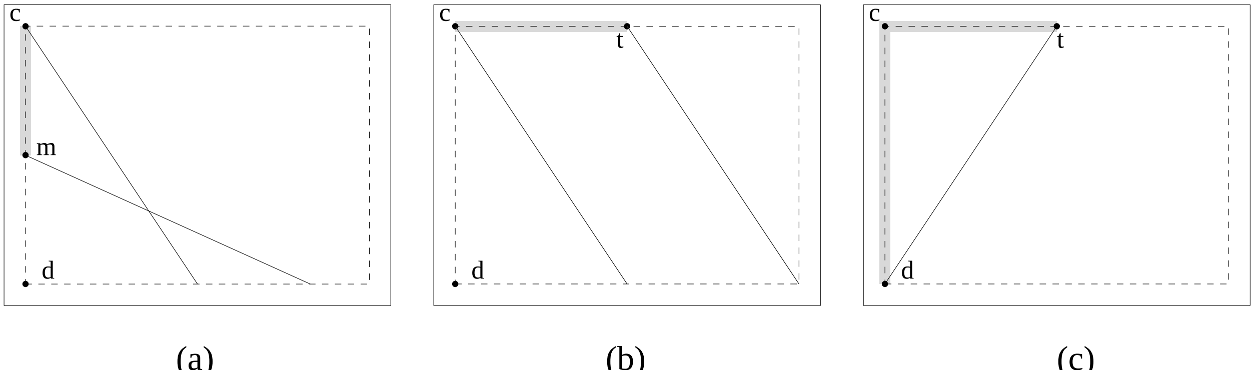


Figure 7: Non-2-colorability of $a \times b$ rectangle-region, where $a < b$ and $a^2 + b^2/4 > 1$

blue all four corners of the square-region, and color red the two end points of the boundary between the two halves. The two blue corners on the red side force two unit-radius arcs inside the blue sub-rectangle-region to be red, but none of the points on those arcs is one unit away from any point on the red side (except the blue corners). (Figure 6a.) This coloring is not as nice as the previous ones, because of the isolated blue points at the two left corners of the red region and the isolated red lines inside the blue region. The isolation of the red lines can be eliminated simply by making the entire region to their left red. (Figure 6b.) It does not seem possible to eliminate the blue corners.

NON-COLORABILITY: There are two cases: (1) ($a \le b$ and) $a^2 + b^2/4 > 1$ and (2) $a < b$ and $a^2 + b^2/4 = 1$.

CASE (1): ($a \le b$ and) $a^2 + b^2/4 > 1$. Suppose the rectangle-region is 2-colored. Create a new rectangle R of dimensions $\alpha \times \beta$ where $\alpha \le \beta$, $\alpha < a$, $\beta < b$, and $\alpha^2 + \beta^2/4 = 1$.

Let c be the top left corner point of R, d be the bottom left corner point, m be the middle point of the left side, and t be the middle point on the top side. Put one end of a rod on c and the other end on the bottom side of R, where it will lie at the center because $\alpha^2 + \beta^2/4 = 1$. Slide the rod downward on the left side and rightward on the bottom side until the left end reaches m (Figure 7a). This is possible because the distance between m and the bottom right corner is $\sqrt{\alpha^2/4 + \beta^2} \ge \sqrt{\alpha^2 + \beta^2/4} = 1$ (which follows because $\alpha \le \beta$). The line segment (c, m) must be monochromatic (by Lemma 3.4). By symmetry, the line segment (d, m) must also be monochromatic.

Starting with the rod in its initial position, slide it rightward until the top end touches t (and the bottom end touches the bottom right corner of R). The line segment (c, t) must be monochromatic (Figure 7b). So the entire L-segment (d, c, t) must be monochromatic. But, the points d and t are exactly one unit apart (Figure 7c). Contradiction.

CASE (2): $a < b$ and $a^2 + b^2/4 = 1$. Suppose the rectangle-region is 2-colored. Let $\epsilon > 0$ be a small value that depends on a and b (to be specified later). Create a rectangle R of dimensions $\alpha \times \beta$ where $\alpha = a - \epsilon$, $\beta = b - \epsilon$, and $\alpha^2/4 + \beta^2 \ge 1$. (Notice the position of the denominator 4!) This latter condition is possible since $a < b$. Put R inside the rectangle-region with the same center and orientation.

Let c be the top left corner point of R, m be the middle point of the left side, and t be the point on the top exactly one unit away from the bottom right corner. Put one end of a rod on c and the other end on the bottom side of R (just right of the middle). Slide the rod downward on the leftward side and right on the bottom side until the left end reaches m

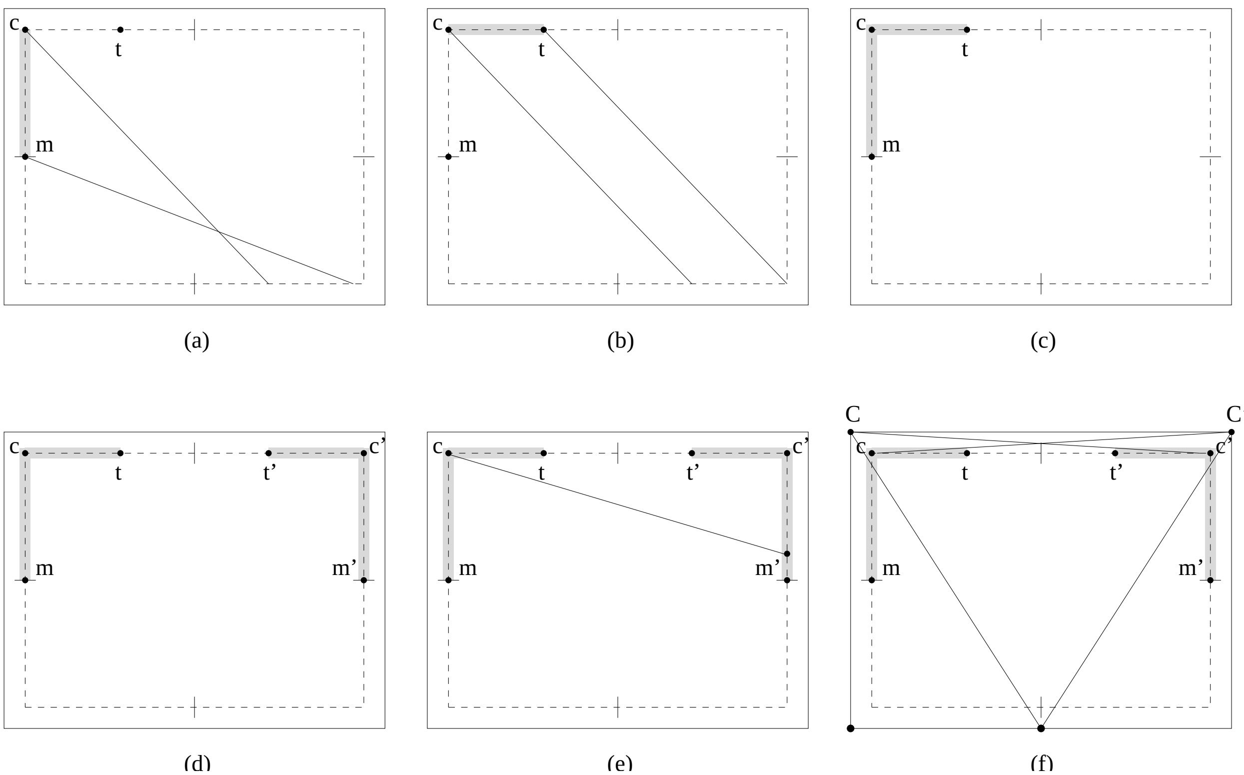


Figure 8: Non-2-colorability of $a \times b$ rectangle-region, where $a < b$ and $a^2 + b^2/4 = 1$

(Figure 8a). This is possible because the distance between m and the bottom right corner is $\sqrt{\alpha^2/4+\beta^2} \geq 1$. The line segment (c, m) must be monochromatic.

As before, put one end of a rod on c and the other end on the bottom side of R (just right of the middle). Now, slide the rod rightward until the top end touches t (and the bottom end touches the bottom right corner of R). The line segment (c, t) must be monochromatic (Figure 8b).

So, the L-segment (m, c, t) must be monochromatic (Figure 8c). Let c' be the top right corner point of R, m' be the middle point of the right side, and t' be the point on the top exactly one unit away from the bottom left corner. Symmetrically, the L-segment (m', c', t') must be monochromatic (Figure 8d). By continuity, c is exactly one unit away from some point on (t', c', m'), since $|(c, t')| = \sqrt{1-\alpha^2} < 1$ and $|(c, m')| = \sqrt{\alpha^2/4+\beta^2} \geq 1$ (Figure 8e). So the two L-segments must be different colors.

Let C be the top left corner point of the original rectangle-region, and let C' be the top right corner point. Now, consider the point exactly in the middle of the bottom half of the original rectangle-region. It is exactly one unit away from both C and C', so the latter two points must be the same color (as each other). By continuity, each corner, C and C', is exactly one unit away from some point on the opposite L-segment on R, since

$$|(C, t')| = |(C', t)| = \sqrt{(\sqrt{1-\alpha^2}+\epsilon)^2+\epsilon^2} < 1,$$

if ϵ is small enough, and obviously $|(C, m')| = |(C', m)| > 1$ since $|(c, m')| \geq 1$ (Figure 8f). So the two L-segments must be the same color (as each other). Contradiction. ▮

For regions where the conditions of colorability involves $<$ rather than $\leq$, one might wonder how much of the boundary can be included and still be 2-colorable. While a rectangle-region of dimensions $a \times b$, where $a < b$, is not 2-colorable for $a^2 + b^2/4 = 1$, it is almost 2-colorable:

Theorem 3.7 *A rectangle-region of dimensions $a \times b$, where $a < b$ and $a^2+b^2/4 = 1$, becomes 2-colorable if of the set of three points {top left corner, middle bottom side, top right corner} one of them is removed, and of the set of three points {bottom left corner, middle top side, and bottom right corner} one of them is removed.*

Proof: Bisect the rectangle-region into two $a \times \frac{b}{2}$ subrectangle-regions, and color the left side red and the right side blue. The points on the boundary between the two subrectangles can be either color, except if a middle point on the top or bottom side exists, it must be a different color than the remaining corner point in its set. ▮

Note 3.8 The non-colorability argument in Theorem 3.6 (for the case $a < b$ and $a^2+b^2/4 = 1$) shows that the region is not 2-colorable if only one point is removed, or even if any other pair of points is removed.

The following are well known, but useful, facts about regular polygons.

Lemma 3.9 *Consider a regular n-gon. Let s be the length of a side; r be the distance from the center to a corner (its radius); and q be the distance from the center to the middle of a side.*

Then

$$r = \frac{q}{\cos(\pi/n)} = \frac{s}{2\sin(\pi/n)}$$

$$q = r\cos(\pi/n) = \frac{s}{2\tan(\pi/n)}$$

$$s = 2r\sin(\pi/n) = 2q\tan(\pi/n)$$

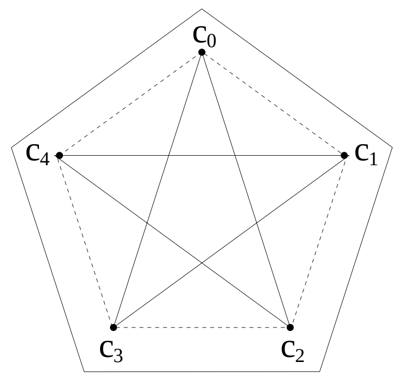


Figure 9: Non-2-colorability of regular n-gon-region with n odd.

Theorem 3.10 *A regular n-gon-region is 2-colorable if and only if the circumradius r satisfies*

$$r \le \frac{1}{\sqrt{\sin(\pi/n)^2+4\cos(\pi/n)^2}} \quad \text{if } n \text{ is even}$$
$$r < \frac{1}{\sqrt{2(1+\cos(\pi/n))}} \quad \text{if } n \text{ is odd}$$

Proof: **CASE (1):** n is odd.

COLORABILITY: If $r < \frac{1}{\sqrt{2(1+\cos(\pi/n))}}$, just use one color. Two farthest apart points of the same color are from a corner to an endpoint of its opposite side. It is sufficient that this distance be less than 1. So we need:

$$\begin{aligned}
& \sqrt{(s/2)^2+(r+q)^2} \;<\; 1 \\
\Longrightarrow\quad & \sqrt{(r\sin(\pi/n))^2+(r+r\cos(\pi/n))^2} \;<\; 1 \\
\Longrightarrow\quad & r\sqrt{\sin^2(\pi/n)+(1+\cos(\pi/n))^2} \;<\; 1 \\
\Longrightarrow\quad & r\sqrt{\sin^2(\pi/n)+1+2\cos(\pi/n)+\cos^2(\pi/n)} \;<\; 1 \\
\Longrightarrow\quad & r\sqrt{2+2\cos(\pi/n)} \;<\; 1 \\
\Longrightarrow\quad & r \;<\; \frac{1}{\sqrt{2(1+\cos(\pi/n))}}
\end{aligned}$$

NON-COLORABILITY: Assume that $r \ge \frac{1}{\sqrt{2(1+\cos(\pi/n))}}$ and suppose that the n-gon-region is 2-colored. Imbed a regular n-gon inside the original n-gon-region, but not necessarily strictly, with circumradius $r = \frac{1}{\sqrt{2(1+\cos(\pi/n))}}$. Let $c_0, c_1, \cdots, c_{n-1}$ be the corners in order. The distance from any corner to a corner on its opposite side is exactly 1, so those two points must have different colors. Take the sequence of opposite corners

$$c_0, c_{(n-1)/2+1}, c_1, c_{(n-1)/2+2}, c_2, c_{(n-1)/2+3}, \cdots, c_{n-2}, c_{(n-1)/2-1}, c_{n-1}, c_{(n-1)/2}, c_0$$

Since n is odd, c_0 must have a different color than itself, which is a contradiction. Figure 9 shows an example for $n = 5$, the pentagon.

CASE (2): n is even.

COLORABILITY: If $r < \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$ then bisect the n-gon-region from the middle of one side to the middle of its opposite side, and color the two sides with different colors. The bisecting line can be either color. Figure 10a shows an example for $n = 6$, the hexagon.

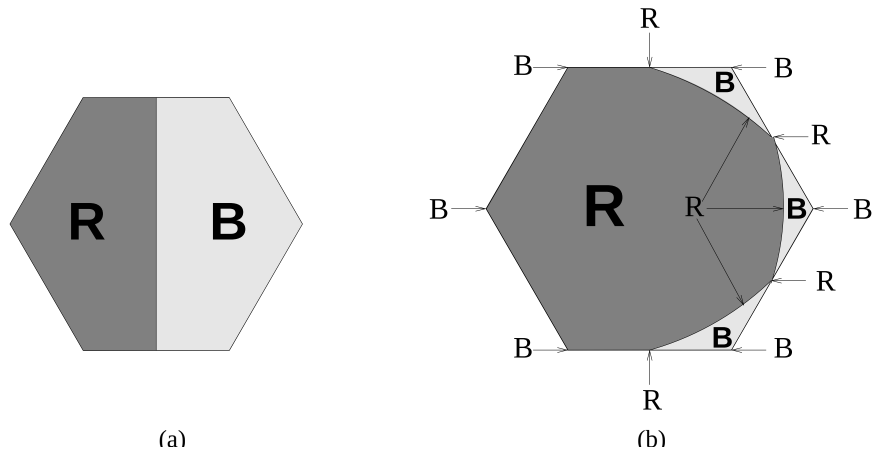


Figure 10: 2-coloring of regular n-gon-region with even number of sides.
(a) Circumradius $r < \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$. (b) Circumradius $r = \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$.

The farthest pairs of points of the same color are from the middle of one of the bisected sides to a corner on its opposite side. It is sufficient that this distance be less than 1. So we need:

$$\begin{aligned} & \sqrt{(s/2)^2 + (2q)^2} \; < \; 1 \\ \Longrightarrow \quad & \sqrt{(r\sin(\pi/n))^2 + (2r\cos(\pi/n))^2} \; < \; 1 \\ \Longrightarrow \quad & r\sqrt{\sin^2(\pi/n) + 4\cos^2(\pi/n)} \; < \; 1 \\ \Longrightarrow \quad & r \; < \; \frac{1}{\sqrt{\sin^2(\pi/n) + 4\cos^2(\pi/n)}} \end{aligned}$$

If $r = \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$ then once again color it by bisecting the n-gon-region from the middle of one side to the middle of its opposite side, and coloring the two sides with different colors. Again, the bisecting line can be either color. However, the midpoints of all of the sides must be one color, say red, and all of the corners must be the other color, say blue. Furthermore, the points that are unit distance away from any corner (which are arcs of a unit-radius circle) must be red. As in the coloring of the square (Figure 6b), the isolation of the red lines is eliminated by coloring the entire region to their left red, except for the blue corners. This completes the coloring. Figure 10b shows an example for $n = 6$, the hexagon.

NON-COLORABILITY: Assume $r > \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$ and suppose the n-gon-region is 2-colored. Imbed a regular n-gon inside the original n-gon-region with circumradius $r = \frac{1}{\sqrt{\sin^2(\pi/n)+4\cos^2(\pi/n)}}$. Let S be a side, say the top. Place a rod with one end at the left corner of S and the other end in the middle of the bottom (opposite) side. Slide the rod rightward until the top end touches the middle of S (and the bottom end touches the right corner of the bottom side). Thus the left half of S including the endpoints must have the same color (Figure 11a). Symmetrically, the right half of S including the endpoints must have the same color. Thus the entire side S including its endpoints must have the same color (Figure 11b). Any two adjacent sides must have the same color (since they share an endpoint), so all of the sides must have the same color. But the middle of a side and the corner of the opposite side are exactly one unit apart (Figure 11c). Contradiction. ∎

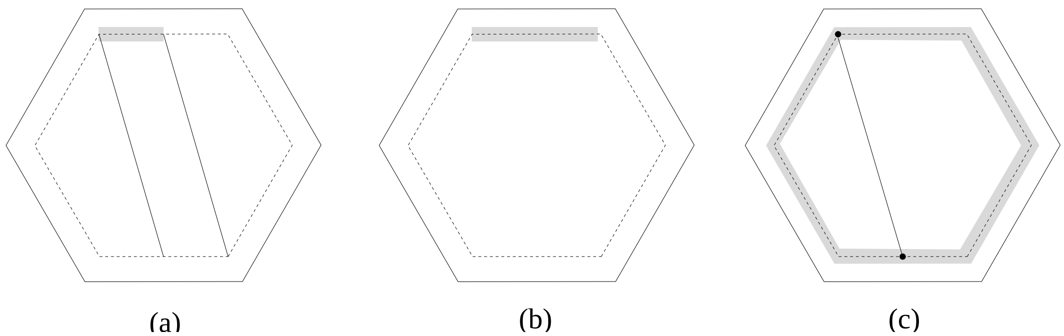


Figure 11: Non-2-colorability of regular n-gon-region with even number of sides.

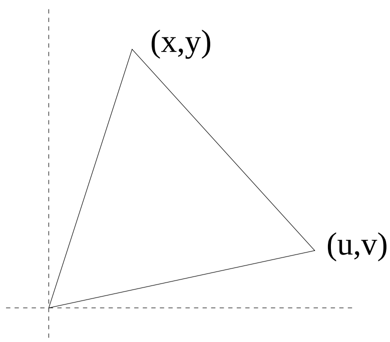


Figure 12:

Corollary 3.11 *An equilateral-triangle-region is 2-colorable if and only its side length* $s < 1$.

For regular n-gon-regions, where n is odd, the conditions of colorability involve a $<$ rather than a $\leq$. It turns out that almost the the entire boundary can be included:

Theorem 3.12 *A regular n-gon-region with n is odd and circumradius* $r = \frac{1}{\sqrt{2(1+\cos(\pi/n))}}$, *is 2-colorable if one corner point is removed.*

Proof: Bisect the n-gon-region from the corner with the missing point to the middle of its opposite side. Color the two halves with different colors. The bisecting line can be either color. ∎

Note 3.13 The non-colorability argument in Theorem 3.10 for n odd shows that the region is not 2-colorable if any other point is removed, or even if any set of points not including a corner point is removed.

4 3-COLORINGS

Definition 4.1 A *tri-rod* (p, q, r) is a unit-side equilateral triangle with corners p, q, r.

Lemma 4.2 *Consider a tri-rod in the upper right quadrant of a cartesian coordinant system, with a corner at the origin* $(0,0)$, *the top corner is at* (x, y), *and right corner is at* (u, v).

(Figure 12.) *Then*

$$\begin{aligned} x &= \tfrac{1}{2}u - \tfrac{\sqrt{3}}{2}\sqrt{1-u^2} = \tfrac{1}{2}\sqrt{1-v^2} - \tfrac{\sqrt{3}}{2}v \\ \text{and} \quad y &= \tfrac{\sqrt{3}}{2}u + \tfrac{1}{2}\sqrt{1-u^2} = \tfrac{\sqrt{3}}{2}\sqrt{1-v^2} + \tfrac{1}{2}v \end{aligned}$$

Proof:

$$\begin{aligned} x &= \cos(\frac{\pi}{3} + \arccos(u)) \\ &= \cos(\frac{\pi}{3})\cos(\arccos(u)) - \sin(\frac{\pi}{3})\sin(\arccos(u)) \\ &= \frac{1}{2}u - \frac{\sqrt{3}}{2}\sqrt{1-u^2} \end{aligned}$$

$$\begin{aligned} y &= \sin(\frac{\pi}{3} + \arccos(u)) \\ &= \sin(\frac{\pi}{3})\cos(\arccos(u)) + \cos(\frac{\pi}{3})\sin(\arccos(u)) \\ &= \frac{\sqrt{3}}{2}u + \frac{1}{2}\sqrt{1-u^2} \end{aligned}$$

∎

The following lemma, which is analogous to Lemma 3.3, is the backbone for all of the non-colorability arguments.

Lemma 4.3 *Let (p, q, r) be a tri-rod, where the disks of radius ϵ, $0 < \epsilon < 1/2$, centered at p, q, and r (called the p-disk, q-disk, and r-disk) are contained in a 3-colored region. Then there are inner open disks of radius $O(\epsilon^2)$ centered at p, q, and r that are each monochromatic (with different colors).*

Proof: Construct inner discs of radius $K\epsilon^2$ about p, q, r, where K is a positive constant chosen small enough to accomodate Case(2) of the following argument. There are two cases:

CASE (1): No color occurs in all three inner disks. There are two subcases:

CASE (1a): One inner disk is monochromatic. We can assume it is centered at p and is red. Obviously no point in either other inner disk can be red, so the other two inner disks, centered at q and r, are bichromatic blue/green. By Lemma 3.3, there are radius $O(\epsilon^4)$ disks within the inner disks centered at q and r that are monochromatic. By Lemma 3.4, moving rod (q, r) inside the radius $O(\epsilon^2)$ inner disks forces them to be monochromatic (with different colors).

CASE (1b): No inner disk is monochromatic. Since no color occurs in all three inner disks, each color occurs in at most two inner disks, which implies that the sum of number colors used over all three inner disks at most six. Since no inner disk is monochromatic, each inner disk contains at least two colors, so each inner disk must contain exactly two colors. (Otherwise the sum of number colors used over the three disks would be more than six.) Thus, each inner disk must be bichromatic, and each of the three possibilities (red/blue, red/green, blue/green) occurs exactly once. (If some pair of colors occurred twice then those two colors would each already occur in two inner disks, so the third inner disk would have to be monochromatic.)

Consider the red/blue disk. (Figure 13.) There exist two points, one red and one blue, much closer to each other than either is to the boundary.[2] The points unit distance from the

[2] Take any two points of different colors (red and blue). Consider the point halfway between them. It must have a different color from one of the original points. Keeping the middle point and the opposite colored point produces two new points half as far apart. Iterate until the separation is small enough.

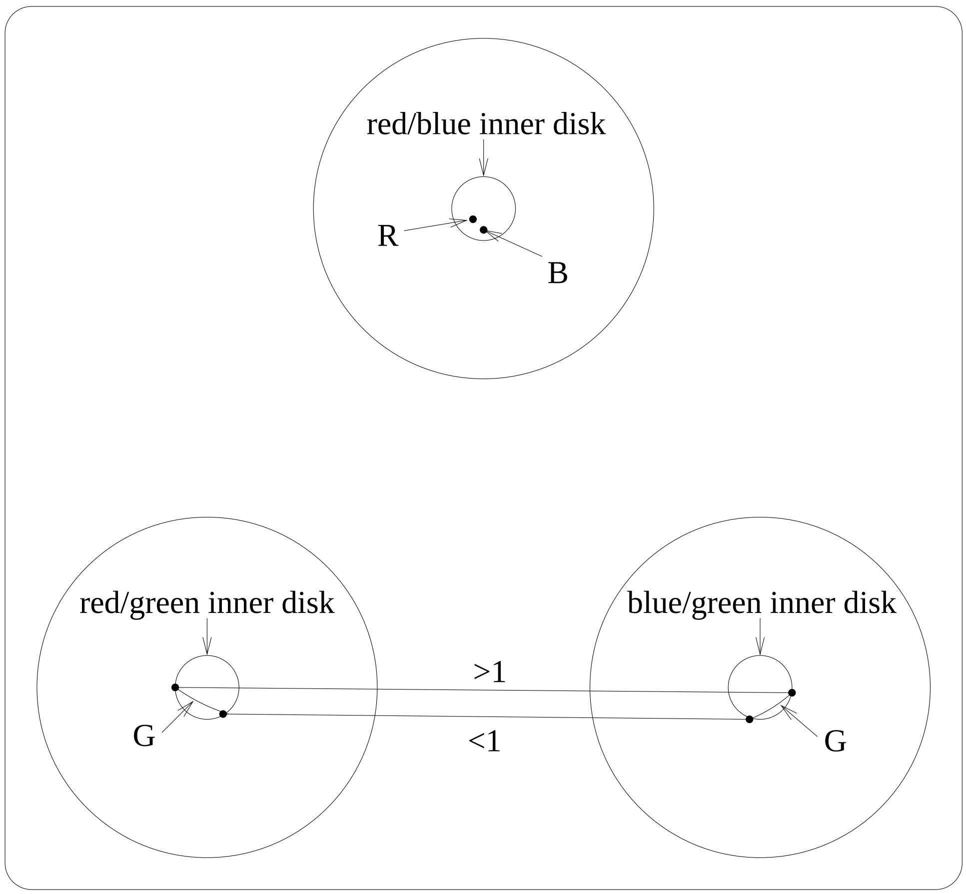

red/blue inner disk
R
B
red/green inner disk
blue/green inner disk
>1
<1
G
G

Figure 13:

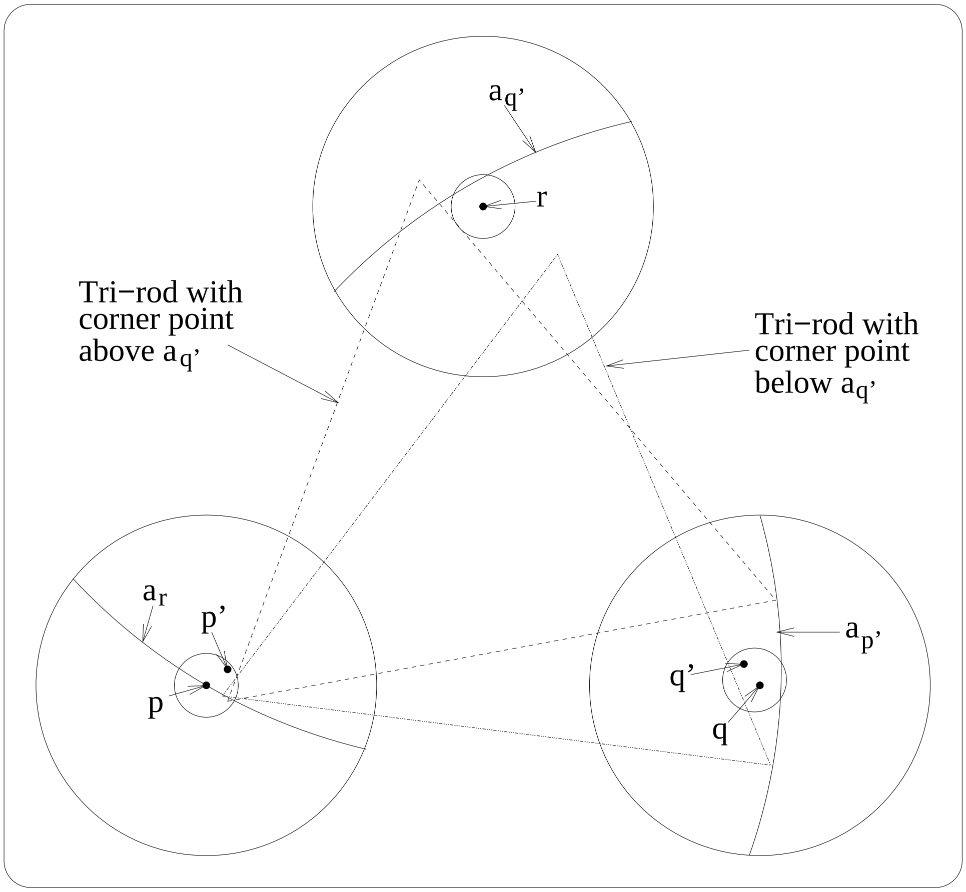


Figure 14:

red point lying in the inner red/green disk form a green arc; the points unit distance from the blue point lying in the inner blue/green disk also form a green arc. Since the red and blue points are arbitrarily close to each other, the two closest points on these two arcs are less than one unit apart and the two furthest points are more than one unit apart. By continuity, two points, one on each arc, must be exactly one unit apart. Contradiction.

CASE (2): Some color, say red, does occur in all three inner disks. (Figure 14.) Since p, q, and r must have different colors, one of them, say r, must be red. Let p' be a red point inside the inner disk centered at p, and q' be a red point inside the inner disk centered at q. (We do not need r' since r is already red.) Let $a_{p'}$ be the intersection of the arc of the unit-radius circle around p' with the q-disk, $a_{q'}$ be the intersection of the arc of the unit-radius circle around q' with the r-disk, and a_r be the intersection of the arc of the unit-radius circle around r with the p-disk. Note that the arcs must be colored only blue and green. We prove below that there is a tri-rod whose three corners lie on these arcs. Figure 14 gives the intuition for this as you can see the two tri-rods, and that one is obtained by sliding the other one. Since there are only two colors available for the three corners this will be a contradiction.

To prove the claim, choose cartesian coordinates (x, y), such that p is $(0, 0)$, q is $(1, 0)$, and r is $(1/2, \sqrt{3}/2)$. We approximate each of the arcs by a straight line; within each ϵ-radius disk the error is only $O(\epsilon^2)$. Arc $a_{p'}$ is approximated by $x = 1 + O(\epsilon^2)$, arc $a_{q'}$ is approximated by $y = \frac{x}{\sqrt{3}} + \frac{1}{\sqrt{3}} + O(\epsilon^2)$, and arc a_r is approximated by $y = -\frac{x}{\sqrt{3}} + O(\epsilon^2)$.

Any given point $\beta = (1 + O(\epsilon^2), d)$ on $a_{p'}$ is unit distance away from some point $\alpha = (O(\epsilon^2), O(\epsilon^2))$ on a_r. By a trivial extension of Lemma 4.2, if (α, β, γ) is the tri-rod with γ near

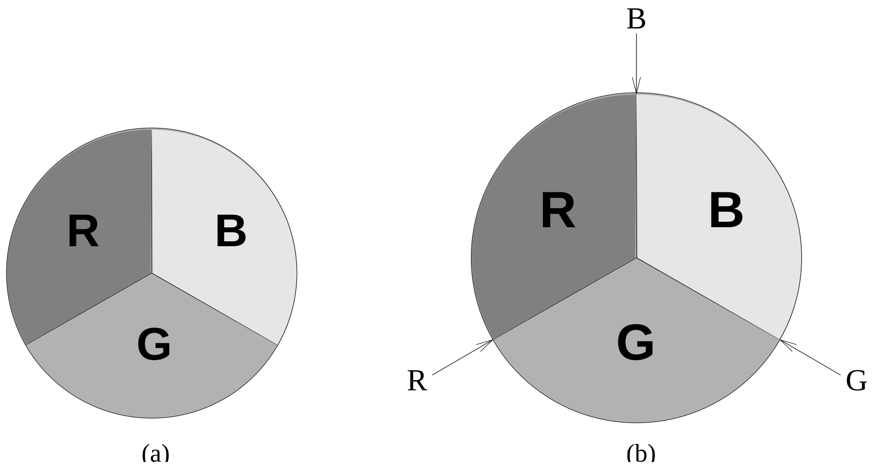


Figure 15: 3-coloring of circle-region for diameter (a) $r < \frac{1}{\sqrt{3}}$ and (b) $r = \frac{1}{\sqrt{3}}$

r, then

$$\gamma = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}d + O(\epsilon^2),\ \frac{\sqrt{3}}{2} + \frac{d}{2} + O(\epsilon^2) \right)$$

In particular, by setting $d = \epsilon/2$, there is a tri-rod with two corners on a_r and $a_{p'}$ and top corner at

$$\left(\frac{1}{2} - \frac{\sqrt{3}\epsilon}{4} + O(\epsilon^2),\ \frac{\sqrt{3}}{2} + \frac{\epsilon}{4} + O(\epsilon^2) \right) ,$$

which is above the arc $a_{q'}$, and, by setting $d = -\epsilon/2$, there is a tri-rod with two corners on a_r and $a_{p'}$ and top corner at

$$\left(\frac{1}{2} + \frac{\sqrt{3}\epsilon}{4} + O(\epsilon^2),\ \frac{\sqrt{3}}{2} - \frac{\epsilon}{4} + O(\epsilon^2) \right) ,$$

which is below the arc $a_{q'}$. So, by continuity, there exists a tri-rod with its corners on all three arcs. This establishes the contradiction. ▮

Lemma 4.4 *Assume a region is 3-colored. If we slide a tri-rod continuously around the inside of the region, the points passed over by a given corner must all be the same color, and the points passed over by different corners must have different colors.*

Proof: Follows from the previous lemma. ▮

Theorem 4.5 *A circle-region is 3-colorable if and only if its radius $r \le \frac{1}{\sqrt{3}} \approx .57735$.*

Proof: **COLORABILITY:** Assume $r \le 1/\sqrt{3}$. Trisect the circle-region and color the three sections different colors. The points on the boundary between two sections can be either color (Figure 15a), except if the radius is exactly equal to $1/\sqrt{3}$ then color the three end-points, which are on the circle, different colors (Figure 15b).

NON-COLORABILITY: Assume $r > 1/\sqrt{3}$. Suppose the region is 3-colored. Put a tri-rod inside of the region so that their centers correspond, and rotate the tri-rod 120 degrees around its center (one third of the way around). (Figure 16a.) Each corner of the tri-rod

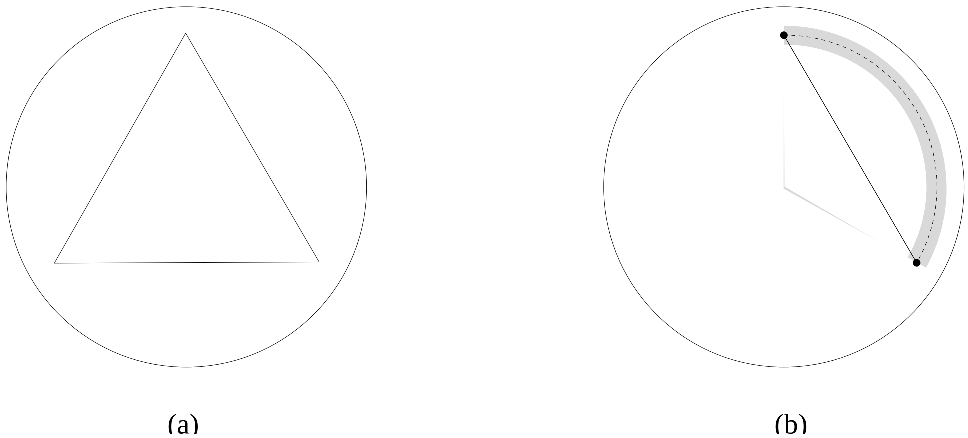


Figure 16: Non-3-colorability of circle-region for $r > \frac{1}{\sqrt{3}}$

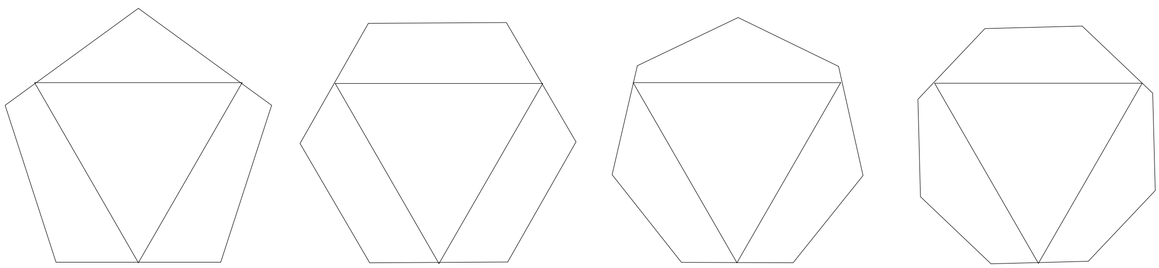


Figure 17: Tri-rod in a regular n-gon-region, for $n = 5, 6, 7, 8$

.

traverses an arc of a circle with radius $1/\sqrt{3}$. By Lemma 4.4 each arc must be monochromatic and have a different color, but the arcs intersect at their endpoints (Figure 16b). Contradiction. ∎

Theorem 4.6 *A regular n-gon-region with $n \geq 5$ is 3-colorable if and only if the inradius*

$$q \leq \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$$

where $\theta = \lfloor \frac{n+1}{3} \rfloor \frac{2\pi}{n}$.

Proof: Place the n-gon onto the cartesian coordinates with center at the origin, and one side parallel to the x-axis in the lower half of the plane. Make the polygon just large enough so that the tri-rod fits exactly inside it with one corner in the middle of the bottom side and the other two corners each touching a side (above the x-axis). Figure 17 shows this for a pentagon, hexagon, septagon, and octogon (with the cartesian coordinates not shown). We will determine the size of such a regular polygon, which will help with both the colorability and non-colorability arguments.

Number the sides of the polygon counting counter-clockwise, starting with the bottom side, which is numbered 0. The top right corner of the tri-rod is about one third the way around the polygon. More precisely, it touches side number $\lfloor \frac{n+1}{3} \rfloor$. So the circular angle from the middle of the bottom side to the middle of this side is $\theta = \lfloor \frac{n+1}{3} \rfloor \frac{2\pi}{n}$. The angle from the x-axis is

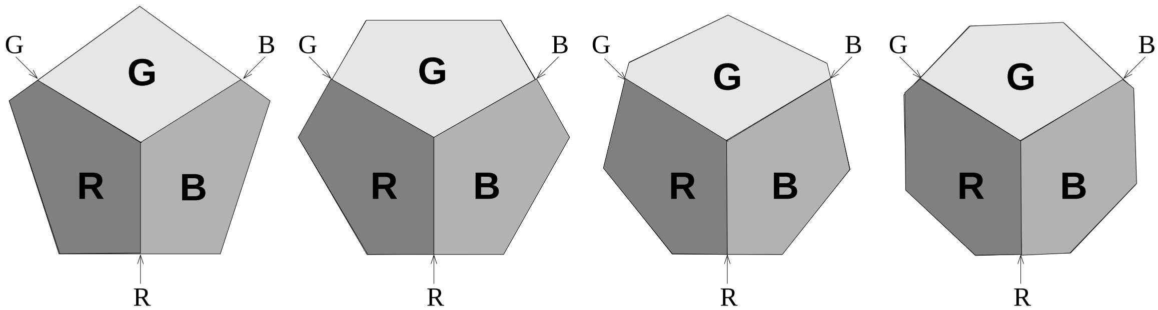


Figure 18: 3-coloring of regular n-gon-region, for $n = 5, 6, 7, 8$ and $q = \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$.

$\theta - \frac{\pi}{2}$, which implies the slope of this side is

$$-\cot(\theta - \frac{\pi}{2}) \;=\; \tan\theta\ .$$

There are two particular points that we know are on that side: The midpoint of the side, which is $(q\cos(\theta - \frac{\pi}{2}), q\sin(\theta - \frac{\pi}{2})) = (q\sin\theta, -q\cos\theta)$, and the corner point of the tri-rod, which is $(1/2, \sqrt{3}/2 - q)$. If these two points are distinct, they define a line whose slope is

$$\frac{-q\cos\theta - (\sqrt{3}/2 - q)}{q\sin\theta - 1/2}$$

So,

$$\begin{aligned}
\tan\theta \;&=\; \frac{-q\cos\theta - (\sqrt{3}/2 - q)}{q\sin\theta - 1/2} \\
\Longrightarrow \quad q\sin\theta\tan\theta - \frac{1}{2}\tan\theta \;&=\; -\,q\cos\theta - \frac{\sqrt{3}}{2} + q \\
\Longrightarrow \quad q(\sin\theta\tan\theta + \cos\theta + 1) \;&=\; \frac{1}{2}\tan\theta - \frac{\sqrt{3}}{2} \\
\Longrightarrow \quad q(\sin^2\theta + \cos^2\theta + \cos\theta) \;&=\; \frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta \\
\Longrightarrow \quad q(1 + \cos\theta) \;&=\; \frac{\sin\theta - \sqrt{3}\cos\theta}{2} \\
\Longrightarrow \quad q \;&=\; \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1 + \cos\theta)}
\end{aligned}$$

If the two points coincide (which happens when $3|n$) then

$$q\sin\theta \;=\; 1/2 \qquad \text{and} \qquad -\,q\cos\theta \;=\; \sqrt{3}/2 - q$$

This simplifies to

$$q \;=\; \frac{1}{\sqrt{3}} \qquad \text{and} \qquad \theta \;=\; \frac{2\pi}{3}$$

which is a solution to the above equation.

COLORABILITY: Assume $q = \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$. Consider the three points where the three corners of a tri-rod intersect three sides of the polygon as above. Draw the three line segments from the center of the polygon to these three points, partitioning the polygon into three subregions, color each subregion a different color. The points on the boundary between two subregions can be any color, except the three intersection points must be different colors. Figure 18 shows the coloring for a pentagon, hexagon, septagon, and octogon. If $q < \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$, shrink the above coloring to the desired size (and there is no extra condition on the three intersection points).

NON-COLORABILITY: Assume $q > \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$ and the n-gon-region is 3-colored. Put an n-gon with inradius exactly $\frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$ strictly inside the original polygon (so that its center at the origin, and one side parallel to the x-axis in the lower half of the plane). We show that a tri-rod can be maneuvered around inside the inner polygon.

Consider the top edge of the tri-rod. On the left side it touches an edge of the inner polygon whose slope is, say m, where $m > 0$, and on the right side it touches a edge whose slope is $-m$. We claim that the as long as those two corners of the tri-rod touch those two edges, respectively, the third corner of the tri-rod will be inside the inner polygon. Let p and q be the original upper left and right corners of the tri-rod, respectively. For any locations the two points will be $p = (-1/2 + \epsilon, \sqrt{3}/2 + m\epsilon)$ for some ϵ and $q = (1/2 + \delta, \sqrt{3}/2 - m\delta)$ for some δ. So the y-coordinate of midpoint of the upper edge of the tri-rod will be

$$\frac{\sqrt{3}}{2} + \frac{m(\epsilon - \delta)}{2}.$$

If $\epsilon - \delta > 0$ then the midpoint has moved up, which implies that the bottom corner of the tri-rod will have moved up also. We can calculate the relation between ϵ and δ because the edge of the tri-rod has length one:

$$\begin{aligned}
& ((\frac{1}{2} + \delta) - (-\frac{1}{2} + \epsilon))^2 + ((\frac{\sqrt{3}}{2} - m\delta) - (\frac{\sqrt{3}}{2} + m\epsilon))^2 = 1 \\
\Longrightarrow \quad & (1 - (\epsilon - \delta))^2 + (-m(\epsilon + \delta))^2 = 1 \\
\Longrightarrow \quad & 1 + 2(\epsilon - \delta) + (\epsilon - \delta)^2 + m^2(\epsilon + \delta)^2 = 1 \\
\Longrightarrow \quad & \epsilon - \delta = \frac{(\epsilon - \delta)^2 + m^2(\epsilon + \delta)^2}{2}
\end{aligned}$$

The right hand side is clearly positive.

Place the tri-rod inside the inner polygon so that its three corner points intersect three sides of the inner polygon as above (Figure 19a). Rotate the tri-rod clockwise so that the two upper points stay on the boundary, until one of them reaches a corner of the inner polygon (Figure 19b). Fix this latter point of the tri-rod, and slide the tri-rod so that the bottom point touches the bottom edge of the inner polygon (Figure 19c). Rotate the tri-rod clockwise so that the previously fixed point and the bottom point stay on the boundary of the inner polygon, until the third point intersects the middle a side, which by symmetry will happen (Figure 19d). By continuing in this fashion, each corner of the tri-rod will intersect the middle of each side of the inner polygon. By Lemma 4.4 each corner must be monochromatic and have a different color, but the corners intersect at their endpoints (one third of the way around the inner polygon). This establishes the contradiction. ▮

Corollary 4.7 *A regular n-gon-region with $n \geq 6$ and $3|n$ is 3-colorable if and only if the inradius*

$$q \leq \frac{1}{\sqrt{3}}$$

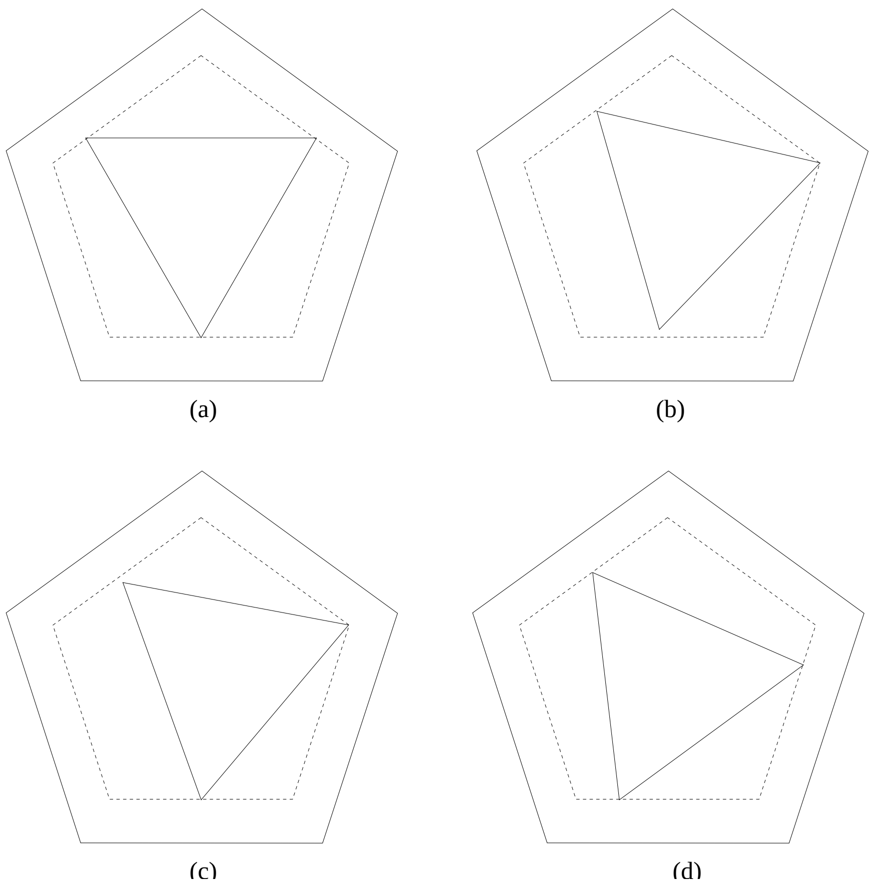


Figure 19: Non-3-colorability of regular polygon region for $q > \frac{\sin\theta - \sqrt{3}\cos\theta}{2(1-\cos\theta)}$.

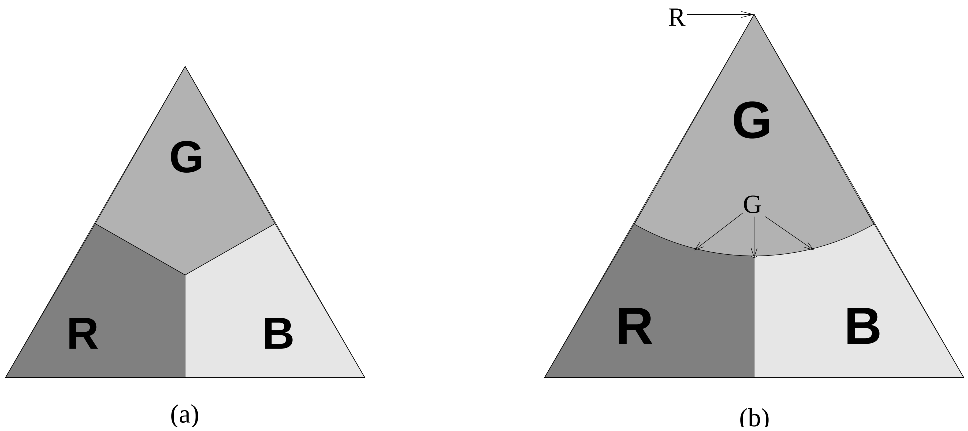


Figure 20: 3-coloring of equilateral-triangle-region for side length (a) $s < \sqrt{3}$ and (b) $s = \sqrt{3}$.

The equilateral triangle and square can be colored in very much the same way: Place a "tri-rod" inside the region as above, and use the three intersection points with the center of the region to separate into three subregions for coloring. The diffence is that the two fartherest apart points in a subregion are no longer the two intersection points inside that subregion: For the triangle it is a corner and the center of the triangle, and for the square it is a corner and one of the intersection points. So the "tri-rod" will have sides smaller than one unit. This necessitates different colorability and non-colorability arguments.

Theorem 4.8 *An equilateral-triangle-region is 3-colorable if and only if its side length $s \leq \sqrt{3}$ (or equivalently its circumradius $r \leq 1$).*

Proof: **COLORABILITY:** Assume $s < \sqrt{3}$ (Figure 20a). Trisect the triangle-region with the line segments from the middle of each side to the center. Color each section a different color. Color the points on the boundary between two sections either color, and color the middle point any color, say green. The farthest two points of the same color are the middle point and the corner point in the green section. This is the circumradius and it must be less than 1. So, by Lemma 3.9,

$$s \;=\; 2r\sin(\pi/n) \;=\; 2r\sin(\pi/3) \;=\; 2r\frac{\sqrt{3}}{2} \;=\; r\sqrt{3} \;<\; \sqrt{3}$$

(by 3.9).

Assume $s = \sqrt{3}$ (Figure 20b). Again trisect the triangle-region with the line segments from the middle of each side to the center, color each section a different color, and color the points on the boundary between two sections either color. Color the corner point of the green section red, and color green the points on the unit-radius arc centered at that point.[3] The isolation of the green arc is eliminated by coloring the sub-region above the arc all green.

NON-COLORABILITY: Assume $s > \sqrt{3}$, and suppose the region is 3-colored. Imbed an equilateral triangle T with side length exactly $\sqrt{3}$ inside of the triangle-region. Place a tri-rod inside of T, and slide the tri-rod so that each corner touches a different corner of T. Thus, the three corners of T must have different colors. But the center of T is exactly distance 1 away from each corner. ∎

[3]Actually, only the points in the red section need to be colored green, but this makes the coloring more symmetrical.

The following result was proved by Bauslaugh using very different techniques.

Theorem 4.9 *([Baus]) An infinite strip is 3-colorable if and only if it has width $a \le \sqrt{3}/2$.*

Proof: **COLORABILITY:** Assume the width $a \le \sqrt{3}/2$. Color with blocks of width $1/2$, rotating colors red, green, blue, red, green, blue, etc. Color the boundary between two blocks the color of the points to its right.

NON-COLORABILITY: Assume the width $a > \sqrt{3}/2$, and suppose the region is 3-colored. Put a strip of width $\sqrt{3}/2$ inside the original strip. Put a tri-rod inside the new strip with two corners on one side of the strip and the third corner on the other side of the strip. Slide the tri-rod rightward one unit. The left corner finishes where the right corner starts, but by Lemma 4.4 the points crossed by these corners must have different colors. Contradiction. ∎

Theorem 4.10 *A rectangle-region of dimensions $a \times b$, with $a \le b$, is 3-colorable if and only if*

$$\begin{array}{lll} & (1) & a \le \frac{\sqrt{3}}{2} \\ or & (2) & \frac{\sqrt{3}}{2} < a \le \frac{2}{\sqrt{5}} \text{ and } b \le 3\sqrt{1-a^2} \\ or & (3) & \frac{2}{\sqrt{5}} < a \le \frac{8}{\sqrt{65}} \text{ and } b \le \sqrt{1-a^2} + \sqrt{1-a^2/4} \end{array}$$

Proof: Note that

$$\max\left(3\sqrt{1-a^2},\ \sqrt{1-a^2}+\sqrt{1-a^2/4}\right) = \begin{cases} 3\sqrt{1-a^2} & \text{if } a \le \frac{2}{\sqrt{5}} \\ \sqrt{1-a^2}+\sqrt{1-a^2/4} & \text{if } a \ge \frac{2}{\sqrt{5}} \end{cases}$$

COLORABILITY:

CASE (1): $a \le \sqrt{3}/2$. Color it like an infinite strip, but chop it to length b.

CASE (2): $\sqrt{3}/2 < a \le 2/\sqrt{5}$ and $b \le 3\sqrt{1-a^2}$. We consider two methods of coloring: *vertical stripes*, which are are simpler for $b < 3\sqrt{1-a^2}$, and *non-stripes*, which are simpler for $b = 3\sqrt{1-a^2}$. We present both methods, partly because they provide insight for the non-colorability argument.

Vertical stripes (Figure 21). Assume $b < 3\sqrt{1-a^2}$. Trisect the rectangle with vertical lines and color the three small rectangles red, blue, and green. Within each small rectangle, two points farthest apart are diagonally opposite corners. So to have a legitimate 3-coloring, we need $\sqrt{a^2+(b/3)^2} < 1$, which is equivalent to

$$b < 3\sqrt{1-a^2}\ .$$

Assume $b = 3\sqrt{1-a^2}$. From the lower left corner draw a red unit-radius circular arc from the top to the horizontal middle, and from the lower right corner draw a green unit-radius circular arc from the top to the horizontal middle. Color their intersection point green. From the upper left corner draw a red unit-radius circular arc from the bottom to the horizontal middle, and from the upper right corner draw a green unit-radius circular arc from the bottom to the horizontal middle. Color their intersection point green. Draw a vertical line in the horizontal middle connecting the two greeen intersection points. This creates a large region to the left, one to the right, and two small regions in the middle (one at the top and one at the bottom). Color the left region red, the right region green, and the two middle regions blue. Color the two points at the top and bottom, one third the way across blue, the two points at the top and bottom, two thirds the way across green, and the two points at the top and bottom right corners blue. This completes the coloring.

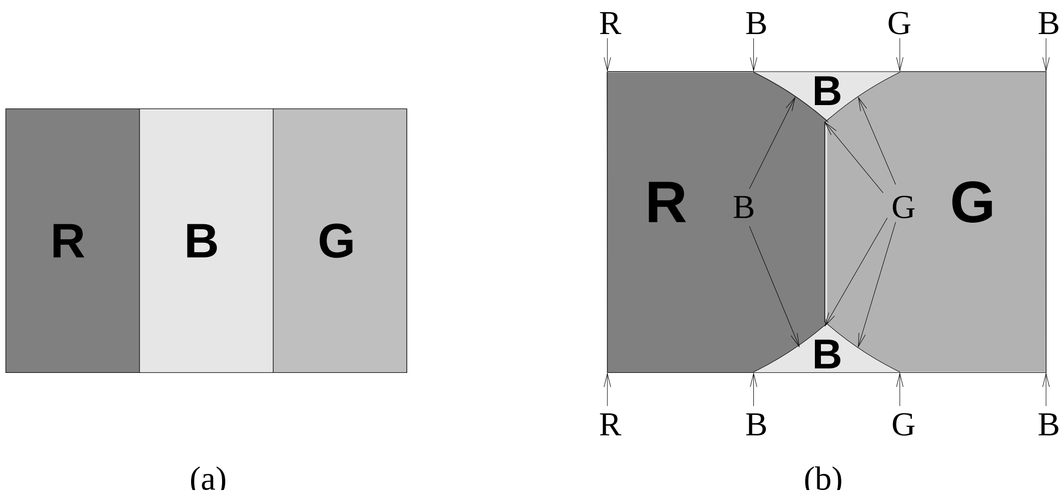


Figure 21: Vertical stripes 3-coloring of $a \times b$ rectangle-region, where $a \leq b$, $\frac{\sqrt{3}}{2} < a \leq \frac{2}{\sqrt{5}}$.
(a) $b < 3\sqrt{1-a^2}$. (b) $b = 3\sqrt{1-a^2}$.

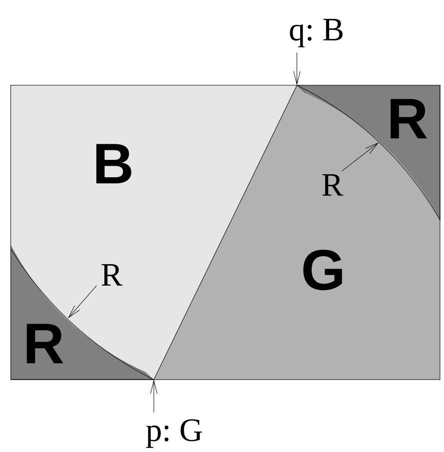


Figure 22: Non-stripes 3-coloring of $a \times b$ rectangle-region, where $a \leq b$, $\frac{\sqrt{3}}{2} < a \leq \frac{2}{\sqrt{5}}$, and $b \leq 3\sqrt{1-a^2}$

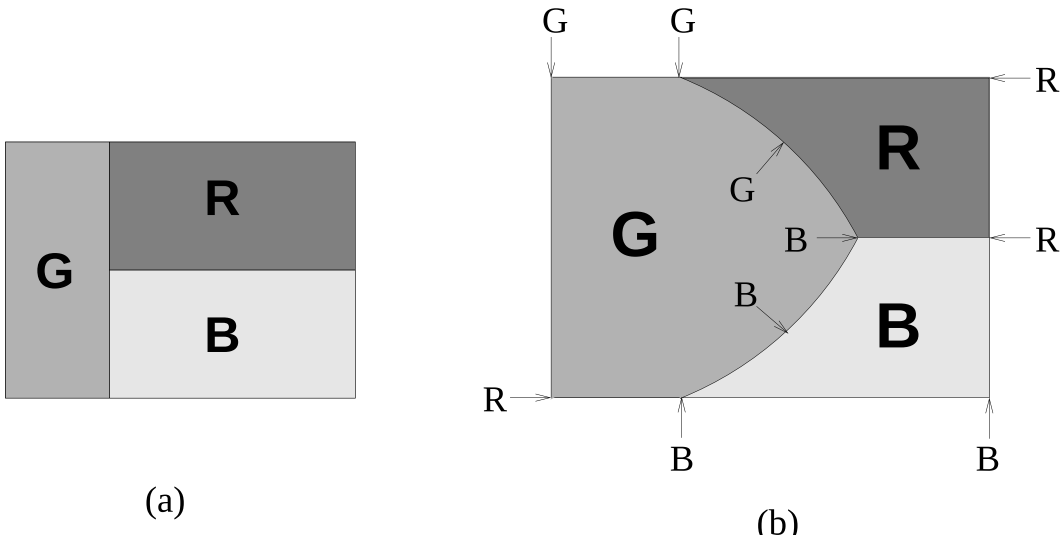


Figure 23: Mixed stripes 3-coloring of $a \times b$ rectangle-region, where $a \le b$, $\frac{2}{\sqrt{5}} \le a \le \frac{8}{\sqrt{65}}$.
(a) $b < \sqrt{1-a^2} + \sqrt{1-a^2/4}$. (b) $b = \sqrt{1-a^2} + \sqrt{1-a^2/4}$.

Non-stripes (Figure 22). The point p on the bottom side, distance $(b - \sqrt{1-a^2})/2$ from the bottom left corner, and the point q on the top side, distance $(b - \sqrt{1-a^2})/2$ from the top right corner, are exactly one unit apart. Color p green and q blue. Consider the unit-radius arc centered at p, near the top right corner of the region. Color the arc and all points outside it (but inside the region) red, except of course for point q. Consider the unit-radius arc centered at q, near the bottom left corner of the region. Color the arc and all points outside it (but inside the region) red, except of course for point p. Cut the remainder of the region in half by the line segment from p to q and color the left side blue and the right side green. The boundary points can be either color. As long as $b \le 3\sqrt{1-a^2}$, no two points of the same color will be exactly one unit apart.

CASE (3): $\frac{2}{\sqrt{5}} < a \le \frac{8}{\sqrt{65}}$ and $b \le \sqrt{1-a^2} + \sqrt{1-a^2/4}$. We use *Mixed stripes.*[4] Let $x < b$ be a value to be determined later. Use green to color the $a \times x$ vertical stripe at the left of the region. Bisect the remaining subrectangle with a horizontal line and color the top red and the bottom blue. (Figure 23a.) For the vertical stripe, we need $\sqrt{a^2 + x^2} < 1$, which is equivalent to $x < \sqrt{1-a^2}$. For each horizontal stripe, we need $\sqrt{(a/2)^2 + (b-x)^2} < 1$, which is equivalent to $x > b - \sqrt{1-a^2/4}$. Thus, we need $b - \sqrt{1-a^2/4} < x < \sqrt{1-a^2}$, which is satisfied for some x if

$$b \;<\; \sqrt{1-a^2} + \sqrt{1-a^2/4}\;.$$

If this last inequality is replaced by equality (Figure 23b), color the same way except that in the red rectangle color the right two points red, in the blue rectangle color the bottom two points blue, in the green rectangle color the top two points green and the bottom left point red. Create a green unit-radius arc from the bottom left point in the red sub-region, and to eliminate the isolation of the green arc color everything to the left of the arc green. For symmetry create a blue unit-radius arc from the top left point in the (already) blue sub-region and color everything to the left of the arc green. So we can 3-color any rectangle with ($a \le b$

[4]Cases (2) and (3) both give acceptable 3-colorings of the rectangle for larger ranges of a than stated in their conditions, but, by the note at the beginning of the proof, for $a \le 2/\sqrt{5}$ Case (2) is better and for $a \ge 2/\sqrt{5}$ Case (3) is better. The two methods are the same for $a = 2/\sqrt{5}$.

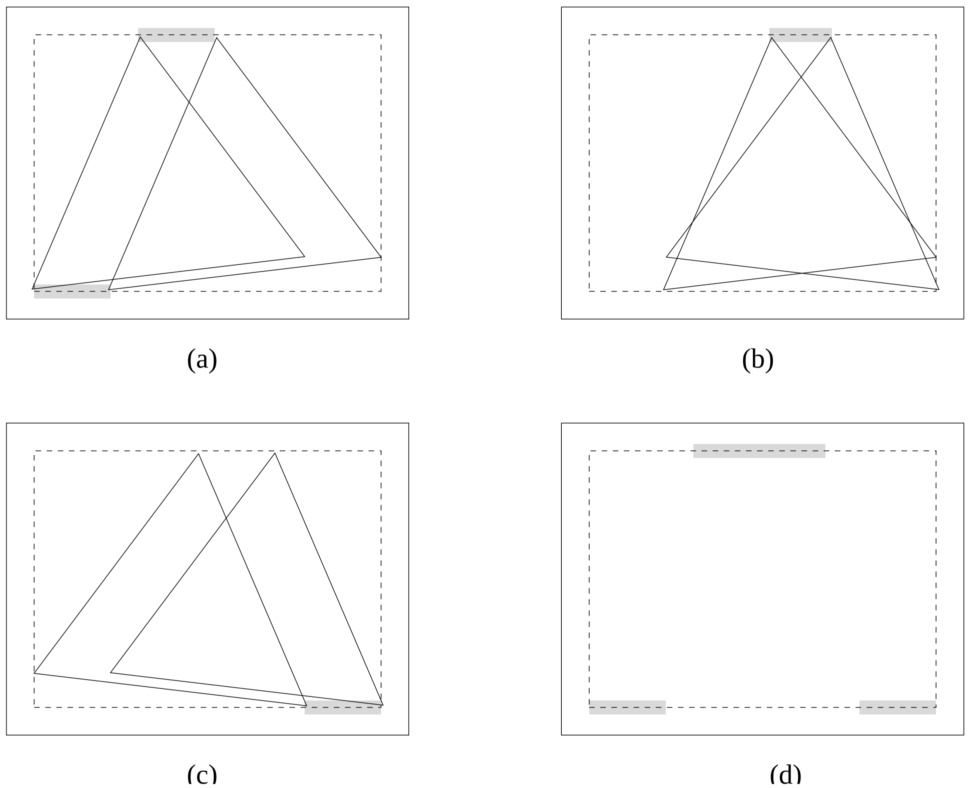


Figure 24: Three segments that must be monochromatic and have different colors in 3-coloring of rectangle for $a > \sqrt{3}/2$, $b > \max(3\sqrt{1-a^2}, \sqrt{1-a^2} + \sqrt{1-a^2/4})$, and $b > 1$.

and)

$$b \leq \sqrt{1-a^2} + \sqrt{1-a^2/4} \ .$$

Note that $a = \sqrt{1-a^2} + \sqrt{1-a^2/4}$ implies $a = \frac{8}{\sqrt{65}}$.

NON-COLORABILITY: We may assume $a < 1$. Assume that $a > \sqrt{3}/2$. By the note at the beginning of the proof, we need to consider only $b > \max\left(3\sqrt{1-a^2}, \sqrt{1-a^2} + \sqrt{1-a^2/4}\right)$. Suppose the rectangle-region is 3-colored. There are two cases: (1) $b > 1$ and (2) $b \leq 1$.

CASE (1): $b > 1$. Let R be an $\alpha \times \beta$ rectangle inside the region, with $\sqrt{3}/2 < \alpha < a$, $1 < \beta < b$, and $\beta = \max(3\sqrt{1-\alpha^2}, \sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4})$. To see that this is possible, first assume that $a \leq 2/\sqrt{5}$. Then, as noted above, $\max(3\sqrt{1-a^2}, \sqrt{1-a^2} + \sqrt{1-a^2/4}) = 3\sqrt{1-a^2}$. Let $\alpha = a - \epsilon$, where ϵ is small enough so that both $\alpha > \sqrt{3}/2$ and $3\sqrt{1-\alpha^2} < b$. Now assume that $a > 2/\sqrt{5}$. Then, as noted above, $\max(3\sqrt{1-a^2}, \sqrt{1-a^2} + \sqrt{1-a^2/4}) = \sqrt{1-a^2} + \sqrt{1-a^2/4}$. Let $\alpha = a - \epsilon$, where ϵ is small enough so that both $\alpha > 2/\sqrt{5}$ and $\sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4} < b$.

We show what is immediately known about the color of points on the top and bottom sides of R: Put a tri-rod inside R with one corner on the bottom-left corner of R and another corner on the top side. Slide the tri-rod rightward until it touches the right side of R, keeping track of points crossed by the left and top corners of the tri-rod (Figure 24a). Now slide the top corner right and the right corner downward until it reaches the bottom right corner of R, keeping track of points crossed by just the top corner of the tri-rod (Figure 24b). Finally, slide the tri-rod leftward until it reaches the left side of R, keeping track of points crossed by just the right corner of the tri-rod (Figure 24c). Each set of points crossed by a corner of the

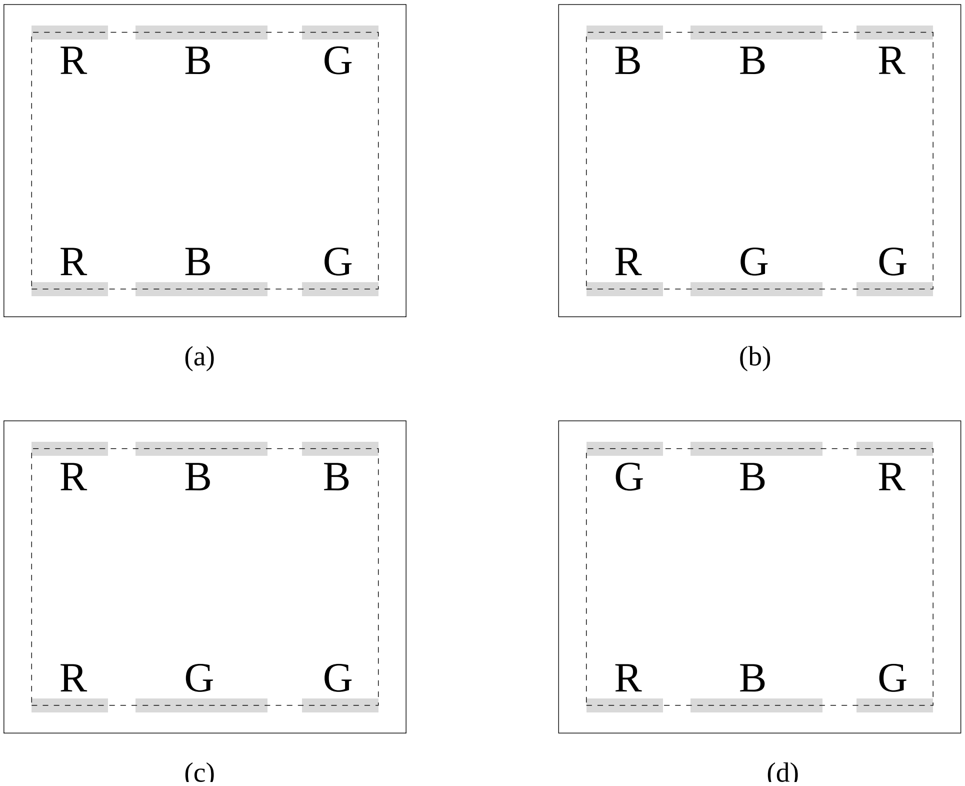


Figure 25: Four non-isomorphic potential 3-colorings of a rectangle for $a > \sqrt{3}/2$, $b > \max(3\sqrt{1-a^2}, \sqrt{1-a^2} + \sqrt{1-a^2/4})$, and $b > 1$.

tri-rod must be monochromatic and have different colors from each other (Lemma 4.4); see Figure 24d. Each line segment on the bottom has length $\beta - (\frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\sqrt{1-\alpha^2})$. The line segment on the top is $\sqrt{1-\alpha^2}$ distant from each side, so it has length $\beta - 2\sqrt{1-\alpha^2}$.

Similarly, by symmetry, there are three line segments, the left of the top side, the middle of the bottom side, and the right of the top side, that are monochromatic and have different colors from each other. Since $\frac{\sqrt{3}}{2} < \alpha < 1$, it may be verified that none of the segments overlap.

Putting everything together produces, up to isomorphism (with respect to renaming colors and left-right reflections) only four types of coloring on the segments: Figure 25. Type (a) corresponds to the vertical stripes coloring, type (b) corresponds to the non-stripes coloring, and type (c) corresponds to the mixed stripes coloring. Type (d) does not correspond to any coloring method used, and does not seem to be a good approach. We will show that, under our assumptions, none of the four types produces a 3-coloring of the rectangle.

Start with type (a). The middle two segments have the same color. The right endpoints of the two segments are exactly distance α (which is < 1) apart. Each segment has length $\beta - 2\sqrt{1-\alpha^2}$, and, since $\beta > 3\sqrt{1-\alpha^2}$, each segment has length at least $\sqrt{1-\alpha^2}$. So the right endpoint of one segment and the left endpoint of the other are at least one unit apart. By continuity, two points, one on each segment, are exactly one unit apart. So type (a) does not produce a 3-coloring. The same reasoning shows that type (d) does not produce a 3-coloring.

To handle types (b) and (c), we extend our knowledge of the coloring of R by piggybacking on the earlier 2-coloring results. Suppose that the middle and right segments on the bottom side have the same color, say green (as shown in Figure 26). We will see that the middle and right segments on the top side must have the same color (as each other), as must all of the

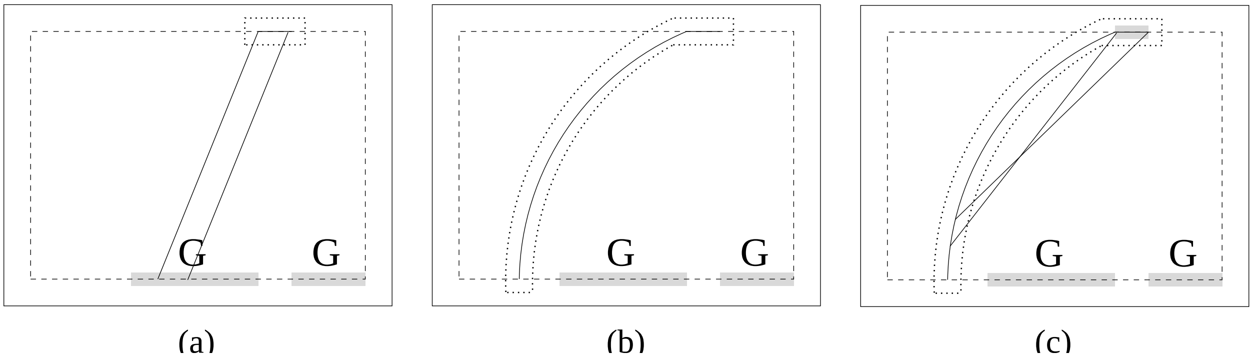


Figure 26:

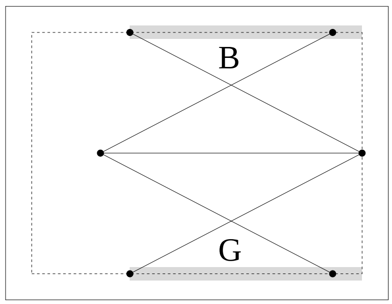


Figure 27:

points in between the two segments. It will follow by symmetry that on the bottom side the points in between the two segments must be green.

Consider the closed line segment S on the top side of R between the middle and right segments (Figure 26a). Each point on S is one unit away from some point on the middle segment of the bottom side, since that segment starts distance $\sqrt{1-\alpha^2}$ from the right side and, as we just showed, has length at least $\sqrt{1-\alpha^2}$. So points on S cannot be green, and since R is strictly inside the rectangle-region neither can points near S. Consider the unit radius arc of points in R centered at the bottom right corner point of R. These points and points nearby cannot be green (Figure 26b), since the corner and points nearby must be green. (This latter fact holds because a tri-rod can be placed inside the rectangle with one corner at the bottom right corner point of R, which itself is inside the full rectangle.) Together the points on and near S and on and near the arc form a region that is 2-colorable (with red and blue). Put one end of a rod on one endpoint of S and the other end on the arc. Slide the top end of the rod along S until it reaches the other endpoint of S, keeping the bottom end of the rod on the arc (Figure 26c). By Lemma 3.4, the points on S must be monochromatic, and since S intersects the middle and right segments on the top side (which are monochromatic), the entire segment from the left endpoint of the middle segment to the right corner of R must be monochromatic.

Now consider type (b) (in Figure 25). The bottom middle and the bottom right segments are both green, so by the above argument the top middle and top right segments must be the same color, but they are not. Thus, type (b) does not produce a 3-coloring.

Finally consider type (c) (Figure 27). By the above argument, all of the points on the top side starting distance $\sqrt{1-\alpha^2}$ from the left and going right must be blue, and all of the points on the bottom side starting distance $\sqrt{1-\alpha^2}$ from the left and going right must be green. Consider the point exactly in the middle of the right side of R. It is exactly one unit away from a blue point on the top side of R, and exactly one unit away from a green point on

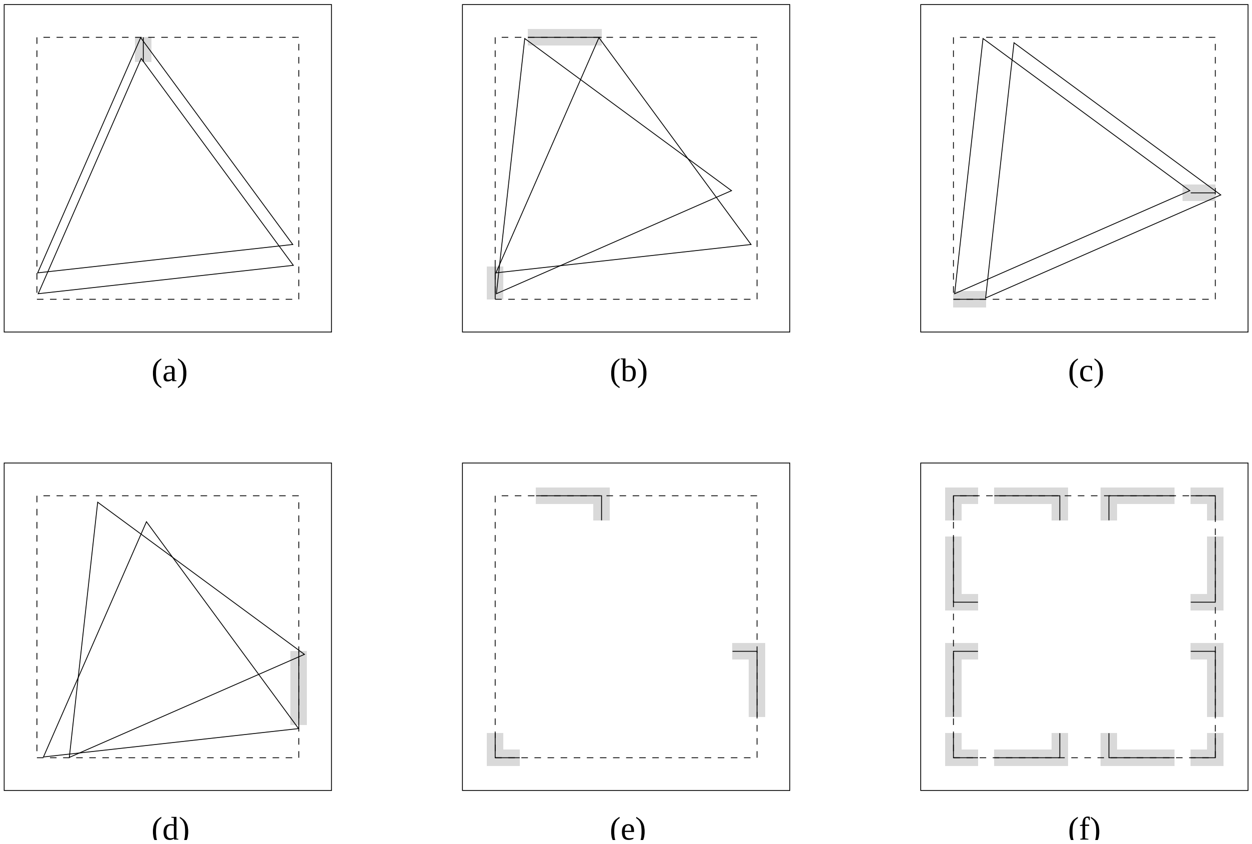


Figure 28:

the bottom side, since $\beta \geq \sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4}$. So it must be red. Now consider the point exactly one unit to the left of this point. It is also exactly one unit away from a point on the top blue segment, and exactly one unit away from a point on the bottom green segment, so it too must be red. Contradiction. Thus, type (c) does not produce a 3-coloring.

CASE (2): $b \leq 1$.

Recall that we are assuming $b > \max(3\sqrt{1-a^2}, \sqrt{1-a^2} + \sqrt{1-a^2/4})$. If $a \leq 2/\sqrt{5} \approx 0.8944$ then (as noted in the beginning of the proof) $b > 3\sqrt{1-a^2}$, so $3\sqrt{1-a^2} < 1$. Solving for a yields $a > \sqrt{8/9} \approx 0.9428$, which is a contradiction.

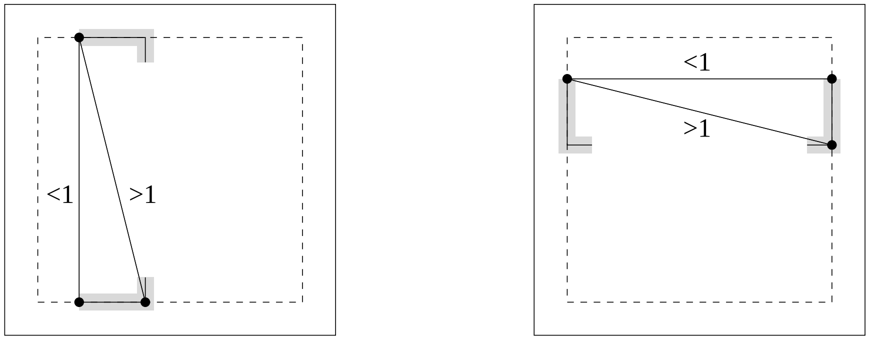
Thus it must be the case that $a > 2/\sqrt{5}$, which (as noted in the beginning of the proof) implies $b > \sqrt{1-a^2} + \sqrt{1-a^2/4}$. Let R be an $\alpha \times \beta$ rectangle inside the region with $2/\sqrt{5} < \alpha < a$, $\alpha \leq \beta < b$, and $\beta = \sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4}$. To see that this is possible, let $\alpha = a - \epsilon$, where ϵ is small enough so that $\sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4} < b$.

Since $\beta < b \leq 1$, $\sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4} < 1$. Solving for α yields $\alpha > \sqrt{\frac{8}{9}\left(\sqrt{13} - \frac{5}{2}\right)} \approx .9913$. Let $\underline{\alpha}$ equal this lower bound for α. Furthermore, α is maximized and β is minimized when $\alpha = \beta$. Solving, $\alpha = \sqrt{1-\alpha^2} + \sqrt{1-\alpha^2/4}$, yields $\alpha = \beta = \frac{8}{\sqrt{65}} \approx 0.9923$. Call this value $\overline{\alpha}$ and $\underline{\beta}$. Thus, there is a very narrow range of values for α and β:

$$.9913 \approx \underline{\alpha} < \alpha \leq \overline{\alpha} = \underline{\beta} \leq \beta < 1 .$$

Because of this, all of the figures here will be drawn as squares (even though they may not be quite squares).

We show what is immediately known about the color of points on all four sides of R, along



(a) (b)

Figure 29:

with some nearby points in the interior. Put a tri-rod inside R with the left corner on the bottom left corner of R and the right corner on the right side of R. Slide the tri-rod upward until the top corner touches the top of R, only keeping track of the points crossed by the top corner (Figure 28a). Now, slide the top corner leftward and the left corner downward until it touches the bottom left corner of R, keeping track of the points crossed by the left and top corners of the tri-rod (Figure 28b). Now, slide the tri-rod rightward until the right corner touches the right side of R, keeping track of the points crossed by the left and right corners of the tri-rod (Figure 28c). Finally, slide the the right corner downward and the left corner leftward until it touches the bottom left corner of R, only keeping track of the points crossed by the right corner of the tri-rod (Figure 28d). Each L-segment crossed by a corner of the tri-rod must be monochromatic and have a different color by Lemma 4.4 (Figure 28e). Following a similar procedure for each corner produces twelve monochromatic L-segments (Figure 28f).

Not only does each group of three associated L-segments have to be composed of three different colors, but there are restrictions between L-segments in different groups. In particular, as we will see, an interior (i.e. non-corner) L-segment must have a different color than either interior L-segment on the opposite side. Furthermore, we will see that, if a corner L-segment and the (interior) L-segment next to it on a side have the same color, then the corner L-segment and the L-segment next to it on the opposite side must have the same color (as each other).

It is possible (depending on the values of α and β) for the corner L-segments to overlap the neighboring interior L-segments on the short sides (i.e., the left and right sides), but not on the long sides (i.e., the top and bottom sides). We never use this information.

We will need to know the exact positions and sizes of the L-segments. This is easy to derive using Lemma 4.2. An interior L-segment on a long side, say the left of the top side as in Figure 28e, starts $\sqrt{1-\alpha^2}$ from the top left corner, goes right until it is $\frac{\beta}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\beta^2}$ from the top left corner, and then goes down $\alpha - (\frac{\sqrt{3}}{2}\beta + \frac{\sqrt{1-\beta^2}}{2})$. An interior L-segment on a short side, say the lower right side as in Figure 28e, starts $\sqrt{1-\beta^2}$ from the bottom right corner, goes up until it is $\frac{\alpha}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\alpha^2}$ from the bottom right corner, and then goes left $\beta - (\frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{1-\alpha^2}}{2})$.

Consider two "opposite" L-segments on the long sides, an interior L-segment on, say, the left of the top side and one on the left of the bottom side as in Figure 29a. Let κ be the length of the line segment on the boundary. Then

$$\kappa \;=\; \left(\frac{\beta}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\beta^2}\right) - \sqrt{1-\alpha^2} \;>\; \frac{\underline{\beta}}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\underline{\beta}^2} - \sqrt{1-\underline{\alpha}^2}\ .$$

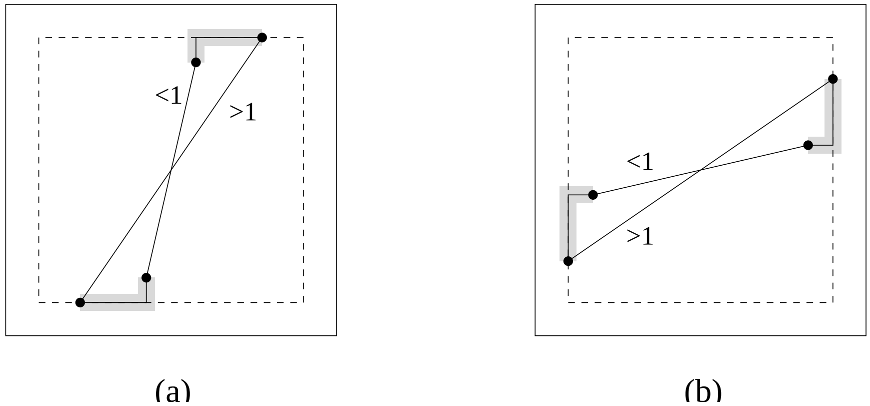


Figure 30:

Call the final value $\underline{\kappa}$. The distance between the left endpoint of the top segment and the right endpoint of the bottom segment is

$$\sqrt{\alpha^2+\kappa^2} \quad > \quad \sqrt{\underline{\alpha}^2+\underline{\kappa}^2} \quad \approx \quad 1.02414 \quad > \quad 1 \; .$$

The two left endpoints are distance $\alpha < 1$ apart. By continuity, there must be two points, one on each segment, exactly one unit apart, so the two segments must have different colors. Similary, consider two opposite L-segments on the short sides, an interior L-segment on, say, the top of the left side one on the top of the right side as in Figure 29b. Let λ be the length of the segment on the boundary. Then

$$\lambda \quad = \quad \left(\frac{\alpha}{2}-\frac{\sqrt{3}}{2}\sqrt{1-\alpha^2}\right)-\sqrt{1-\beta^2} \quad > \quad \frac{\underline{\alpha}}{2}-\frac{\sqrt{3}}{2}\sqrt{1-\underline{\alpha}^2}-\sqrt{1-\underline{\beta}^2} \; .$$

Call the final value $\underline{\lambda}$. The distance between opposite endpoints of the two line segments is

$$\sqrt{\alpha^2+\lambda^2} \quad > \quad \sqrt{\underline{\alpha}^2+\underline{\lambda}^2} \quad \approx \quad 1.02520 \quad > \quad 1 \; .$$

The two top endpoints are distance $\beta < 1$ apart. Again, by continuity, there must be two points, one on each segment, exactly one unit apart, so the two segments must have different colors.

Now consider two "caddy-corner" L-segments on the long sides, an interior L-segment on, say, the left of the bottom side and one on the right of the top side as in Figure 30a. Consider the two closest points (which are the interior endpoints of the vertical segments). We obtain upper bounds on the horizontal and vertical distances, which we call $\overline{h}$ and $\overline{v}$, between the two points. The horizontal distance between them is

$$h \quad = \quad \sqrt{3}\sqrt{1-\beta^2} \quad \leq \quad \sqrt{3}\sqrt{1-\underline{\beta}^2} \quad = \quad \overline{h} \; ,$$

and the vertical distance is

$$v \quad = \quad 2\left(\frac{\sqrt{3}}{2}\beta+\frac{\sqrt{1-\beta^2}}{2}\right)-\alpha \quad = \quad \sqrt{3}\beta+\sqrt{1-\beta^2}-\alpha \quad < \quad \sqrt{3}\cdot 1+\sqrt{1-\underline{\beta}^2}-\underline{\alpha} \quad = \quad \overline{v} \; .$$

The distance between the two points is

$$\sqrt{h^2+v^2} \quad < \quad \sqrt{\overline{h}^2+\overline{v}^2} \quad \approx \quad .89107 \quad < \quad 1 \; .$$

On the other hand, the left endpoint of the bottom segment and the right endpoint of the top segment are obviously more than one unit apart. So, by continuity, there must be two points, one on each L-segment, exactly one unit apart.

Consider two caddy-corner L-segments on the short sides, an interior L-segment on, say, the bottom of the left side and one on the top of right side as in Figure 30b. Consider the two closest points (which are the endpoints of the horizontal segments). We obtain upper bounds on the vertical and horizontal distances, which we call $\overline{V}$ and $\overline{H}$, between the two points. The vertical distance between them is

$$V \quad = \quad \sqrt{3}\sqrt{1-\alpha^2} \quad < \quad \sqrt{3}\sqrt{1-\underline{\alpha}^2} \quad = \quad \overline{V} \ ,$$

and the horizontal distance is

$$H \quad = \quad 2(\frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{1-\alpha^2}}{2}) - \beta \quad = \quad \sqrt{3}\alpha + \sqrt{1-\alpha^2} - \beta \quad < \quad \sqrt{3}\cdot\overline{\alpha} + \sqrt{1-\underline{\alpha}^2} - \underline{\beta} \quad = \quad \overline{H} \ .$$

The distance between the two points is

$$\sqrt{V^2+H^2} \quad < \quad \sqrt{\overline{V}^2+\overline{H}^2} \quad \approx \quad .88761 \quad < \quad 1 \ .$$

The points on opposite corners are obviously more than one unit apart. And, again, by continuity, there must be two points, one on each L-segment, exactly one unit apart.

We now show that, if a corner L-segment and the interior L-segment next to it, call it T, on a side have the same color, then the corner L-segment and the L-segment next to it on the opposite side have the same color (as each other) as do all of the points in between the two L-segments. The argument is the same as above with $b > 1$ (see Figure 26). We need to check that if the two segments are on a long side (length β) then T starts $\sqrt{1-\alpha^2}$ from its nearby corner (which is unit distance from the opposite corner) and has length at least $\sqrt{1-\alpha^2}$, and if the two segments are on a short side (length α) then T starts $\sqrt{1-\beta^2}$ from its nearby corner (which is unit distance from the opposite corner) and has length at least $\sqrt{1-\beta^2}$. In both cases, T starts exactly the desired distance from its nearby corner. If the two segments are on a long side, T has length $\frac{\beta}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\beta^2} - \sqrt{1-\alpha^2}$. So we need,

$$\frac{\beta}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\beta^2} - \sqrt{1-\alpha^2} \quad \geq \quad \sqrt{1-\alpha^2} \ .$$

Substituting $\underline{\alpha}$ for α and $\underline{\beta}$ for β, decreases the left side and increases the right side. It yields, $.25722 \geq .13150$, which suffices. If the two segments are on a short side, T has length $\frac{\alpha}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\alpha^2} - \sqrt{1-\beta^2}$. So we need,

$$\frac{\alpha}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\alpha^2} - \sqrt{1-\beta^2} \quad \geq \quad \sqrt{1-\beta^2} \ .$$

Again, substituting $\underline{\alpha}$ for α and $\underline{\beta}$ for β, decreases the left side and increases the right side. It yields, $.25774 \geq .12404$, which suffices.

Putting everything together reduces the possible colorings to a manageable number. Since there are four corners, two of them must be the same color. Either they are (a) diagonally opposite, or (b) on the a same side.

CASE (2a): Two diagonally opposite corners are the same color: say the the bottom left and the top right are red. Then from the bottom left corner counting clockwise, its two associated L-segments are, say, blue and green (Figure 31a). Since the top side has a blue L-segment, from the red L-segment on the top right corner counting clockwise its two associated

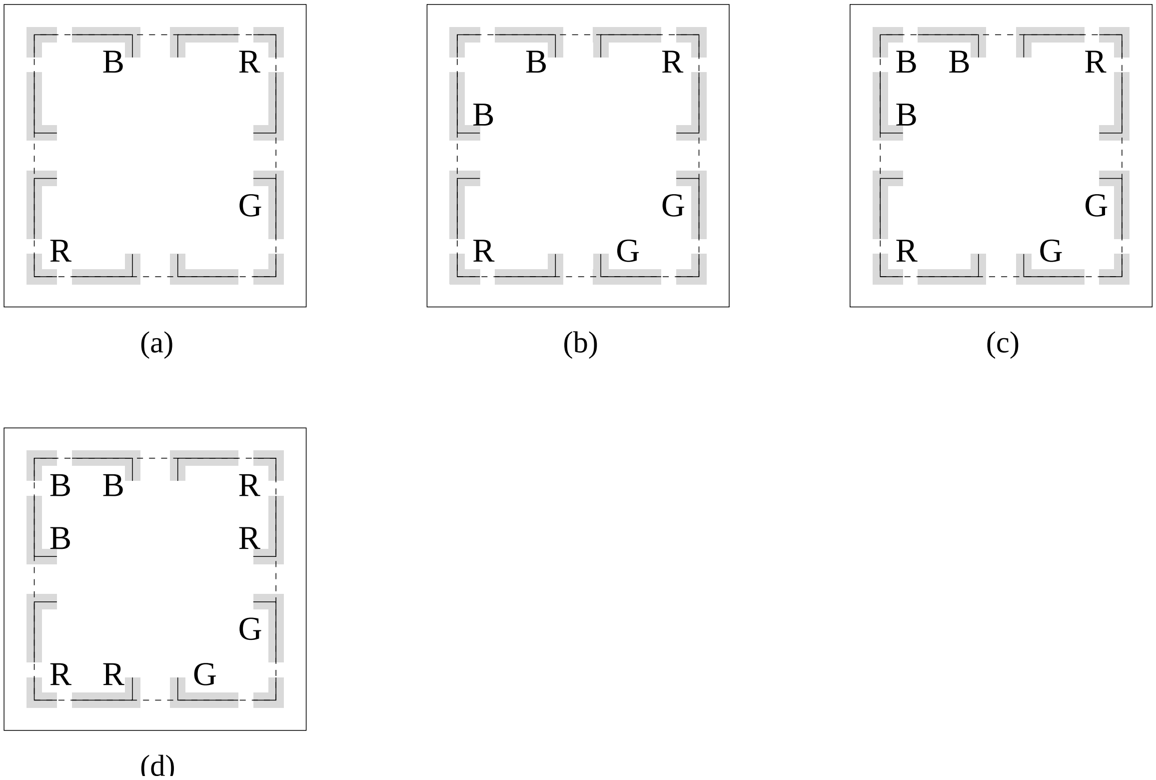


Figure 31:

L-segments must be green and blue (Figure 31b). The top left corner must be blue, since if it were red or green the right side or bottom side would have to have a blue L-segment (Figure 31c). Since the top left corner and the L-segment next to it on the top are both blue, the bottom left corner and the L-segment next to it on the bottom must be the same color. Similarly, since the top left corner and the L-segment next to it on the left side are both blue, the top right corner and the L-segment next to it on the right side must be the same color. These two corners are both red, so the two nearby L-segments must also be red. (Figure 31d). But these two L-segments are in the same group, so they cannot be the same color. Contradiction.

CASE (2b): Two corners on a same side are the same color: say both corners on the left side are red. Then from the bottom left red corner counting clockwise, its two associated L-segments are, say, blue and green (Figure 32a). Since the top side has a blue L-segment, from the top left red corner counting clockwise its two associated L-segments must be blue and green (Figure 32b). Since the right side has both a blue and a green L-segment, the two L-segments on the left side must be red (Figure 32c). So the remaining L-segment on the top side must be blue and the bottom right corner must be green (Figure 32d), and the remaining L-segment on the bottom side must be green and the top right corner must be blue (Figure 32e).

Let m be the point in the middle of the side opposite the four red segments. We will show that m is distance exactly 1 from a blue point, a green point, and a red point, which is impossible. There are actually two possible orientations, depending on whether the four red L-segments are on a short side (with length α) as shown in Figure 33a (and Figure 32e), or on a long side (with length β) as shown in Figure 33b.

Assume the former (Figure 33a). Consider the closest point to m of the upper interior L-segment on the left side (which is the endpoint of the horizontal segment). The vertical

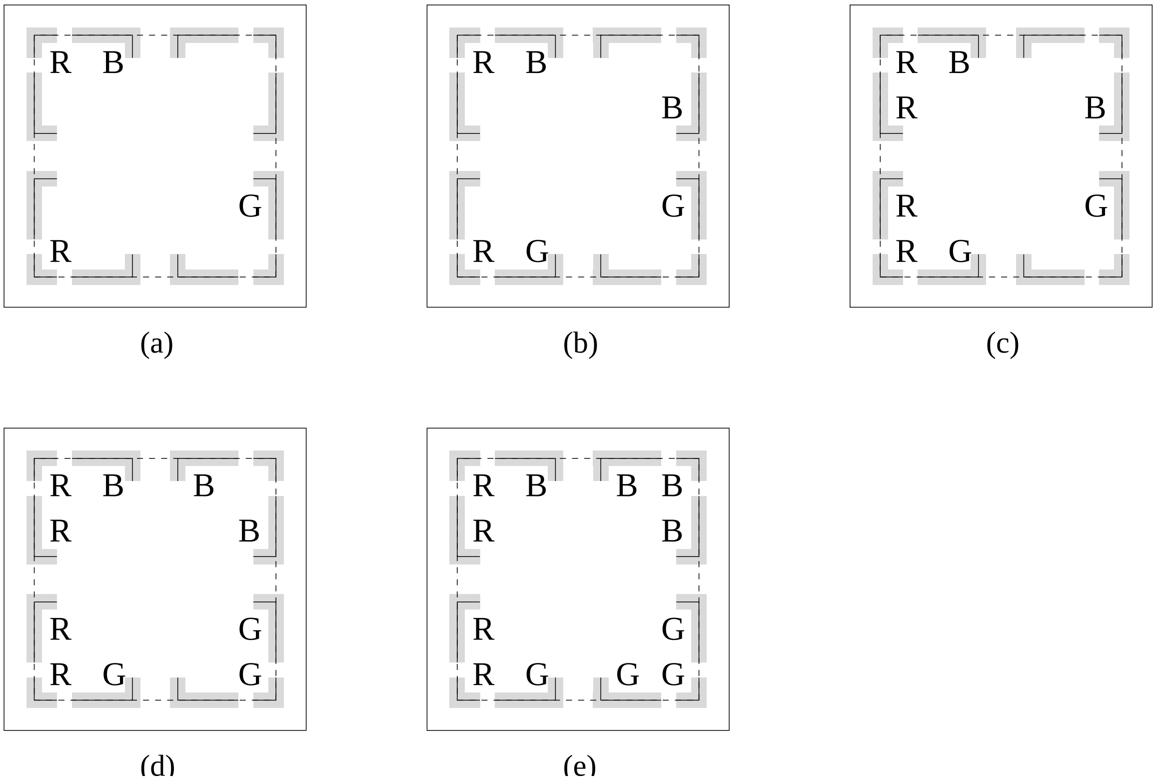


Figure 32:

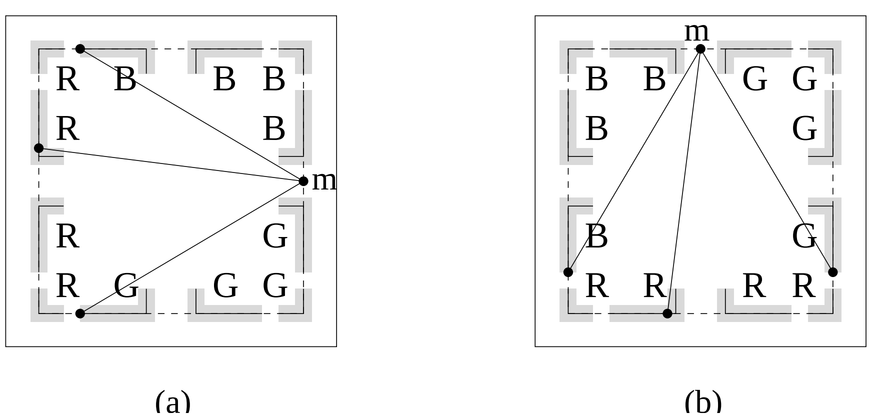


Figure 33:

distance between the two points is

$$v \;=\; \frac{\alpha}{2} \;-\; \left(\frac{\alpha}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\alpha^2}\right) \;=\; \frac{\sqrt{3}}{2}\sqrt{1-\alpha^2}\ ,$$

and the horizontal distance is

$$h \;=\; \frac{\sqrt{3}}{2}\alpha \;+\; \frac{\sqrt{1-\alpha^2}}{2}\ .$$

The square of the distance between the two points is

$$\begin{aligned} v^2 + h^2 \;&=\; \left(\frac{\sqrt{3}}{2}\sqrt{1-\alpha^2}\right)^2 \;+\; \left(\frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{1-\alpha^2}}{2}\right)^2 \;=\; 1 \;-\; \frac{\alpha^2}{4} + \sqrt{3}\alpha\sqrt{1-\alpha^2} \\ &\leq\; 1 \;-\; \frac{\underline{\alpha}^2}{4} + \sqrt{3}\overline{\alpha}\sqrt{1-\underline{\alpha}^2} \;\approx\; .9806 \;<\; 1\ . \end{aligned}$$

So, the distance between the two points is less than 1. Clearly, the distance between the farthest point of the upper interior L-segment on the left side to m is greater than 1. So, by continuity, there must be some point on the L-segment exactly distance 1 from m. So, m cannot be red.

Consider the left interior L-segment on the top side. Its leftmost point is the farthest point from m. Its (horizontal) distance to the top right corner is $\beta - \sqrt{1-\alpha^2}$. The (vertical) distance of m to top right corner is $\alpha/2$. So, in order for the distance from m to the left side of the L-segment to be at least 1, we need

$$(\beta - \sqrt{1-\alpha^2})^2 \;+\; (\alpha/2)^2 \;\;\geq\;\; 1\ ,$$

which is equivalent to

$$\beta \;\geq\; \sqrt{1-\alpha^2} \;+\; \sqrt{1-\alpha^2/4}\ .$$

This is satisfied by the initial condition on β (at the beginning of Case (2)). Clearly, the distance between the closest point of the left interior L-segment on the top side to m is less than 1. So, by continuity, there must be some point on the L-segment exactly distance 1 from m. Similarly, there must be some point on the left interior L-segment on the bottom side exactly distance 1 from m. So, m cannot be blue or green. Contradiction.

Now assume that the four red L-segments are on a long side, say the bottom side (Figure 33b). Consider the closest point to m of the left interior L-segment on the bottom side (which is the endpoint of the horizontal segment). The vertical distance between the two points is

$$V \;=\; \frac{\beta}{2} \;-\; \left(\frac{\beta}{2} - \frac{\sqrt{3}}{2}\sqrt{1-\beta^2}\right) \;=\; \frac{\sqrt{3}}{2}\sqrt{1-\beta^2}\ ,$$

and the horizontal distance is

$$H \;=\; \frac{\sqrt{3}}{2}\beta \;+\; \frac{\sqrt{1-\beta^2}}{2}\ .$$

The square of the distance between the two points is

$$\begin{aligned} V^2 + H^2 \;&=\; \left(\frac{\sqrt{3}}{2}\sqrt{1-\beta^2}\right)^2 \;+\; \left(\frac{\sqrt{3}}{2}\beta + \frac{\sqrt{1-\beta^2}}{2}\right)^2 \;=\; 1 \;-\; \frac{\beta^2}{4} + \sqrt{3}\beta\sqrt{1-\beta^2} \\ &\leq\; 1 \;-\; \frac{\underline{\beta}^2}{4} + \sqrt{3}\overline{\beta}\sqrt{1-\underline{\beta}^2} \;\approx\; .9684 \;<\; 1\ . \end{aligned}$$

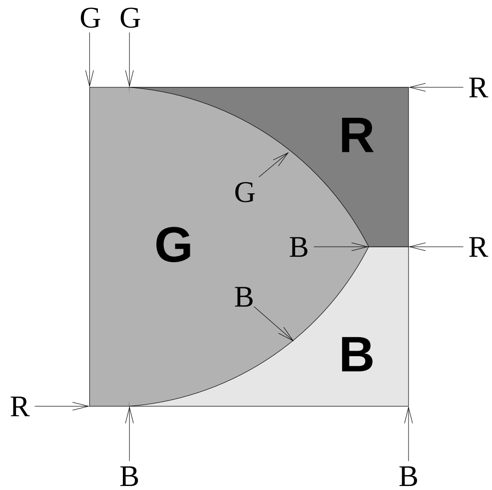


Figure 34: 3-coloring of square-region for side length $s = \frac{8}{\sqrt{65}}$

So, the distance between the two points is less than 1. Clearly, the distance between the farthest point of the left interior L-segment on the bottom side to m is greater than 1. So, by continuity, there must be some point on the L-segment exactly distance 1 from m. So, m cannot be red.

Consider the lower interior L-segment on the left side. Its lowest point is the farthest point from m. Its (vertical) distance to the top left corner is $\alpha - \sqrt{1-\beta^2}$. The (horizontal) distance of m to top left corner is $\beta/2$. So, in order for the distance from m to the left side of the L-segment to be at least 1, we need

$$(\alpha - \sqrt{1-\beta^2})^2 \ + \ (\beta/2)^2 \quad \geq \quad 1 \ ,$$

which is equivalent to

$$\alpha \quad \geq \quad \sqrt{1-\beta^2} \ + \ \sqrt{1-\beta^2/4} \ .$$

We show that this condition is satisfied for $\alpha \leq \frac{8}{\sqrt{65}}$. Let $f(x) = \sqrt{1-x^2/4} + \sqrt{1-x^2}$. Then $\beta = f(\alpha)$ by the definition of β at the beginning of Case(2), and we want to show that $\alpha \geq f(\beta) = f(f(\alpha))$. In the range $[f^{-1}(1), 1]$, $f(x)$ is monotonically decreasing with derivative $f'(x) < -1$, and has fixed point $x_0 = \frac{8}{\sqrt{65}}$. By the chain rule $f \circ f(x)$ has derivative > 1. Since $\alpha \leq x_0$, this suffices. Clearly, the distance between the closest point of the lower interior L-segment on the left side to m is less than 1. So, by continuity, there must be some point on the L-segment exactly distance 1 from m. Similarly, there must be some point on the lower interior L-segment on the right side exactly distance 1 from m. So, m cannot be blue or green. Contradiction.

This completes the proof. ∎

Corollary 4.11 *A square-region is 3-colorable if and only if the length of a side* $s \leq \frac{8}{\sqrt{65}} \approx 0.9923$.

Figure 34 shows the coloring of a square with side length $s = \frac{8}{\sqrt{65}}$.

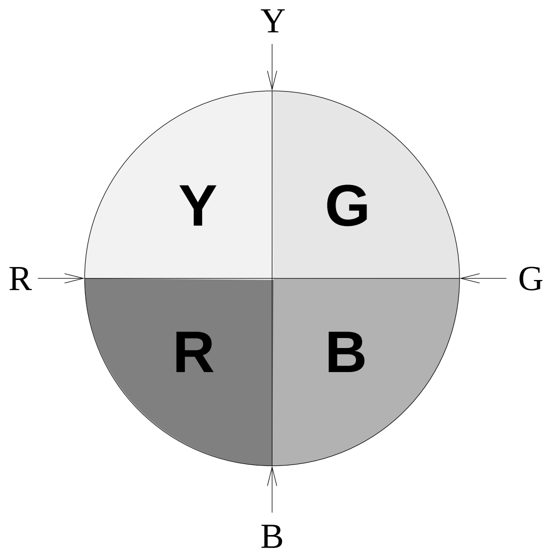


Figure 35: 4-coloring of circle-region for $r = \frac{1}{\sqrt{2}}$

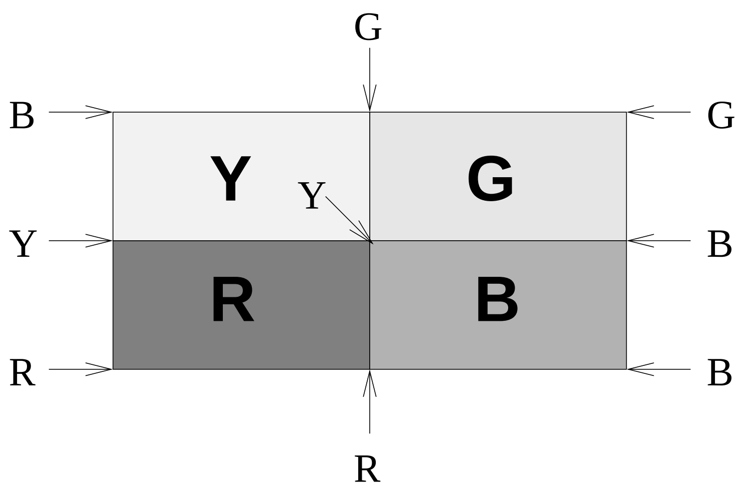


Figure 36: 4-coloring of rectangle-region for $a^2 + b^2 = 4$

5 4-COLORINGS

We can not know the true bounds for 4-colorings without determining if the plane is 4-colorable. Here are some obvious 4-colorings; much more clever 4-colorings are certainly plausible.

The circle-region of radius ≤ 1 and $a \times b$ rectangle-region for $a^2 + b^2 \leq 4$, can both be 4-colored by partitioning into four quadrants. The only difficulty is when the radius of a circle is exactly 1, or when the rectangle has $a^2 + b^2 = 4$, where one must be careful about the "corner" points (Figures 35, 36). A regular polygon-region where $4|n$ can be similarly 4-colored. An equilateral triangle-region with side ≤ 2 can be 4-colored by partitioning into four smaller equilateral triangles; when the side = 2, one must again be careful about the "corner" points (Figure 37). The infinite strip of width $\leq 2\sqrt{2}/3$ can be 4-colored with blocks of width 1/3 by rotating colors red, blue, green, yellow, red, blue, green, yellow, etc.

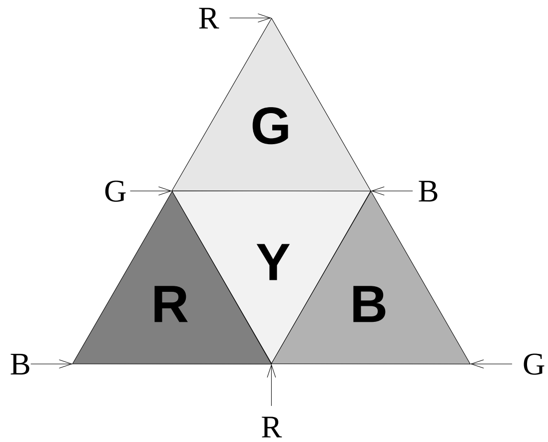


Figure 37: 4-coloring of triangle-region for side = 2

6 OPEN PROBLEMS

- Other than the few colorings with isolated points, all of the colorings are "nice": they consist of few monochromatic regions that are easy to construct. What is a natural, formal definition of a "nice" coloring? Can you prove that some colorings require isolated points (including the maximum sized squares for 2-colorings and 3-colorings); in other words prove that they do not have nice colorings.

- How big a region can you 4-color, 5-color, or 6-color? Obviously non-colorability results would imply (partial) solutions to the chromatic number of the plane problem, so this would be unlikely. But, what if you restrict the type of coloring? Can you find a size so that a region of that size is 4-colorable if and only if the plane is 4-colorable?

- Can you generalize the results to R^n ($n \geq 3$)? In particular, do the rod certainly has the same properties in 3-space as it has in 2-space. What about the tri-rod? What about a tetrahedron?

- Can you generalize the results to regions that are not simply connected, for example the annulus? The basic non-colorability lemmas (Lemmas 3.3 and 4.4) used in this paper still apply.

- By a compactness argument, if a region cannot be k-colored there must be a finite subgraph of the region (an obstructionist subgraph) that cannot be k-colored. Can you find (nice) finite, obstructionist subgraphs for the regions discussed in this paper? For 2-coloring regular n-gons for n odd, our non-colorability arguments did produce finite, obstructionist subgraphs, which, in fact, had only n vertices. For regions where the size that cannot be k-colored has a strict inequality (for example, a circle-region cannot be 2-colored if it has radius strictly greater than 1/2), there will have to be a sequence of subgraphs, which likely will become more complicated as the size of the region decreases. Most of our results for 2 and 3 colorings are like that (the only exceptions being 2-coloring a regular n-gon for n odd, as just discussed, and 2-coloring a nonsquare rectangle).

  Bohannon, et al. ([Boha] do create finite obstructionist subgraphs, and the bounds match for their 2-colorings. Can you produce finite obstructionist subgraphs with matching bounds for our 3-colorings? It is not clear how to use our techniques, because the proof of the tri-rod lemma (Lemma 3.3) is nonconstructive.

7 APPENDIX

We summarize the coloring results of Bohannon, et al. ([Boha]). They only consider circles and rectangles. For 2-colorings, they obtain tight bounds.

Their 3-coloring of the circle-region is optimal. For rectangles, they only use the vertical stripes method, so their 3-coloring is optimal only for $a \leq \frac{2}{\sqrt{5}}$. Their non-colorability results for finite regions are never tight. Here are their 3-coloring results:

Boundary of region	*3-colorable if*	*not 3-colorable if*
circle	radius $\leq \frac{1}{\sqrt{3}}$	radius $> \frac{\sqrt{3}}{2}$
$a \times b$ rectangle, $a \leq b$		
$a \leq \frac{\sqrt{3}}{2}$	always	
$a > \frac{\sqrt{3}}{2}$	$b \leq 3\sqrt{1-a^2}$	$b \geq \frac{5}{2}$
square	side $\leq \frac{3}{\sqrt{10}}$	side $\geq \frac{5}{2}$

Their 4-coloring of the rectangle-region use four vertical stripes for most of the range, and a different method for the more squarish rectangles. Neither method is as good as cutting region into four quadrants. Here are their 4-coloring results:

Boundary of region	*4-colorable if*
circle	radius $\leq \frac{1}{\sqrt{2}}$
$a \times b$ rectangle, $a \leq b$	
$a \leq \frac{2\sqrt{2}}{3}$	always
$\frac{2\sqrt{2}}{3} < a \leq \frac{\sqrt{15}}{4}$	$b \leq 4\sqrt{1-a^2}$
$\frac{\sqrt{15}}{4} \leq a \leq 1$	$b \leq 1$
square	side ≤ 1

8 ACKNOWLEDGEMENTS

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