The Orchard Visibility Problem and some Variants

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January 14, 2008

Abstract

Imagine that you are standing at the center of a circular orchard, with trees centered at all of the lattice points except there is no tree at the origin itself (where you are standing). How large must the radius of the trees be in order to completely block your view (in every direction).

Let R be the radius of the orchard, and r be the radius of the trees. It turns out that if R (the radius of the orchard) is an integer then you can see out if and only if $r < 1/\sqrt{R^2+1}$. Allen [Alle] attributes this problem to Polya and solves it using trigometric techniques. He generalizes the result to orchards whose radius is not an integer. We give an alternative proof of these results based on the Stern-Brocot wreath.

We generalize the results to parallelogram lattices. We also consider the problem of what radius the trees need to have to block the view between some pair of trees. For parallelogram lattices, the ratio between the radius needed to block the view between all trees and the radius needed to block the view between some pair of trees asymptotically approaches 2 (as the radius of the orchard increases).

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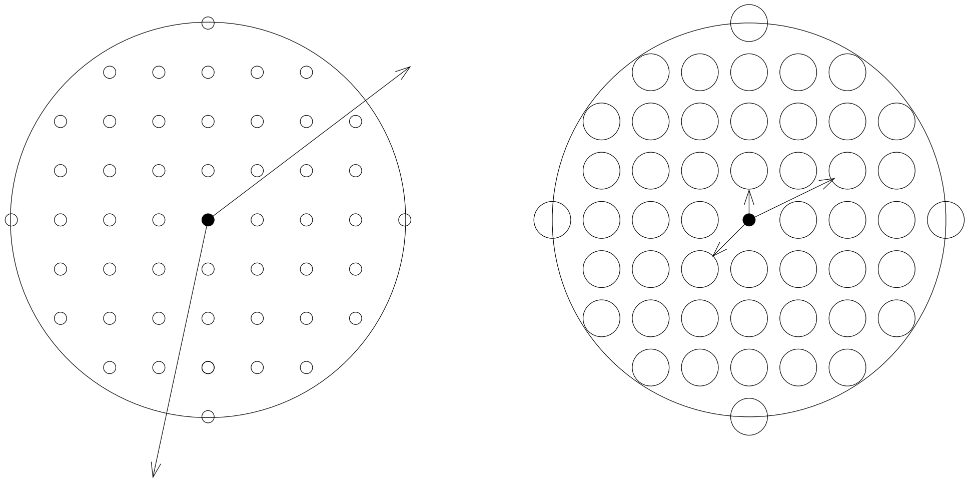

Figure 1: Two orchards of radius 4.

1 Introduction

Formally, a *tree of radius* r is a circle of radius r. A *circular orchard of radius* R consists of trees centered at all of the lattice points within the circle of radius R around the origin, including the boundary points, except there is no tree at the origin. A *potential line-of-sight* is a ray with its endpoint at the origin. A *line-of-sight* is a potential line-of-sight that does not intersect any tree.

We are interested in the problem of how large the tree radius (r) has to be in order to guarantee that there is no line-of-sight in an orchard of radius R. Figure 1 shows two orchards of radius 4. In the first, the trees are narrow enough so that one can see out (in many directions). A couple of lines-of-sight are shown. In the second, the trees are too wide to see out. Several potential lines-of-sight are shown where they are blocked by trees.

2 Previous work

For orchard radius R an integer, Polya bounds the tree radius using a method of Speiser. Yaglom and Yaglom (Yagl) use Minkowski's Convex Body Theorem to show for an orchard of radius 50, if the tree radius $r < 1/\sqrt{50^2+1}$ there is a line-of-sight, and if the tree radius $r > 1/50$ there is no line-of-sight. Honsberger (Hons) gives a more detailed version of the proof in his book *Mathematical Gems*. The proof can be generalized to show for R an integer, if the tree radius $r < 1/\sqrt{R^2+1}$ there is a line-of-sight, and if the tree radius $r > 1/R$ there is no line-of-sight. The Honsberger reference is cited by Weisstein (Weis) with these bounds. Allen [Alle] solves the problem using trigometric techniques, and generalizes the result to orchards whose radius is not an integer.

For completeness we give the proof from Yaglom and Yaglom (Yagl) and Honsberger (Hons) generalized to integers. Let C be the origin $(0,0)$. We use $|L|$ to represent the length of line L. We use *tree* (a,b) to refer to the tree centered at point (a,b).

Theorem 2.1 *An orchard with integer radius R and tree radius $r < 1/\sqrt{R^2+1}$ has a*

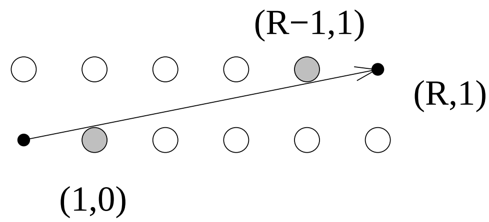

Figure 2: Two consecutive trees in an orchard.

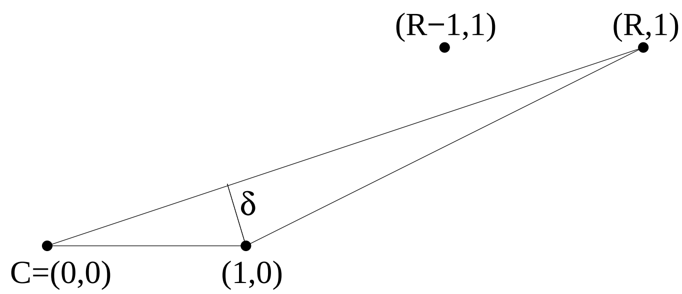

Figure 3: Distance δ to potential line-of-sight.

line-of-sight.

Proof: (Yaglom and Yaglom) Consider the two trees $(1,0)$ and $(R-1,1)$ (see Figure 2). They are equidistant from ray $(R,1)$, and it is clear that no other tree is closer. We can calculate the distance, δ, from $(1,0)$ to the ray by calculating the area of the triangle $(C,(1,0),(R,1))$ in two different ways (see Figure 3). Letting $(C,(R,1))$ be its base makes its height δ, so its area

$$A = \frac{1}{2}|(C,(R,1))|\delta = \frac{\delta}{2}\sqrt{R^2+1}\ .$$

Letting $(C,(1,0))$ be its base makes its height 1, so its area

$$A = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}\ .$$

Therefore

$$\delta = \frac{1}{\sqrt{R^2+1}}\ .$$

So there will be a line-of-sight if the tree radius is less than $1/\sqrt{R^2+1}$. ∎

The complementary bound on the tree radius uses Minkowski's convex body theorem:

Theorem 2.2 *(Minkowski 1889) Any convex region that is symmetric about the origin and has area greater than 4, covers a lattice point other than the origin.*

(Honsberger [Hons] and Yaglom and Yaglom [Yagl] give a proof of Minkowski's convex body theorem using a result of Blichfeldt (1914).)

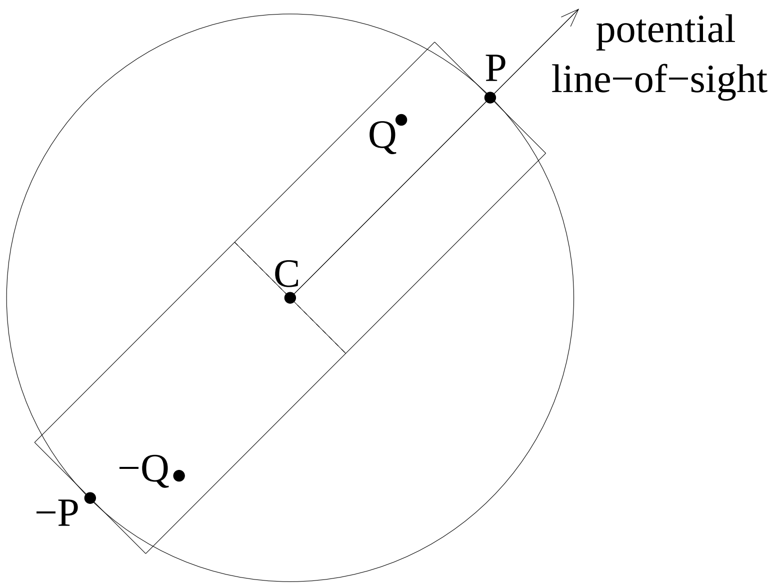

Figure 4: Potential line-of-sight.

Lemma 2.3 *An orchard with integer radius R and tree radius $r > 1/R$ has no line-of-sight.*

Proof: (Yaglom and Yaglom) If $R = 1$ the lemma clearly holds. Assume that $R \geq 2$. In this case, if $r \geq \frac{1}{2}$ the eight trees closest to the origin would be touching each other, guaranteeing that there would be no line-of-sight. So, we can assume $r < \frac{1}{2}$.

Assume that the tree radius $r = 1/R + \epsilon < \frac{1}{2}$ for some $\epsilon > 0$. Consider a potential line-of-sight. Let P be the point where it intersects the edge of the orchard (i.e., the circle of radius R). Consider the rectangle that is symmetric about the origin with one side tangent to P with length $2/R + \epsilon$. This implies that its opposite side is tangent to $-P$ (with the same length), and the two other sides have lengths $2R$. See Figure 4. The rectangle has area $2R(2/R + \epsilon) = 4 + 2R\epsilon > 4$. By Minkowski's convex body theorem there is a lattice point Q inside the rectangle, on the same side as the potential line-of-sight. (If the lattice point Q' is on the opposite side of the rectangle as the potential line-of-sight, then by symmetry the lattice point $Q = -Q'$ is on the same side of the rectangle.) Any point on the same side of the rectangle as P is within distance $1/R + \epsilon/2$ of the potential line-of-sight. Since the tree radius is $1/R + \epsilon$, there is no line-of-sight.

There is still a technicality: What if the lattice point is inside the sliver of the rectangle that is just outside of the orchard? The distance, D, of any such point from the origin would have to satisfy

$$\begin{aligned} R \ < \ D \ < \ \sqrt{R^2 + (1/R + \epsilon/2)^2} \ < \ \sqrt{R^2 + (1/R + \epsilon)^2} \ &= \ \sqrt{R^2 + r^2} \\ < \ \sqrt{R^2 + (\frac{1}{2})^2} \ < \ \sqrt{R^2 + 1} \qquad & \text{since } r < \tfrac{1}{2} \end{aligned}$$

Thus

$$R^2 < D^2 < R^2 + 1$$

Since D^2 is an integer for any lattice point, there can be no lattice point satisfying the condition. ∎

This argument can be strengthened considerably by slightly narrowing the rectangles and extending them further outside of the orchard. It will not, however, yield the tight bound we desire.

3 The Stern-Brocot Wreath

A lattice point (a, b) is *irreducible* if the $\gcd(a, b) = 1$. The Stern-Brocot tree generates every irreducible lattice point in the upper-right quadrant exactly once in the following way[1]: Start with the two points

$$(0,1),\ (1,0)\ .$$

Produce their sum $(1, 1)$ and list it in between them:

$$(0,1),\ (1,1),\ (1,0)\ .$$

Now for each consecutive pair of points, list their sum in between them:

$$(0,1),\ (1,2),\ (1,1),\ (2,1),\ (1,0)\ .$$

Do it again:

$$(0,1),\ (1,3),\ (1,2),\ (2,3),\ (1,1),\ (3,2),\ (2,1),\ (3,1),\ (1,0)\ .$$

Etc.

The points are more easily produced as vertices in an infinite binary tree, so that points produced at the same time are on the same level of the tree. See Figure 5.

Starting with the four points $(0, 1)$, $(1, 0)$, $(0, -1)$, and $(-1, 0)$ and listing sums as above with wrap-around produces all of the irreducible points in the plane. This is known as the *Stern-Brocot wreath*, which has many interesting properties.

Fact 3.1 *If (a, b) and (c, d) are consecutive points at some stage of the construction of the Stern-Brocot wreath then*

$$bc - ad \ =\ 1$$

This follows by mathematical induction (and is proved in [Grah]). We will use this property later.

Fact 3.2 *If P and Q are consecutive points at some stage of the construction of the Stern-Brocot wreath, then the area of the triangle (C, P, Q) is $1/2$ (where C is the origin).*

To see this, note that the area of the triangle $(C, (0, 1), (1, 0))$ (or any two other neighboring points from the first level of the construction of the wreath) is $1/2$. From there we can proceed inductively: See Figure 6. Assume it holds for neighboring points P and Q. Then, the four points $C, P, P + Q, Q$ form a parallelogram with area 1. The line $(C, P + Q)$ cuts the parallelogram in half. So each of the two triangles $(C, P, P + Q)$ and $(C, Q, P + Q)$ has area $1/2$. We will use this (well known) property later.

[1] The theorem was independently discovered by Stern (1858) and Brocot (1861). It is closely related to Farey series, which were discovered by John Farey (1816). Normally, the Stern-Brocot tree and Farey series are presented in terms of irreducible fractions, but for our purposes recasting as irreducible lattice points is more natural. See Graham, et al. [Grah] for references, a proof, and further discussion.

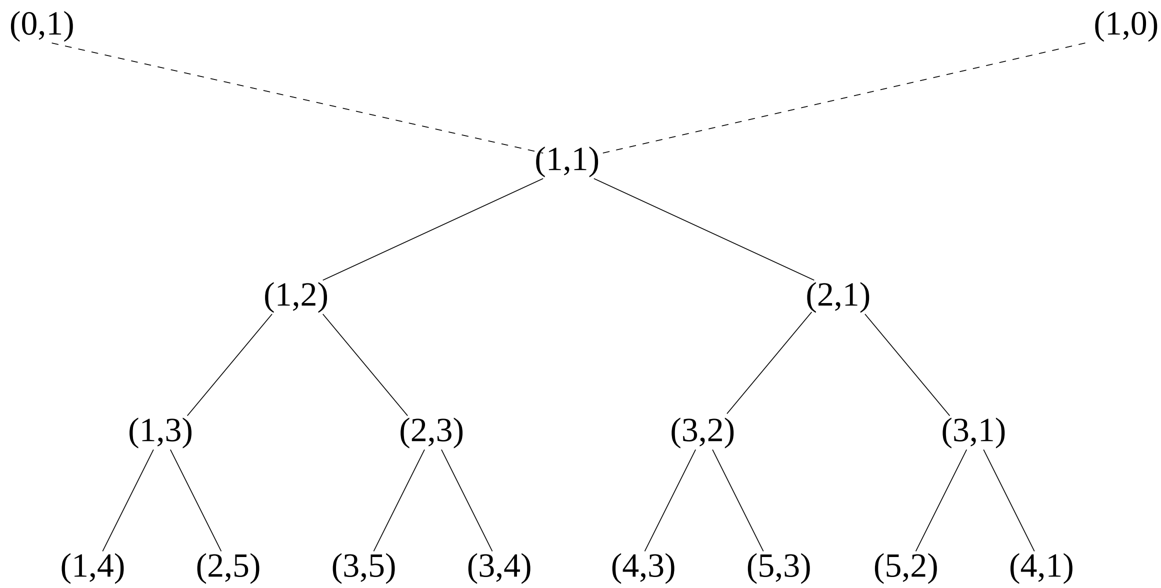

Figure 5: Stern-Brocot tree.

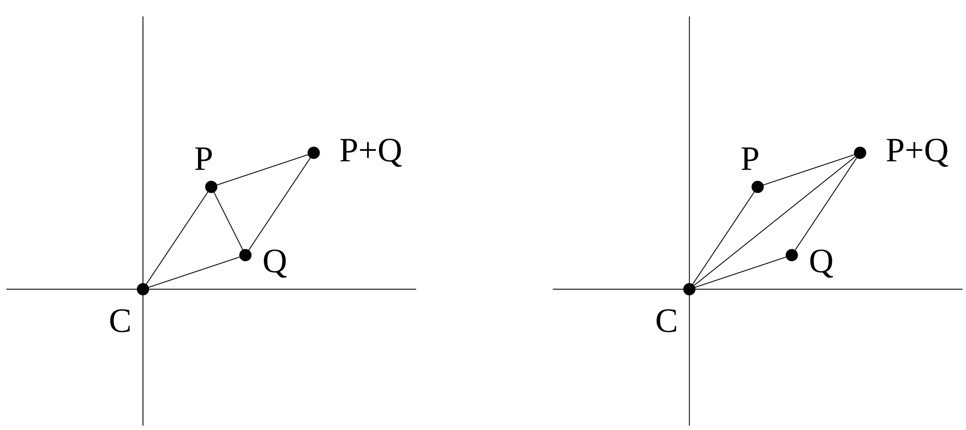

Figure 6: Area of triangles.

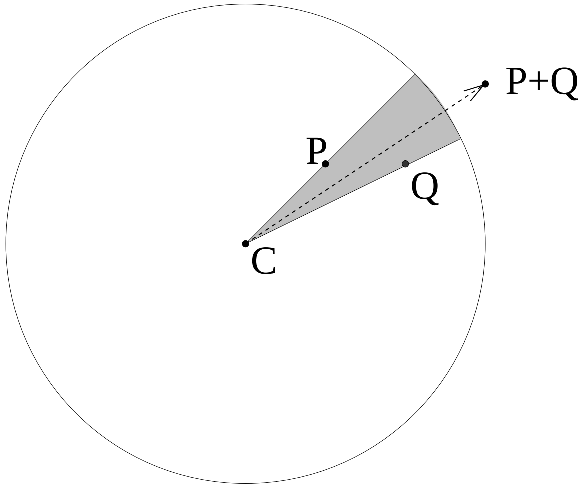

Figure 7: Arc (P, C, Q) with ray $(C, P+Q)$.

4 Exact and Approximate Bounds on Tree Radius

Only trees at irreducible points block potential lines-of-sight: the tree at irreducible point (a, b) blocks the view of all trees at points (ka, kb) for integer $k \geq 2$. We will see that in an orchard with minimum tree radius (to block all potential lines-of-sight) the converse also holds: every tree at an irreducible point blocks some potential lines-of-sight.

Definition 4.1 Two points P and Q are *consecutive inside an orchard* if they are consecutive at some stage of the construction of the Stern-Brocot wreath, both are inside the orchard, but $P+Q$ is outside the orchard.

Lemma 4.2 *Let W be an irreducible point outside of an orchard that is closest to the orchard. Then the orchard has a line-of-sight if and only if the radius of the trees is less than* $1/|W|$.

Proof: Let P and Q be two consecutive (irreducible) points inside the orchard. We will show that there is no line-of-sight in the arc (P, C, Q) if and only if the tree radius $r \geq 1/|P+Q|$. Trees P and Q block the potential lines-of-sight for all points within the arc (P, C, Q) as long as they block the line-of-sight exactly in the middle between P and Q, which is the ray $(C, P+Q)$ (see Figure 7).

In order to calculate the distance, δ, from the point P to the ray $(C, P+Q)$, consider the triangle $(C, P+Q, P)$, where $(C, P+Q)$ is the base. It has area $(1/2)|P+Q|\delta$. Since P and $P+Q$ are consecutive points in the construction of the Stern-Brocot wreath, the triangle also has area $1/2$ (by Fact 3.2). Thus, $\delta = 1/|P+Q|$. Similarly, the distance from Q to $(C, P+Q)$ is also $1/|P+Q|$. So tree radius $1/|P+Q|$ is sufficient to block sight in the arc (P, C, Q).

We need to make sure that no tree of smaller radius could block the line-of-sight for $(C, P+Q)$. Any such tree would have to be strictly inside the intersection of the orchard with the infinite strip bounded between the two lines $(-Q, P)$ and $(-P, Q)$ (see Figure 8). We show that no such tree exists.

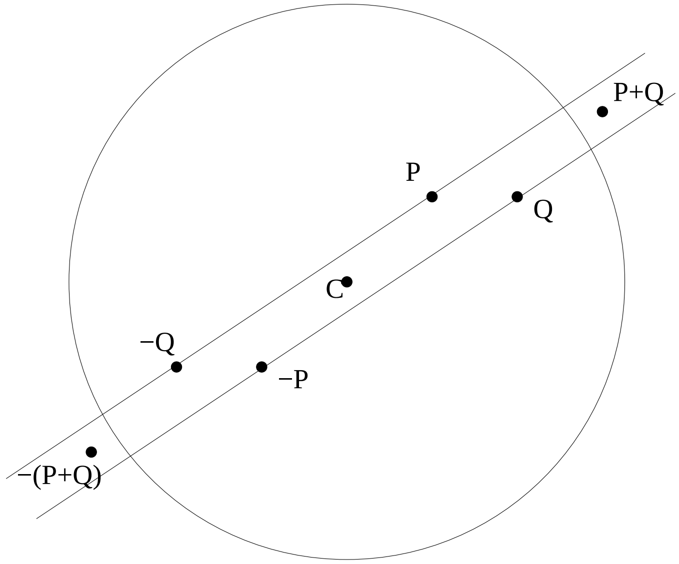

Figure 8: Lines bounding trees that can block line-of-sight.

Let $P = (a, b)$ and $P + Q = (c, d)$. Then the slope of the two lines is d/c. (The value of c cannot be 0, since $(0, 1)$ and $(0, -1)$ are initial points.) So the equation for the first line is

$$(y - b) = \frac{d}{c}(x - a) \quad \Rightarrow \quad y = \frac{d}{c}x + \frac{bc - ad}{c} \ ,$$

and the equation for the second line is

$$y = \frac{d}{c}x + \frac{ad - bc}{c} \ .$$

Thus any point (x, y) between the two lines must satisfy

$$\frac{d}{c}x + \frac{ad - bc}{c} \ < \ y \ < \ \frac{d}{c}x + \frac{bc - ad}{c} \ ,$$

which implies

$$ad - bc \ < \ cy - dx \ < \ bc - ad \ .$$

But, since P and $P + Q$ are consecutive in the Stern-Brocot wreath, $ad - bc = -1$ and $bc - ad = 1$ (by Fact 3.1); so, $cy - dx$ must equal 0. Thus (x, y) is on the line $y = (d/c)x$, but the only irreducible points on this line are $(c, d) = P + Q$ and $(-c, -d) = -(P + Q)$.[2]

Thus, for each pair of consecutive points P and Q inside the orchard, $P + Q$ is the closest point outside the orchard, but inside the arc (P, C, Q). So the minimum $|P + Q|$ over all consecutive points P and Q inside the orchard is the distance of the closest irreducible point to the orchard, and tree radius $1/|P + Q|$ is necessary and sufficient to block sight in the orchard. ∎

[2]Alternatively, this fact can be proven using Pick's Theorem.

Let $S_1 < S_2 < S_3 < \cdots$ be the distances of irreducible points from the center of the orchard. These are the square roots of the sums of squares of the irreducible pairs, i.e., numbers of the form $\sqrt{a^2+b^2}$ where the pair (a,b) is irreducible. The first few S_i are

$$\sqrt{1},\ \sqrt{2},\ \sqrt{5},\ \sqrt{10},\ \sqrt{13},\ \sqrt{17},\ \sqrt{25},\ \sqrt{26},\ \sqrt{29},\ \ldots$$

Corollary 4.3 *Let $S_{i-1} \leq R < S_i$. An orchard of radius R has a line-of-sight, if and only if, $r < 1/S_i$.*

Theorem 4.4 *Let R be a positive integer. An orchard of radius R has a line-of-sight, if and only if, $r < 1/\sqrt{R^2+1}$.*

Proof: For any integer R, the point $(R,1)$ is irreducible and there is no integer between R^2 and R^2+1. ∎

There is not always a unique closest tree (not counting the eight symmetries of the square), even when R is an integer. For example, for $R = 8$, there are two (non-isomorphic) closest trees at distance $\sqrt{65}$ from the origin: $(8,1)$ (between consecutive trees $(1,0)$ and $(7,1)$), and $(7,4)$ (between consecutive trees $(2,1)$ and $(5,3)$).

It is a "lucky accident" that the orchard visibility problem was originally studied for R an integer: it is the only case where this method solves the problem explicitly. We can, however, use Lemma 4.2 to get reasonably tight bounds on the tree radius for general orchard radius: Since $(I,1)$ is irreducible, for every positive integer I, the distance D of the closest tree to the orchard must satisfy

$$R \ < \ D \ < \ R+1 \ .$$

This implies:

Corollary 4.5 *The minimum tree radius r that blocks sight in an orchard of radius R satisfies*

$$\frac{1}{R+1} \ < \ r \ < \ \frac{1}{R} \ .$$

5 Closest trees

Instead of trying to block the total view, what if the goal is to try to block the view between some two consecutive trees inside the orchard? In other words, what are the two "closest" irreducible points in orchard? Since the distance between two consecutive, irreducible points is inversely proportional to their sum, the problem is to find an irreducible point that is furthest from the orchard, but is still the sum of two consecutive points inside the orchard.

Once again it is easy to bound the radius. The furthest such tree from the orchard is the sum of two trees within the orchard, so it has distance less than $2R$. Conversely, let m be the largest integer such that $(m,1)$ is inside the orchard. This tree is within one unit of the radius since $(m+1,1)$ is outside the orchard, and similarly $(m-1,1)$

is within two units of the radius. Since $(k, 1)$ is irreducible, for every positive integer k, there exist two consecutive trees whose sum has distance greater than $2R - 3$ from the origin. Thus, the distance $\bar{D}$ of the furthest tree from the orchard that is the sum of two consecutive trees inside an orchard of radius $R \geq 2$ must satisfy

$$2R - 3 \; < \; \bar{D} \; < \; 2R \, .$$

This implies:

Corollary 5.1 *The minimum tree radius $\bar{r}$ that blocks sight between some two trees in an orchard of radius $R \geq 2$ satisfies*

$$\frac{1}{2R} \; < \; \bar{r} \; < \; \frac{1}{2R-3} \, .$$

Thus the ratio ρ of the radius needed to block the view between some two trees, and the radius needed to block the view between all trees in an orchard of radius $R \geq 2$ satisfies

$$\frac{1}{2} \; < \; \rho \; < \; \frac{R+1}{2R-3} \, .$$

So,

$$\rho \; = \; \frac{1}{2} + O(\frac{1}{R}) \, .$$

This problem is solvable exactly for orchard radius slightly larger than an integer, namely $R = \sqrt{m^2+1}$, so that the two points $(m-1, 1)$ and $(m, 1)$ are a "closest" pair of consecutive points. Their sum is $(2m-1, 2)$, so it suffices to have tree radius

$$\frac{1}{\sqrt{4m^2 - 4m + 5}} \; = \; \frac{1}{\sqrt{4R^2 - 4\sqrt{R^2-1} + 1}} \, .$$

6 Parallelogram lattices

Parallelogram lattices are generated by taking any two points (U, V) in the plane (that are not colinear with the origin) and forming all linear combinations $((iU + jV)$ for integers $i, j)$. See Figure 9. Each parallelogram lattice is a linear transformation of the standard lattice.

A lattice point P is *irreducible* if there is no lattice point on the open line segment (C, P), where C is still the origin. We can form a *generalized Stern-Brocot wreath* starting with the four points (U, V), $(U, -V)$, $(-U, -V)$, $(-U, V)$ to produce all of the irreducible points. The two earlier facts about Stern-Brocot wreaths have corresponding generalizations.

Fact 6.1 *Let U and V generate a parallelogram lattice. If P and Q are consecutive points at some stage of the construction of the generalized Stern-Brocot wreath then*

$$\det(P, Q) \; = \; \det(U, V) \, .$$

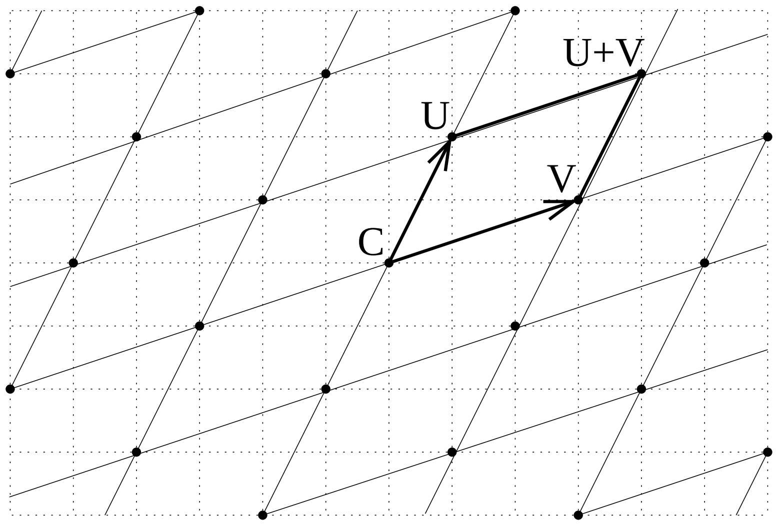

Figure 9: Lattice constructed from vectors U and V.

Fact 6.2 *Let U and V generate a parallelogram lattice. Let A be the area of the* underlying parallelogram $(C, U, U+V, V)$. *If P and Q are consecutive points at some stage of the construction of the generalized Stern-Brocot wreath, then the area of the triangle (C, P, Q) is $A/2$.*

We will need a third fact:

Fact 6.3 *Let U and V generate a parallelogram lattice. If P and Q are irreducible points then*

$$\det(P, Q) \;=\; m \det(U, V)\ ,$$

for some integer $m \neq 0$.

Lemma 4.2 can be generalized for parallelogram lattices with the same basic proof.

Lemma 6.4 *Consider the orchard corresponding to the parallelogram lattice generated by points (U, V). Let A be the area of its underlying parallelogram $(C, U, U+V, V)$. Let W be an irreducible point outside of the orchard that is closest to the orchard. Then the orchard has a line-of-sight if and only if the radius of the trees is less than $A/|W|$.*

Proof: Let P and Q be two consecutive (irreducible) points inside the orchard. We will show that there is no line-of-sight in the arc (P, C, Q) if and only if the tree radius $r \geq A/|P+Q|$. Trees P and Q block the potential lines-of-sight for all points within the arc (P, C, Q) as long as they block the line-of-sight exactly in the middle between P and Q, which is the ray $(C, P+Q)$.

Let δ be the distance from the point P to the ray $(C, P+Q)$. The triangle $(C, P+Q, P)$ has area $(1/2)|P+Q|\delta$. It also has area $A/2$ (by Fact 6.2). Thus $\delta = A/|P+Q|$. Similarly, the distance from Q to $(C, P+Q)$ is $A/|P+Q|$. So tree radius $A/|P+Q|$ is sufficient to block sight in the arc (P, C, Q).

Any tree of smaller radius could block the line-of-sight for $(C, P+Q)$ would have to be strictly between the two lines $(-Q, P)$ and $(-P, Q)$ (see Figure 8). Let $P = (a, b)$

and $P+Q=(c,d)$. Reasoning similarly to the proof of Lemma 4.2, any such point $S=(x,y)$ must satisfy

$$\begin{array}{rl} & ad-bc \ < \ cy-dx \ < \ bc-ad \\ \Rightarrow & \det(P,P+Q) \ < \ \det(P+Q,S) \ < \ \det(P+Q,P) \\ \Rightarrow & \det(U,V) \ < \ \det(P+Q,S) \ < \ \det(V,U) \qquad \text{by Fact 6.1} \\ \Rightarrow & \det(U,V) \ < \ m\det(U,V) \ < \ -\det(U,V) \qquad \text{by Fact 6.3} \\ \Rightarrow & -1 \ < \ -m \ < \ 1\ . \end{array}$$

This implies, $m=0$ (since m is an integer), so S cannot be irreducible (by Fact 6.3).

Thus, for each pair of consecutive points P and Q inside the orchard, $P+Q$ is the closest point outside the orchard, but inside the arc (P,C,Q). So the minimum $|P+Q|$ over all consecutive points P and Q inside the orchard is the distance of the closest irreducible point to the orchard, and tree radius $A/|P+Q|$ is necessary and sufficient to block sight in the orchard. ∎

Let $S_1 < S_2 < S_3 < \cdots$ be the distances of irreducible points from the center of a parallelogram orchard.

Corollary 6.5 *Let $S_{i-1} \le R < S_i$. A parallelogram orchard of radius R, whose underlying parallelogram has area A, has a line-of-sight, if and only if, $r < A/S_i$.*

This method does not provide an explicit formula for the minimum tree radius r to block lines-of-sight for general parallelogram lattices, even when R is an integer. We can, however, use Lemma 6.4 to get reasonably tight bounds on the tree radius for any orchard radius: Assume, that U is the smaller of the two vectors that generate the parallelogram lattice (i.e., $|U| \le |V|$). Since, for every positive integer m, (mU,V) is an irreducible lattice point, the distance D of the closest tree to the orchard must satisfy

$$R \ < \ D \ < \ R+|U|\ .$$

This implies:

Theorem 6.6 *The minimum tree radius r that blocks sight in an orchard of radius R satisfies*

$$\frac{A}{R+|U|} \ < \ r \ < \ \frac{A}{R}\ .$$

What about trying to block the view between some two consecutive trees inside the orchard? As before, the problem is to find an irreducible point that is furthest from the orchard, but is still the sum of two consecutive points inside the orchard.

The furthest irreducible point from the orchard is the sum of two (irreducible) points within the orchard, so it has distance less than $2R$ from the origin. Conversely, let m be the largest integer such that (mU,V) is inside the orchard. Then $((m-1)U,V)$ is also inside the orchard, and $((m+1)U,V)$ is outside the orchard. So trees (mU,V) and $((m-1)U,V)$ are within $|U|$ and $2|U|$, respectively, of the radius. Since (kU,V) is irreducible, for every positive integer k, some two consecutive trees have sum with distance greater than $2R-3|U|$. Thus, the distance $\bar{D}$ of the furthest tree from the orchard that is the sum of two consecutive trees inside an orchard of radius $R \geq 2|U|$ must satisfy

$$2R - 3|U| \; < \; \bar{D} \; < \; 2R \; .$$

This implies:

Theorem 6.7 *The minimum tree radius $\bar{r}$ that blocks sight between some two trees in an orchard of radius $R \geq 2|U|$ satisfies*

$$\frac{A}{2R} \; < \; \bar{r} \; < \; \frac{A}{2R - 3|U|} \; .$$

Thus the ratio ρ of the radius needed to block the view between some two trees, and the radius needed to block the view between all trees in an orchard of radius $R \geq 2|U|$ satisfies

$$\frac{1}{2} \; < \; \rho \; < \; \frac{R + |U|}{2R - 3|U|} \; .$$

Fixing the lattice and letting the size of the orchard grow gives:

Lemma 6.8 *For any fixed parallelogram lattice, the ratio between the tree radius needed to block line-of-sight between some two trees and to block the total view is*

$$\frac{1}{2} + O(\frac{1}{R}) \; .$$

7 Open Problems

There are several obvious open problems:

- Can you find an explicit solution for the tree radius needed to block all lines-of-sight for the equilateral triangle lattice with side length 1, when the orchard has integer radius? (This is the parallelogram lattice generated by $(1, 0)$ and $(1/2, \sqrt{3}/2)$.)
- Can you generalize the results to hexagons (which are lattices, but not parallelogram lattices). In particular, what can you say about hexagons with side length 1.
- Can you generalize the results to three dimensions (or higher)?

8 Acknowledgements

The author thanks Bill Gasarch for suggesting the problem, and Tom Draper for pointing out Pick's theorem, which, although it is not used here, led the author to a simplified proof.

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