Recursive Runtimes

Note Title 9/25/2007

Note: ℓ = level

① $T(n) = n + 2T(\frac{n}{2}) \in O(n \log n)$

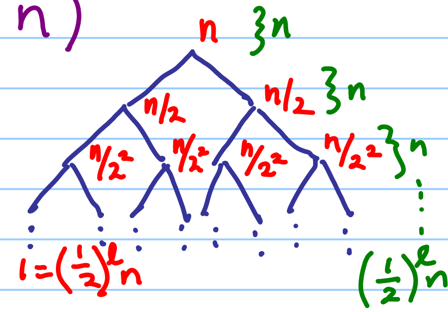
Merge Sort

a. recursive calls = 2

b. size of data at each level = $(\frac{1}{2})^{\ell} \cdot n$

c. # of levels: $(\frac{1}{2})^{\ell} n = 1$; $n = (\frac{2}{1})^{\ell}$; $\ell = \log_2 n$

d. work @ each level = n $\quad \sum_{i=0}^{\log_2 n} n$



② $T(n) = 1 + T(\frac{n}{2}) \in O(\log n)$

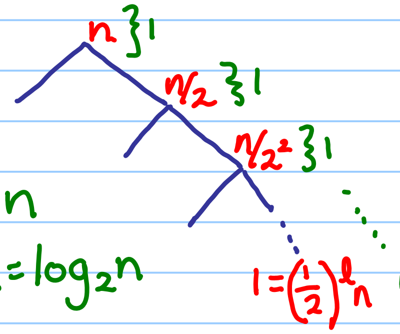
Binary Search

a. recursive calls = 2

b. size of data at each level = $(\frac{1}{2})^{\ell} \cdot n$

c. # of levels: $(\frac{1}{2})^{\ell} \cdot n = 1$; $n = (\frac{2}{1})^{\ell}$; $\ell = \log_2 n$

d. work @ each level = 1 $\quad \sum_{i=0}^{\log_2 n} 1$



③ $T(n) = n + 3T(\frac{n}{4}) \in O(n)$

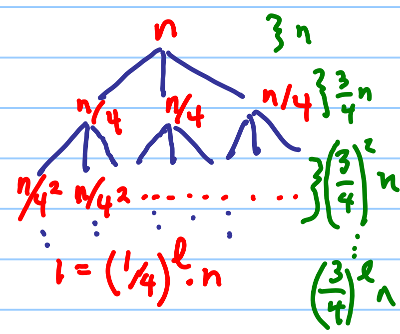
Mystery I

a. recursive calls = 2

b. size of data at each level = $(\frac{1}{4})^{\ell} \cdot n$

c. # of levels: $(\frac{1}{4})^{\ell} n = 1$; $n = (\frac{4}{1})^{\ell}$; $\ell = \log_4 n$

d. work @ each level = $(\frac{3}{4})^{\ell} \cdot n$ $\quad \sum_{i=0}^{\log_4 n} (\frac{3}{4})^{i} \cdot n$



④ $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + n \in O(n \log n)$

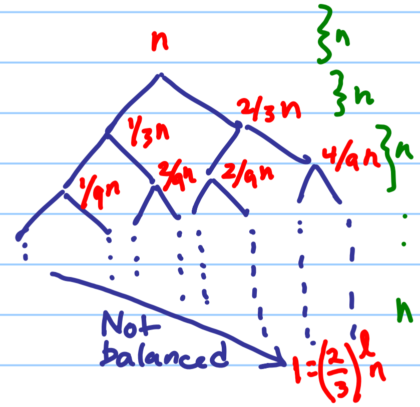
Mystery II

a. recursive calls = 2

b. size of data at each level $\leq (\frac{2}{3})^{\ell} \cdot n$

c. # levels: $(\frac{2}{3})^{\ell} \cdot n = 1$; $n = (\frac{3}{2})^{\ell}$; $\ell = \log_{\frac{3}{2}} n$

d. work @ each level = n $\quad \sum_{i=0}^{\log_{3/2} n} n$



⑤ $T(n) = 2T\left(\frac{2n}{3}\right) + n \in O\left(4n^{\log_{\frac{3}{2}} 2} - n\right) \in O\left(n^{1.7}\right)$

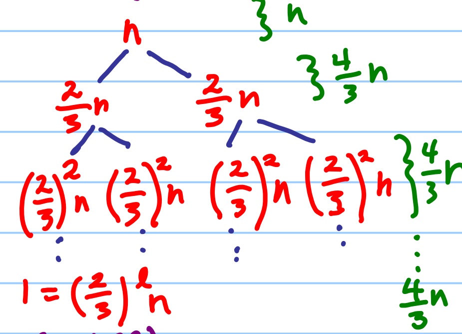
Mystery III

a. recursive calls = 2

b. size of data at each level $= \left(\frac{2}{3}\right)^{\ell} \cdot n$

c. # levels: $\left(\frac{2}{3}\right)^{\ell} \cdot n = 1$; $n = \left(\frac{3}{2}\right)^{\ell}$; $\ell = \log_{3/2} n$

d. Work @ each level $= \frac{4}{3}n$ $\quad \sum_{i=0}^{\log_{3/2} n} \frac{4}{3}n$



⑥ $T(n) = 6T\left(\frac{n}{4}\right) + 1 \in O\left(\frac{6}{5}n^{\log_4 6} - \frac{1}{5}\right) \in O\left(n^{1.29}\right)$

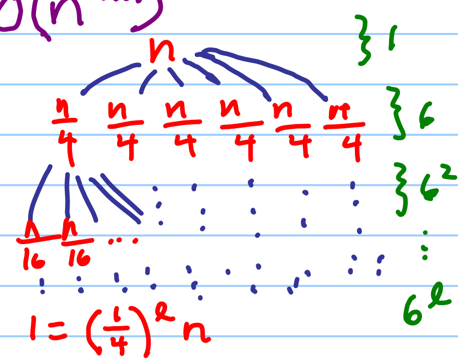
Mystery IV

a. recursive calls = 6

b. size of data at each level $= \left(\frac{1}{4}\right)^{\ell} \cdot n$

c. # levels: $\left(\frac{1}{4}\right)^{\ell} \cdot n = 1$; $n = \left(\frac{4}{1}\right)^{\ell}$; $\ell = \log_4 n$

d. Work @ each level $= 6^{\ell}$ $\quad \sum_{i=0}^{\log_4 n} 6^{i}$



These examples reveal patterns.

Master Theorem yields a general formula for these patterns.

$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$

Compare $f(n)$ to $n^{\log_b a}$.

But what about ϵ?

$f(n)$ must be polynomially $\left\{\begin{array}{l} \text{I) smaller than} \\ \text{II) the same as} \\ \text{III) larger than} \end{array}\right\}$ $n^{\log_b a}$

Case I: $\Theta\left(n^{\log_b a}\right)$ if $f(n) < n^{\log_b a}$

Case II: $\Theta\left(n^{\log_b a} \log n\right)$ if $f(n) = n^{\log_b a}$

Case III: $\Theta(f(n))$ if $f(n) > n^{\log_b a}$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$a = 3$
$b = 4$

$$3 \cdot \underbrace{f\left(\frac{n}{4}\right)} \le c \cdot f(n)$$

$$3 \cdot \frac{n}{4} \le c \cdot n$$

$c < 1$

Apply Master Theorem to these

6 examples:

	$f(n)$	$n^{\log_b a}$	Θ
$T(n) = n + 2T\left(\frac{n}{2}\right)$	n	$n^{\log_2 2} = n$	II $\Theta(n \log n)$
$T(n) = 1 + T\left(\frac{n}{2}\right)$	1	1	II $\Theta(\log n)$
$T(n) = n + 3T\left(\frac{n}{4}\right)$	n	$n^{\log_4 3}$	III $\Theta(n)$
$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$	n	—	—
$T(n) = 2T\left(\frac{2n}{3}\right) + n$	n	$n^{\log_{3/2} 2} = n^{1.7}$	I $\Theta(n^{1.7})$
$T(n) = 6T\left(\frac{n}{4}\right) + 1$	1 ✓	$n^{\log_4 6}$ ✓	I $\Theta(n^{1.29})$ $\cancel{\Theta(n \log n)}$