Taking Stock of Our Situation: Pricing and Randomness

Dianne P. O'Leary[1]

Many fascinating mathematical problems arise in the study of economics and financial markets. In this case study, we consider two computational techniques that are useful in this field:

- Simulation using Markov chains,
- Stochastic differential equations.

The first recorded options trade was by the Greek philosopher Thales (5th century BC), who believed that there would be a bumper crop of olives because of the good weather. He contracted time on all of the olive presses, essentially betting his deposit that he could make a profit by subletting his reserved time. Therefore he had an **option** on olive press time.

In this case study we won't get as far as studying options, but we will study several models of how the prices of assets change over time. As an example, we will discuss prices of stocks.

Random Walks and Markov Chains

Let's make a simple model of how the price of a stock behaves over time. Suppose the price is initially $p_0 = \$1.00$, and at each time $t = 0, 1, \ldots,$ the price p_t behaves as follows:

- If $p_t = (1+\lambda)^{10}$, then $p_{t+1} = (1+\lambda)^9$.
- If $p_t = (1+\lambda)^{-10}$, then $p_{t+1} = (1+\lambda)^{-9}$ with probability 0.99, and the price remains unchanged with probability 0.01.

[1] This case study is a supplement to *Scientific Computing with Case Studies*, Dianne P. O'Leary, SIAM Press, Philadelphia, 2009.

- Otherwise:
  - With probability α the price becomes $p_{t+1} = (1+\lambda)p_t$.
  - With probability $1-\alpha$ the price becomes $p_{t+1} = (1+\lambda)^{-1}p_t$.

In words, the sequence of stock prices forms a random walk on the domain

$$\{(1+\lambda)^j, j = -10, \ldots, 0, \ldots, 10\}.$$

The behavior of the price is described by a Markov chain with one node corresponding to each point in the domain. (See Chapter 19 for information on Markov chains.)

CHALLENGE 1.

(a) What is the probability that $p_3 = (1+\lambda)^3$? What is the probability that $p_3 = (1+\lambda)$?

(b) Use MATLAB to compute and graph the stationary probabilities for the Markov chain for $\alpha = 0.5, 0.6, 0.7$, and 0.8. (Remember to check that you are using the correct eigenvector, and remember to normalize the eigenvector so that the entries sum to 1.)

(c) Let $\lambda = 0.1$. For each of the α values in (b), use MATLAB to compute the mean value of the price after a long time has passed (i.e., when the probabilities have reached their stationary values). Also compute the variance in the price after a long time has passed. Describe the trend in words.

(d) What value of α makes the mean value of the price equal to \$1.00?

Stochastic Differential Equations

This part of the case study depends on Sections 1, 2, 4, 5, and 6 of "An Algorithmic introduction to the numerical simulation of stochastic differential equations," by Desmond J. Higham, *SIAM Review* 43, 2001, pp. 525-546. Check his website for the software and a list of important typos.

We are interested in his equation (4.2), in which X denotes the stock price. I'll call this equation (4.2H). In the next challenge, we will predict the stock price using a simple model, the Black-Scholes model, for which an exact solution is known.

CHALLENGE 2.
Solve (4.2H) with

$$f(X(t)) = 3X(t),$$
$$g(X(t)) = X(t),$$
$$X(0) = 1.$$

(a) Modify Higham's program `emstrong.m` to use $M = 2000$ and $T = 0.5$. Graph the 2000 paths $X(t)$, for $0 \le t \le T$, for the finest time step, on a single plot. Compute the mean and variance of the error and the mean and variance of the approximate solution values at T for the finest time step. Describe (in words) the behavior of the sampled paths. Is the convergence rate consistent with Higham's claims?

Add to Higham's documentation so that someone who has not read his paper would still understand something about what problem is being solved, what the method is, and what the least squares fit tells you. Make sure you acknowledge Higham as the source of the programs.

(b) Repeat, modifying `milstrong.m` to use the same parameters as `emstrong`, and to solve the same test problem as in (a), but keeping its loop structure the same. Make sure you compare with the Black-Scholes solutions, not the "reference solutions" that Higham uses.

(c) Make sure your documentation is detailed enough to describe the differences between the two programs. In particular, document the difference between having p on the outer loop vs. having s on the outer loop. Give a 1 paragraph discussion of the advantages/disadvantages of having s on the outer loop.

Finally, we'll use the history of the price of a stock to fit two parameters in the stochastic differential equation model. Numerically, this is a difficult problem.

To set up this problem, you or your instructor should use `generate_data.m` to generate a set of $(t, X(t))$ values.

CHALLENGE 3. Given the data from `generate_data.m` for the history of a stock's price, we want to find parameters α and β so that the model in (4.2H) fits the data, with

$$f(X(t)) = \alpha X(t),$$
$$g(X(t)) = \beta X(t),$$
$$X(0) = 1.$$

Estimate α and β for the given data. (You might make use of `lsqnonlin`.) Discuss the effects of the randomness in the function evaluation and what you did to make this effect small. How confident are you of your solution? What is the sensitivity of the model to small changes in the parameters?