Partial Solution to
Image Deblurring: I Can See Clearly Now
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Solution to Challenge 1.

Observe that if $\mathbf{y}$ is a $p \times 1$ vector and $\mathbf{z}$ is a $q \times 1$ vector then

$$\left\| \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \right\|_2^2 = \sum_{i=1}^{p} y_i^2 + \sum_{i=1}^{q} z_i^2 = \|\mathbf{y}\|_2^2 + \|\mathbf{z}\|_2^2.$$

Therefore,

$$\left\| \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{K} \\ \alpha \mathbf{I} \end{bmatrix} \mathbf{f} \right\|_2^2 = \left\| \begin{bmatrix} \mathbf{g} - \mathbf{K}\mathbf{f} \\ \alpha \mathbf{f} \end{bmatrix} \right\|_2^2 = \|\mathbf{g} - \mathbf{K}\mathbf{f}\|_2^2 + \|\alpha \mathbf{f}\|_2^2 = \|\mathbf{g} - \mathbf{K}\mathbf{f}\|_2^2 + \alpha^2 \|\mathbf{f}\|_2^2 .$$

Solution to Challenge 2.

First note that

$$\mathbf{Q} \equiv \begin{bmatrix} \mathbf{U}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^T \end{bmatrix}$$

is an orthogonal matrix since $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$, and recall that the 2-norm of a vector is invariant under multiplication by an orthogonal matrix: $\|\mathbf{Q}\mathbf{z}\|_2 = \|\mathbf{z}\|_2$. Therefore,

$$\begin{aligned}
\left\| \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{K} \\ \alpha \mathbf{I} \end{bmatrix} \mathbf{f} \right\|_2^2 &= \left\| \begin{bmatrix} \mathbf{g} - \mathbf{K}\mathbf{f} \\ \alpha \mathbf{f} \end{bmatrix} \right\|_2^2 \\
&= \left\| \mathbf{Q} \begin{bmatrix} \mathbf{g} - \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T \mathbf{f} \\ \alpha \mathbf{f} \end{bmatrix} \right\|_2^2 \\
&= \left\| \begin{bmatrix} \mathbf{U}^T \mathbf{g} - \boldsymbol{\Sigma}\mathbf{V}^T \mathbf{f} \\ \alpha \mathbf{V}^T \mathbf{f} \end{bmatrix} \right\|_2^2 \\
&= \left\| \begin{bmatrix} \hat{\mathbf{g}} - \boldsymbol{\Sigma}\hat{\mathbf{f}} \\ \alpha \hat{\mathbf{f}} \end{bmatrix} \right\|_2^2 \\
&= \left\| \begin{bmatrix} \hat{\mathbf{g}} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Sigma} \\ \alpha \mathbf{I} \end{bmatrix} \hat{\mathbf{f}} \right\|_2^2
\end{aligned}$$

Solution to Challenge 3.

Let's write the answer for a slightly more general case: $\mathbf{K}$ of dimension $m \times n$ with $m \geq n$.

$$\left\| \begin{bmatrix} \hat{\mathbf{g}} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Sigma} \\ \alpha \mathbf{I} \end{bmatrix} \hat{\mathbf{f}} \right\|_2^2$$

$$= \|\hat{\mathbf{g}} - \mathbf{\Sigma}\hat{\mathbf{f}}\|_2^2 + \alpha^2 \|\hat{\mathbf{f}}\|_2^2$$
$$= \sum_{i=1}^{n} (\hat{g}_i - \sigma_i \hat{f}_i)^2 + \sum_{i=n+1}^{m} \hat{g}_i^2 + \alpha^2 \sum_{i=1}^{n} \hat{f}_i^2$$

Setting the derivative with respect to f_i to zero we obtain

$$-2\sigma_i(\hat{g}_i - \sigma_i \hat{f}_i) + 2\alpha^2 \hat{f}_i = 0$$

so

$$\hat{f}_i = \frac{\sigma_i \hat{g}_i}{\sigma_i^2 + \alpha^2}\,.$$

Solution to Challenge 4.

From Challenges 2 and 3 above, with $\alpha = 0$, the solution is

$$\hat{f}_i = \frac{\sigma_i \hat{g}_i}{\sigma_i^2} = \frac{\hat{g}_i}{\sigma_i}\,.$$

Note that $\hat{g}_i = \mathbf{u}_i^T \mathbf{g}$. Now, since $\mathbf{f} = \mathbf{V}\hat{\mathbf{f}}$, we have

$$\mathbf{f} = \mathbf{v}_1 \hat{f}_1 + \ldots + \mathbf{v}_n \hat{f}_n$$

and the formula follows.

Solution to Challenge 5.

See the posted program. Some comments:

- One common bug in the TSVD section: zeroing out pieces of $\mathbf{S}_A$ and $\mathbf{S}_B$. This does not zero the smallest singular values, and although it is very efficient in time, it gives worse results than doing it correctly.
- The data was generated by taking an original image $\mathbf{F}$ (posted on the webpage), multiplying by $\mathbf{K}$, and adding random noise using the MATLAB statement `G = B * F * A' + .001 * rand(256,256)`. Note that the noise prevents us from completely recovering the initial image.
- In this case, the best way to choose the regularization parameter is by eye: choose a detailed part of the image and watch its resolution for various choices of the regularization parameter, probably using bisection to find the best parameter. Figure 1 shows data and two reconstructions. Although both algorithms yield images in which the text can be read, noise in the data appears in the background of the reconstructions. Some nonlinear reconstruction algorithms reduce this noise.
- In many applications, we need to choose the regularization parameter by automatic methods rather than by eye. If the noise-level is known, then the discrepancy principle is the best: choose the parameter to make the residual $\mathbf{K}\mathbf{f} - \mathbf{g}$ close in norm to the expected norm of the noise. If the noise-level is not known, then generalized cross validation and the L-curve are popular methods. See [1] for discussion of such methods.

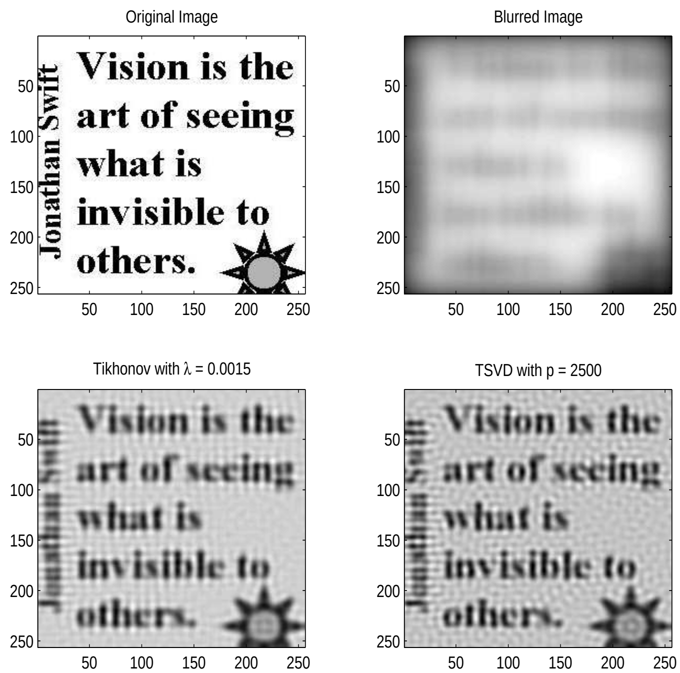

Figure 1: The original image, blurred image, and results of the two algorithms

References

[1] P. C. Hansen, Rank-Deficient and Discrete Ill-Posed Problems, SIAM Press, 1998, Chapter 7.