MAPL 600 / CMSC 760 Fall 2007
Take-Home Exam 1

1. (10) On the website `http://www.cs.umd.edu/users/oleary/SCSCwebpage/cs_direct` you will find a program `problem4.m` that tests various reordering methods for applying a sparse direct method to a linear system of equations. The methods include `specnd.m`, an implementation of spectral partitioning, available in a freely-available Matlab package. Run the program as is, using either the package (that you search for and download from the web) or your own implementation, following Demmel's notes. Also run the program on the matrix obtained from `load('west0479.mat')`. Discuss the results, comparing the algorithms and identifying their strengths and weaknesses.

The best answer plots the time, storage, and residual norm as a function of problem size. For these matrices, approx. minimum degree and minimum degree both worked well, and Cuthill-McKee was somewhat less effective. Spectral partitioning is rather slow and did not produce as sparse a factor as the other methods.

2. (10) Does SOR converge for strictly diagonally dominant matrices for $1 \leq \omega < 2$? Give a proof if the answer is yes and a counterexample if the answer is no.

SOR can diverge for strictly diagonally dominant matrices for ω arbitrarily close to 1. Consider the matrix

$$\begin{bmatrix} 1+\epsilon & 1 \\ -1 & 1+\epsilon \end{bmatrix}$$

The SOR iteration matrix with $\omega = 1+\epsilon$ is

$$\begin{bmatrix} -\epsilon & 1 \\ \epsilon & -1-\epsilon \end{bmatrix}$$

which has an eigenvalue close to $-1-2\epsilon$ (to be precise, $\lambda = -(1+2\epsilon \pm \sqrt{1+4\epsilon})/2$) and thus is divergent for $\epsilon > 0$.

3. Let $\mathbf{A}$ be an $n \times n$ matrix. Denote the rows of $\mathbf{A}$ by $\mathbf{a}_1^T, \ldots \mathbf{a}_n^T$ and the elements of the vector $\mathbf{b}$ by $b_1, \ldots, b_n$. Assume $\mathbf{a}_i^T \mathbf{a}_i = 1$ for $i = 1, \ldots, n$. Consider the following algorithm for solving $\mathbf{Ax} = \mathbf{b}$:

Choose an arbitrary $\mathbf{x}^0$.
For $k = 0, 1, \ldots,$

$\bar{\mathbf{x}}^{(1)} = \mathbf{x}^{(k)}$
For $j = 1, 2, \ldots, n$

$$\bar{\mathbf{x}}^{(j+1)} = \bar{\mathbf{x}}^{(j)} - \mathbf{a}_j \mathbf{a}_j^T \bar{\mathbf{x}}^{(j)} + b_j \mathbf{a}_j$$

End for
$\mathbf{x}^{(k+1)} = \bar{\mathbf{x}}^{(n+1)}$

End for

3a. (10) Show that this is a stationary iterative method. (Hint: It is sufficient to express it in the form $\mathbf{x}^{(k+1)} = \mathbf{G}\mathbf{x}^{(k)} + \mathbf{c}$.)

$$x^{(k+1)} = (I - a_n a_n^T) \dots (I - a_1 a_1^T) x^{(k)} + \sum_{i=1}^{n-1} (I - a_n a_n^T) \dots (I - a_{i+1} a_{i+1})^T b_i a_i + b_n a_n$$

Note: If you use product notation ($\prod$), tell your reader whether the factors are given left-to-right or right-to-left; this is necessary since matrix multiplication is not commutative.

3b. (10) Give a sufficient condition on the matrix $\mathbf{A}$ to ensure convergence.

Each of the matrices $P \equiv (I - aa^T)$ has one eigenvalue equal to zero (corresponding to eigenvector a) and $n-1$ others equal to 1 (corresponding to vectors in the subspace orthogonal to a). Therefore,

$$||(I - aa^T)z||^2 = z^T z - z^T a \leq ||z||^T$$

for all vectors z. Therefore, $||Gz|| \leq ||z||$. Let k be the first index such that $a_k z \neq 0$. Then $||Gz|| = ||P_n \dots P_k z|| \leq ||P_n \dots P_{k+1}||||P_k z|| < 1||z||$. If A *is nonsingular* then $a_k^T z \neq 0$ for some index k and therefore $||Gz|| < ||z||$ for all $z \neq 0$. It therefore follows that $\rho(G) < 1$. (Nonsingularity of A is necessary and sufficient.) This method is called Kaczmarz's method. More information about it and its relatives can be found in Saad p. 259 or Y. Censor, "Row action methods...," *SIAM Review* 23 (1981) p.444.

4. This problem involves the *five-point operator matrix* discussed in class, except that we add to it the matrix $\sigma\mathbf{I}$.

Consider the partial differential equation

$$-u_{xx} - u_{yy} + \sigma u = f(x, y),$$

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0.$$

Let v_{ij} be our approximation to the value $u(ih, jh)$, $i, j = 0, \dots, m$, with $h = 1/m$. Then using finite difference approximations to the second derivatives, we can write a linear system $\mathbf{A}\mathbf{v} = \mathbf{b}$ for the unknown values v_{ij} as follows:

$$\frac{-v_{i+1,j} + 2v_{ij} - v_{i-1,j}}{h^2} + \frac{-v_{i,j+1} + 2v_{ij} - v_{i,j-1}}{h^2} + \sigma v_{ij} = f(ih, jh),$$

for $i, j = 1, \ldots, m$. (Note that we are abusing notation somewhat by labeling the elements of the vector v with two subscripts. You can think of the elements as stacked in a column vector, but programming is somewhat simpler if you store v as a 2-dimensional array.)

Let $f(x, y)$ be the function corresponding to the true solution $u(x, y) = x^3 + y^3 + \cos(xy)$.

4a. (10) For what values of σ is the matrix strictly diagonally dominant? Justify your answer.

The main diagonal element in each row is $4/h^2 + \sigma$. The sum of the off-diagonal elements is at most $4/h^2$, so diagonal dominance is strict for any $\sigma > 0$ or $\sigma < -8/h^2$.

4b. (10) Is the matrix reducible? Justify your answer.

Sketch of an Algebraic Proof: (The geometric one – based on the graph of the matrix – is easier.) Suppose the matrix is reducible. Then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} F & G \\ 0 & H \end{bmatrix}$$

where H is a square matrix and 0 is a block of zeros. Among all of the variables corresponding to the columns in H, let v_{ij} be the one with minimal first index, and, among these, the one with minimal second index. Then its first index i must be 1; otherwise, since there is a nonzero in the i,j row of the matrix in the column for $v_{i-1,j}$, there would be a smaller first index. Similarly, we can show that the index j must be 1. Now, all variables with nonzeros in row $1, 1$ of the matrix must be in the H block, as must all variables with nonzeros in those rows, iterating this argument we find that all variables must be, and therefore F is null and the matrix is irreducible.

4c. (10) Write a Matlab program to solve this system with $\sigma = 1000$, $m = 20$, using Chebyshev semi-iteration based on the Jacobi splitting. Run three cases, using .31, .61, and .91 as bounds on $\rho(\mathbf{G}_J)$. Start each iteration with the guess $\mathbf{v}^{(0)} = 0$ and do 25 iterations, printing v_{11} and v_{55} at each iteration (so I can easily detect bugs) and printing whatever information you need to get an estimate of the convergence rate of the Chebyshev iteration. Discuss your results: What happens for each of the three bounds on $\rho(\mathbf{G}_J)$? How much work is performed per iteration? (If your work per iteration or your storage is more than $O(m^2)$, you will lose points.)

See attached program and output. Try to read the program even if you are not a Matlab expert. The spectral radius of the Jacobi iteration matrix for this problem is $4\cos(\pi/m)/(4+\sigma/m^2) \approx .61$, so that value gives the best result. The convergence rate can be estimated by looking at ratios of successive residuals.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This matlab program solves a finite difference
% approximation to a 2-dimensional pde using Chebyshev
% semi-iteration applied to the Jacobi iteration.

% The spectral radius of the Jacobi iteratation matrix is
% 4*cos(pi/m) / (4 + sigma/(m*m))
% DPO 09/1996

disp('here I am')
% initialize

m = 20;
h = 1/m;
for param = [.31 .61  .91]
disp(param)
z = 1/param;
sigma = 1000;
vbarnew = zeros(m+1,m+1);

% Compute the true solution and the right hand side.

b = zeros(m+1,m+1);
for i=2:m,
  x = (i-1)*h;
for j=2:m,
  y = (j-1)*h;
  b(i,j) = -(6*x + 6*y  - (x^2+y^2)*cos(x*y)) + ...
            (x^3 + y^3 + cos(x*y))*sigma ;
end
end

rnorm = norm(b,'fro');
normb = rnorm;



%  The "vector" v is stored in a 2-dimensional array of
%  size m+1 x m+1, including the boundary values.

dii = 4/(h^2) + sigma;
scale = 1/(h^2);

% Perform the iteration
```

```
disp('iteration     v_{11}    v_{55}   resid norm rnorm/rnormold')
for k = 1:50,

%  Apply Jacobi to vbar^k:

v = vbarnew;

%  Compute p = b + (L+U) v

p = b;
p(2:m,2:m) = p(2:m,2:m) + ...
                 scale * (v(3:m+1,2:m) + v(1:m-1,2:m) + ...
                 v(2:m,3:m+1) + v(2:m,1:m-1)) ;

% Compute p = D^{-1} ( b + (L+U) v)

p = p / dii;



% compute omega


if (k == 1)
   c = 1;
   cnew = z;
   omega = 1;
   vbar = zeros(m+1,m+1);
else
   cold = c;
   c = cnew;
   cnew = 2*z*c - cold;
   omega = 1 + cold/cnew;
end


% form vbar^{k+1}

vbarold = vbar;
vbar = vbarnew;
vbarnew = omega *(p - vbarold) + vbarold;

% assess the progress
rnormold = rnorm;
% This next piece of code is inefficient and should be
% incorporated into the computation of p above.
```

```
r = b;
v = vbarnew;
r(2:m,2:m) = r(2:m,2:m) - v(2:m,2:m)*dii +  ...
                scale * (v(3:m+1,2:m) + v(1:m-1,2:m) + ...
                v(2:m,3:m+1) + v(2:m,1:m-1)) ;
rnorm = norm(r, 'fro');

disp(sprintf('%4.0f        %f   %f  %7.2e     %7.2e ', ...
      k, vbarnew(2,2), vbarnew(6,6), rnorm, rnorm/rnormold))

end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
>> hmwk1
here I am
    0.3100

iteration     v_{11}    v_{55}   resid norm rnorm/rnormold
   1        0.384481   0.394778  1.60e+04     5.79e-01
   2        0.528233   0.670354  8.54e+03     5.35e-01
   3        0.570961   0.818726  4.82e+03     5.64e-01
   4        0.590518   0.905281  2.74e+03     5.69e-01
   5        0.600248   0.956235  1.57e+03     5.72e-01
   6        0.605096   0.986175  8.98e+02     5.73e-01
   7        0.607591   1.003685  5.15e+02     5.74e-01
   8        0.608884   1.013878  2.96e+02     5.75e-01
   9        0.609569   1.019789  1.70e+02     5.75e-01
  10        0.609933   1.023207  9.81e+01     5.76e-01
  11        0.610129   1.025181  5.65e+01     5.76e-01
  12        0.610236   1.026318  3.26e+01     5.77e-01
  13        0.610294   1.026973  1.88e+01     5.77e-01
  14        0.610326   1.027350  1.09e+01     5.77e-01
  15        0.610343   1.027568  6.27e+00     5.78e-01
  16        0.610353   1.027692  3.62e+00     5.78e-01
  17        0.610358   1.027764  2.09e+00     5.78e-01
  18        0.610362   1.027806  1.21e+00     5.78e-01
  19        0.610363   1.027829  7.00e-01     5.78e-01
  20        0.610364   1.027843  4.05e-01     5.78e-01
  21        0.610365   1.027851  2.34e-01     5.79e-01
  22        0.610365   1.027856  1.36e-01     5.79e-01
  23        0.610365   1.027858  7.84e-02     5.79e-01
  24        0.610365   1.027860  4.54e-02     5.79e-01
  25        0.610365   1.027861  2.63e-02     5.79e-01
```

```
  26        0.610365   1.027861  1.52e-02     5.79e-01
  27        0.610365   1.027861  8.81e-03     5.79e-01
  28        0.610365   1.027862  5.10e-03     5.79e-01
  29        0.610365   1.027862  2.95e-03     5.79e-01
  30        0.610365   1.027862  1.71e-03     5.79e-01
  31        0.610365   1.027862  9.91e-04     5.79e-01
  32        0.610365   1.027862  5.74e-04     5.79e-01
  33        0.610365   1.027862  3.32e-04     5.79e-01
  34        0.610365   1.027862  1.93e-04     5.79e-01
  35        0.610365   1.027862  1.12e-04     5.79e-01
  36        0.610365   1.027862  6.46e-05     5.79e-01
  37        0.610365   1.027862  3.74e-05     5.79e-01
  38        0.610365   1.027862  2.17e-05     5.79e-01
  39        0.610365   1.027862  1.26e-05     5.79e-01
  40        0.610365   1.027862  7.28e-06     5.79e-01
  41        0.610365   1.027862  4.22e-06     5.79e-01
  42        0.610365   1.027862  2.45e-06     5.79e-01
  43        0.610365   1.027862  1.42e-06     5.79e-01
  44        0.610365   1.027862  8.21e-07     5.79e-01
  45        0.610365   1.027862  4.76e-07     5.79e-01
  46        0.610365   1.027862  2.76e-07     5.79e-01
  47        0.610365   1.027862  1.60e-07     5.79e-01
  48        0.610365   1.027862  9.26e-08     5.79e-01
  49        0.610365   1.027862  5.37e-08     5.79e-01
  50        0.610365   1.027862  3.11e-08     5.79e-01
    0.6100

iteration     v_{11}     v_{55}   resid norm rnorm/rnormold
   1        0.384481   0.394778  1.60e+04     5.79e-01
   2        0.617791   0.784009  5.55e+03     3.47e-01
   3        0.624505   0.940454  1.82e+03     3.28e-01
   4        0.613236   0.996890  5.99e+02     3.29e-01
   5        0.610674   1.017309  2.00e+02     3.34e-01
   6        0.610436   1.024676  6.71e+01     3.36e-01
   7        0.610389   1.027140  2.26e+01     3.37e-01
   8        0.610367   1.027794  7.61e+00     3.36e-01
   9        0.610364   1.027888  2.54e+00     3.34e-01
  10        0.610365   1.027871  8.39e-01     3.30e-01
  11        0.610365   1.027860  2.71e-01     3.23e-01
  12        0.610365   1.027860  8.40e-02     3.10e-01
  13        0.610365   1.027861  2.43e-02     2.89e-01
  14        0.610365   1.027861  6.42e-03     2.65e-01
  15        0.610365   1.027862  1.93e-03     3.00e-01
  16        0.610365   1.027862  7.92e-04     4.11e-01
  17        0.610365   1.027862  3.03e-04     3.83e-01
  18        0.610365   1.027862  1.03e-04     3.39e-01
```

```
  19        0.610365   1.027862  3.64e-05     3.54e-01
  20        0.610365   1.027862  1.25e-05     3.43e-01
  21        0.610365   1.027862  4.02e-06     3.23e-01
  22        0.610365   1.027862  1.34e-06     3.32e-01
  23        0.610365   1.027862  4.39e-07     3.29e-01
  24        0.610365   1.027862  1.54e-07     3.50e-01
  25        0.610365   1.027862  5.63e-08     3.66e-01
  26        0.610365   1.027862  2.18e-08     3.87e-01
  27        0.610365   1.027862  8.65e-09     3.97e-01
  28        0.610365   1.027862  3.37e-09     3.90e-01
  29        0.610365   1.027862  1.25e-09     3.70e-01
  30        0.610365   1.027862  4.23e-10     3.39e-01
  31        0.610365   1.027862  1.37e-10     3.23e-01
  32        0.610365   1.027862  4.62e-11     3.38e-01
  33        0.610365   1.027862  1.90e-11     4.11e-01
  34        0.610365   1.027862  1.03e-11     5.45e-01
  35        0.610365   1.027862  9.06e-12     8.76e-01
  36        0.610365   1.027862  6.90e-12     7.61e-01
  37        0.610365   1.027862  6.15e-12     8.91e-01
  38        0.610365   1.027862  5.85e-12     9.52e-01
  39        0.610365   1.027862  5.81e-12     9.92e-01
  40        0.610365   1.027862  5.44e-12     9.37e-01
  41        0.610365   1.027862  5.42e-12     9.97e-01
  42        0.610365   1.027862  5.27e-12     9.72e-01
  43        0.610365   1.027862  5.37e-12     1.02e+00
  44        0.610365   1.027862  5.27e-12     9.82e-01
  45        0.610365   1.027862  5.37e-12     1.02e+00
  46        0.610365   1.027862  5.27e-12     9.82e-01
  47        0.610365   1.027862  5.37e-12     1.02e+00
  48        0.610365   1.027862  5.27e-12     9.82e-01
  49        0.610365   1.027862  5.37e-12     1.02e+00
  50        0.610365   1.027862  5.27e-12     9.82e-01
    0.9100

iteration     v_{11}     v_{55}   resid norm rnorm/rnormold
   1        0.384481   0.394778  1.60e+04     5.79e-01
   2        0.858181   1.089076  4.91e+03     3.08e-01
   3        0.841140   1.432956  1.16e+04     2.37e+00
   4        0.645776   1.367164  8.42e+03     7.25e-01
   5        0.563858   1.152035  2.79e+03     3.32e-01
   6        0.575865   0.993294  1.92e+03     6.89e-01
   7        0.600610   0.944597  2.28e+03     1.19e+00
   8        0.612348   0.967521  1.32e+03     5.78e-01
   9        0.614820   1.008788  4.06e+02     3.07e-01
  10        0.613506   1.037286  4.32e+02     1.07e+00
  11        0.611346   1.045693  3.97e+02     9.20e-01
```

12	0.609949	1.040051	1.97e+02	4.95e-01
13	0.609682	1.030321	7.16e+01	3.64e-01
14	0.609999	1.024068	8.64e+01	1.21e+00
15	0.610317	1.023442	6.65e+01	7.70e-01
16	0.610448	1.026021	2.88e+01	4.33e-01
17	0.610450	1.028369	1.45e+01	5.02e-01
18	0.610407	1.029000	1.62e+01	1.12e+00
19	0.610369	1.028500	1.08e+01	6.64e-01
20	0.610352	1.027911	4.26e+00	3.95e-01
21	0.610353	1.027671	2.98e+00	7.00e-01
22	0.610360	1.027696	2.92e+00	9.80e-01
23	0.610366	1.027797	1.71e+00	5.84e-01
24	0.610368	1.027872	6.66e-01	3.91e-01
25	0.610367	1.027896	5.85e-01	8.78e-01
26	0.610366	1.027887	5.12e-01	8.75e-01
27	0.610365	1.027868	2.63e-01	5.14e-01
28	0.610365	1.027857	1.07e-01	4.05e-01
29	0.610365	1.027856	1.17e-01	1.10e+00
30	0.610365	1.027859	8.54e-02	7.27e-01
31	0.610365	1.027861	3.81e-02	4.47e-01
32	0.610365	1.027863	2.30e-02	6.03e-01
33	0.610365	1.027863	2.10e-02	9.12e-01
34	0.610365	1.027862	1.37e-02	6.52e-01
35	0.610365	1.027862	6.46e-03	4.73e-01
36	0.610365	1.027862	4.23e-03	6.55e-01
37	0.610365	1.027862	3.63e-03	8.59e-01
38	0.610365	1.027862	2.28e-03	6.28e-01
39	0.610365	1.027862	1.07e-03	4.67e-01
40	0.610365	1.027862	7.49e-04	7.03e-01
41	0.610365	1.027862	6.43e-04	8.59e-01
42	0.610365	1.027862	3.78e-04	5.88e-01
43	0.610365	1.027862	1.67e-04	4.42e-01
44	0.610365	1.027862	1.37e-04	8.21e-01
45	0.610365	1.027862	1.14e-04	8.33e-01
46	0.610365	1.027862	6.01e-05	5.25e-01
47	0.610365	1.027862	2.63e-05	4.37e-01
48	0.610365	1.027862	2.61e-05	9.95e-01
49	0.610365	1.027862	1.98e-05	7.59e-01
50	0.610365	1.027862	8.98e-06	4.53e-01