AMSC/CMSC 460 Final Exam Answers , Fall 2001

1. Mark each of the following statements T (True) or F (False).
Grading: 2 * (number correct) - 1.5 * (number incorrect) + 0 * (number blank).

a. F Suppose $A$ is an $n \times n$ nonsingular matrix with no zero elements. It takes fewer floating point operations to form $A^{-1}$ and solve a linear system by multiplying $A^{-1}b$ than to factor the matrix $A = LU$ and use forward and backward substitution to solve the system.

b. F The condition number of a matrix is always less than or equal to 1.

c. T $\|A\|_1 = 5$ if

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

d. F As you increase the number $n$ of data points, equally spaced in the interval $[0, 1]$, the error in the the composite Simpson's rule approximation to the integral of a smooth function is of order $n$ to the power 4.

e. T For any numerical integration routine with a nonrandom set of evaluation points, we can find a function whose integral is 1, but for which the routine reports an integral of zero.

f. F When adding numbers, relative error bounds add.

g. T The QR factorization usually gives a more accurate solution to a least squares problem than forming and solving the normal equations.

h. F Backward error measures the distance between the true solution and the computed solution.

i. F The software `ode15s` should not be used for stiff systems of differential equations.

j. F In general, a cubic spline interpolant will oscillate more than a polynomial interpolant.

2a. (10) Suppose we have data points $(t_i, y_i)$, $i = 1, \ldots, 20$, and we want to model the function $y(t)$ by a polynomial of degree 3 (i.e., a cubic polynomial) using a least squares fit. Give the dimensions $m \times n$ of the matrix that you would form in solving this least squares problem.

**Answer:** m = 20

n = 4

2b. (10) Compute $f[1,2,3]$ if $f(1) = 4$, $f(2) = 1$, and $f(3) = -5$.

**Answer:**

$$f[1,2,3] = \frac{\frac{4-1}{1-2} - \frac{-5-1}{3-2}}{1-3} = \frac{-3+6}{-2} = -\frac{3}{2}$$

3. Suppose $A$ is a matrix of size $100 \times 50$,
$B$ is a matrix of size $50 \times 50$, and
$x$ is a vector of size $50 \times 1$.

3a. (10) How many floating point multiplications does it take to form $A * (B * x)$?

**Answer:** (50*50)to form $B * x$, and then $100 * 50$ to form $A$ times this.

3b. (10) How many floating point multiplications does it take to form $(A * B) * x$?

**Answer:** (100*50*50) to form $A * B$ and then $100 * 50$ to form this times $x$. (This is 34 times the work of 3a.)

4a. (10) Suppose we have 20 data points $(x_i, y_i)$, with $x_1 < x_2 < \ldots < x_n$. Give a formula that approximates

$$Q = \int_{x_1}^{x_n} y(x)\, dx\,.$$

**Answer:** Use Trapezoidal, since we don't know whether the spacing is equal.

$$Q \approx \sum_{i=1}^{19} \frac{x_{i+1} - x_i}{2} (y_{i+1} + y_i)\,.$$

4b. (10) Suppose that $f(x)$ is given as a Matlab m-file. How could you use `ode45` to approximate

$$\int_1^8 f(x)\,dx\,?$$

**Answer:** Solve the differential equation

$$y' = f(x), \quad y(1) = 0\,.$$

The integral will be $y(8)$, so we need the last value from the y vector in `[x,y] = ode45('f',[1,8],0)`.

5. (20) A scientist has solved a linear system on a machine with "machine epsilon" equal to $10^{-16}$. The matrix was of size $100 \times 100$, the condition number was $10^3$, the residual was $\|b - Ax\| = 10^{-12}$, and the algorithm was Gaussian elimination with partial pivoting (i.e., the Matlab backslash operator). The data has bounds $\|b\| = 70$, $\|x_{comp}\| = 10$, and $\|A\| = 30$. The values $b_i$ come from measured data, and the difference between $b_i$ and its true value $d_i$ is bounded by $|b_i - d_i| \le 10^{-5}$, $i = 1, \ldots, 100$. Give the scientist forward and backward error bounds on the results she computed compared to the problem she really wanted to solve: $Ay = d$. Explain to her what these bounds mean about the quality of her results.

**Answer:**

1. Backward error: The residual $d - Ax$ is bounded by $100 * 10^{-5}$ (since for the norm, we need to add $10^{-5}$ 100 times), so she has solved the problem $Ay = d + e$, where $\|e\| < 10^{-3}$.

2. Forward error:

$$\frac{\|x - x_{true}\|}{\|x_{true}\|} \le \kappa \frac{\|e\|}{\|d\|} \le 10^3 \frac{10^{-3}}{70} = \frac{1}{70}\,.$$

   This is how close the computed answer is to the true answer.

6. (25) Suppose we have an increasing function $y = f(x)$, and suppose that we have stored two vectors `x` and `y` in Matlab, with `y(i) = f(x(i))`. Suppose further that `x(1) = -4, y(1) = -2, x(5) = 6, y(5) = 10`. Use the Matlab functions for cubic spline interpolation to determine a point $x$ for which $f(x) \approx 0$.
Note: Up to full credit for "inverse interpolation." Up to 15 points for interpolation plus root finding.
Note: $f$ is increasing if $f(x) > f(z)$ whenever $x > z$.
**Answer:** The approximate zero is `z = spline(y,x,0)`.

7. Refer to the listing of `fzero` that is attached after this page.
7a. (15) Mark changes on the listing to add another output variable `count`, which is a count of the number of function evaluations that were performed. (Don't forget to change the documentation.)
**Answer:** Initialize count to zero, add one to it after each call to feval, and add documentation explaining the purpose of count and documenting who changed the program and when.

7b. (10) Why did the author use `p` and `q` rather than just defining `d` directly?
**Answer:** If $s$ is too close to 1, for example, then $p/q$ can overflow, so we want to avoid making the division until we check (in the line following "Is interpolated point acceptable") that it is safe.