AMSC/CMSC 460 Final Exam , Fall 2002

Show all work. You may leave arithmetic expressions in any form that a calculator could evaluate. By putting your name on this paper, you agree to abide by the university's code of academic integrity in completing the exam. Use no books, calculators, cellphones, communication with others, scratchpaper, etc. You may use one sheet of paper, in your own handwriting, with notes about the course material.

Name ____________________________________

Student number ______________________________

Problem 1:	(20)	
Problem 2:	(20)	
Problem 3:	(20)	
Problem 4:	(35)	
Problem 5:	(20)	
Problem 6:	(20)	
Problem 7:	(15)	

Total: ____________ (150)

1. Mark each of the following statements T (True) or F (False).
Grading: 2 * (number correct) - 1.5 * (number incorrect) + 0 * (number blank).

a. __________ Gaussian integration uses equally spaced points.

b. __________ The formula $y_{n+1} = y_n + \frac{h_n}{2}(f_{n+1} + f_n)$ represents an implicit method for solving ode's.

c. __________ Euler's method is stable for a larger range of h than backward Euler.

d. __________ Suppose A is an $n \times n$ nonsingular matrix with no zero elements. It takes fewer floating point operations to form A^{-1} and solve a linear system by multiplying $A^{-1}b$ than to factor the matrix $A = LU$ and use forward and backward substitution to solve the system.

e. __________ The cost of evaluating the discrete Fourier Transform of a vector of length n using Matlab's `fft` function is of order $n^2 \log_2 n$.

f. __________ When adding numbers, absolute error bounds add.

g. __________ The software `ode15s` should not be used for stiff systems of differential equations.

h. __________ Adaptive numerical integration routines use more points in regions where the function is changing rapidly.

i. __________ The spacing between floating point numbers is uniform.

j. __________ In order to get the most accurate answer when adding a set of positive numbers, add them in order from smallest to largest.

2. (20) Suppose we have data points (t_i, y_i), $i = 1, \ldots, 10$, and we want to model the function $y(t)$ by a polynomial of degree 3 (i.e., a cubic polynomial) using a least squares fit. We will find the coefficients `c` of the polynomial by the Matlab expression `c = A \ b`. Write down the matrix `A` and the vector `b` to make this work.

3. Suppose A is a matrix of size 40×200,
B is a matrix of size 200×50, and
x is a vector of size 50×1.

3a. (10) How many floating point multiplications does it take to form $A * (B * x)$?

3b. (10) How many floating point multiplications does it take to form $(A * B) * x$?

4.

a. (5) Let $A = \begin{bmatrix} 3 & 6 \\ -5 & 2 \end{bmatrix}$. Then $\|A\|_1 =$ ______________.

b. (5) As you increase the number n of data points, equally spaced in the interval $[0,1]$, the error in the composite Simpson's rule approximation to the integral of a smooth function is of order n to the power ______________.

c. (5) As you increase the number n of data points, equally spaced in the interval $[0,1]$, the convergence rate for cubic spline interpolation on functions with 6 continuous derivatives is of order n to the power ______________.

d. (5) The formula for Newton's method for solving the equation $f(x) = 0$ is

e. (5) The cost to solve a linear system $Ax = b$ when A is a lower triangular matrix of size $n \times n$ is of order n to the power ______________.

f. (5) The cost to compute the coefficients for polynomial interpolation with n data points using the Newton basis is of order n to the power ______________.

g. (5) The matrix f_y, (the matrix that determines the stability) for the system

$$u' = 2u - 2\cos v \ , \ v' = u + tv^2$$

is

5. (20) Without citing any theorems proved in the notes or the book, prove that there is a unique polynomial of degree less than or equal to 2 that interpolates the data (1,2), (3,1), and (4,-2). Hint: you need to show existence (by writing down the polynomial) and then show uniqueness.

6. (20) Write Matlab code to use inverse interpolation to find a zero of the function $f(x)$ given the data $(x, f(x) =$ (1,-1), (2,2), (3,7), and (4,14). (You may use Matlab functions from the book.)

7. (15) List 5 important things to include in documentation for a Matlab function that someone else needs to use.

AMSC/CMSC 460 Final Exam Answers , Fall 2002

1. Mark each of the following statements T (True) or F (False).
Grading: 2 * (number correct) - 1.5 * (number incorrect) + 0 * (number blank).

a.	F p155	Gaussian integration uses equally spaced points.
b.	T p354	The formula $y_{n+1} = y_n + \frac{h_n}{2}(f_{n+1} + f_n)$ represents an implicit method for solving ode's.
c.	F ode p9,11	Euler's method is stable for a larger range of h than backward Euler.
d.	F never invert	Suppose A is an $n \times n$ nonsingular matrix with no zero elements. It takes fewer floating point operations to form A^{-1} and solve a linear system by multiplying $A^{-1}b$ than to factor the matrix $A = LU$ and use forward and backward substitution to solve the system.
e.	F int part2 p5	The cost of evaluating the discrete Fourier Transform of a vector of length n using Matlab's `fft` function is of order $n^2 \log_2 n$.
f.	T error p12	When adding numbers, absolute error bounds add.
g.	F ode p18	The software `ode15s` should not be used for stiff systems of differential equations.
h.	T int part2	Adaptive numerical integration routines use more points in regions where the function is changing rapidly.
i.	F error p10	The spacing between floating point numbers is uniform.
j.	T error notes	In order to get the most accurate answer when adding a set of positive numbers, add them in order from smallest to largest.

2. (20) Suppose we have data points (t_i, y_i), $i = 1, \ldots, 10$, and we want to model the function $y(t)$ by a polynomial of degree 3 (i.e., a cubic polynomial) using a least squares fit. We will find the coefficients `c` of the polynomial by the Matlab expression `c = A \ b`. Write down the matrix `A` and the vector `b` to make this work.

Answer: (least squares notes p9) Let

$$p(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 \,.$$

Then

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_{10} & t_{10}^2 & t_{10}^3 \end{bmatrix},$$

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{bmatrix}.$$

3. Suppose A is a matrix of size 40×200,
B is a matrix of size 200×50, and
x is a vector of size 50×1.

3a. (10) How many floating point multiplications does it take to form $A * (B * x)$?

Answer: (matrix computation notes) 50*200 for $B * x$ plus 40*200 for $A * (B * x)$.

3b. (10) How many floating point multiplications does it take to form $(A * B) * x$?

Answer: 40*50*200 for $A * B$ plus 40*50 for $(A * B) * x$.

4.

a. (5) Let $A = \begin{bmatrix} 3 & 6 \\ -5 & 1 \end{bmatrix}$. Then $\|A\|_1 =$ ____________.

Answer: 8 (p185)

b. (5) As you increase the number n of data points, equally spaced in the interval $[0, 1]$, the error in the composite Simpson's rule approximation to the integral of a smooth function is of order n to the power ____________.

Answer: -4 (p147)

c. (5) As you increase the number n of data points, equally spaced in the interval $[0, 1]$, the convergence rate for cubic spline interpolation on functions with 6 continuous derivatives is of order n to the power ____________.

Answer: -4 (piecewise interpolation notes p10)

d. (5) The formula for Newton's method for solving the equation $f(x) = 0$ is

Answer: (p283) $x^{(k+1)} = x^{(k)} - f(x^{(k)})/f'(^{(k)})$.

e. (5) The cost to solve a linear system $Ax = b$ when A is a lower triangular matrix of size $n \times n$ is of order n to the power ____________.

Answer: 2 (p211)

f. (5) The cost to compute the coefficients for polynomial interpolation with n data points using the Newton basis is of order n to the power ____________.

Answer: 2 (p94)

g. (5) The matrix f_y, (the matrix that determines the stability) for the system

$$u' = 2u - 2\cos v \ , \ v' = u + tv^2$$

is

Answer:

$$\begin{bmatrix} 2 & 2\sin v \\ 1 & 2tv \end{bmatrix}$$

5. (20) Without citing any theorems proved in the notes or the book, prove that there is a unique polynomial of degree less than or equal to 2 that interpolates the data (1,2), (3,1), and (4,-2). Hint: you need to show existence (by writing down the polynomial) and then show uniqueness.

Answer: Let

$$p(x) = 2\frac{(x-3)(x-4)}{(1-3)(1-4)} + 1\frac{(x-1)(x-4)}{(3-1)(3-4)} - 2\frac{(x-1)(x-3)}{(4-1)(4-3)}.$$

Then $p(1) = 2$, $p(3) = 1$, and $p(4) = -2$, so the polynomial exists.

Suppose there are two polynomials $p \neq q$ and each of them solves the interpolation problem.

So the polynomials satisfy $p(1) = q(1) = 2$, $p(3) = q(3) = 1$, and $p(4) = q(4) = -2$ so $p - q$, also a polynomial of degree at most 2, has 3 distinct roots. Therefore, $p - q$ must be the zero polynomial, so $p = q$.

6. (20) Write Matlab code to use inverse interpolation to find a zero of the function $f(x)$ given the data $(x, f(x) =$ (1,-1), (2,2), (3,7), and (4,14). (You may use Matlab functions from the book.)

Answer: We will interpolate the data (y, x) by a polynomial $p(x)$, and then evaluate $p(0)$ to approximate the root. Here is one way:

```
y=0;
p = 1*(y-2)*(y-7)*(y-14)/((-1-2)*(1-7)*(1-14)) ...
  + 2*(y+1)*(y-7)*(y-14)/((2+1)*(2-7)*(2-14)) ...
  + 3*(y+1)*(y-2)*(y-14)/((7+1)*(7-2)*(7-14)) ...
  + 4*(y+1)*(y-2)*(y-7)/((14+1)*(14-2)*(14-7));
```

7. (15) List 5 important things to include in documentation for a Matlab function that someone else needs to use.
Answer: See `locate` on p.108.

- name, date, date and list of any modifications.
- purpose of function.
- description of each input parameter.
- description of each output parameter.
- description of method and references.