AMSC/CMSC 660 Scientific Computing I
Fall 2006
UNIT 4: Nonlinear Systems and the Homotopy Method
Part 2 **Dianne P. O'Leary**

Case study: Octahedral variable geometry truss

Note: This case study concerns solving a system of polynomial equations.

As you know, we should be using a special purpose homotopy method for solving this, but just to illustrate the tools we have, we'll use a general purpose method.

Reference: This problem is taken from Layne T. Watson, "Probability-one homotopies in computational science," *J. of Computational and Applied Mathematics* 140 (2002) 785-807.

The problem

Imagine that we have attached 3 trusses together to form a triangle ABC. This will be the base of a platform.

Now take 3 more trusses and form a triangle $\bar{A}\bar{B}\bar{C}$ from them. But assume that these trusses have variable length. This will be the top of the platform.

We will use 6 more trusses to attach the top to the bottom.

- Vertex A will be attached to $\bar{A}$ and $\bar{C}$.
- Vertex B will be attached to $\bar{A}$ and $\bar{B}$.
- Vertex C will be attached to $\bar{B}$ and $\bar{C}$.

Let θ_A be the angle between $A\bar{A}$ and vertical. Define θ_B and θ_C in a similar way.

Problem: Given the lengths of the 12 trusses, determine the three angles.

Formulation 1

There are 9 fixed-length trusses, and the 9 parameters $a, b, \ldots, i$ are computed from these.

There are 3 variable-length trusses, and we let these lengths be L_{ab}, L_{bc}, L_{ac}.

The resulting equations are of the form

$$
\begin{aligned}
L_{ab}^2 - (a\cos\theta_a + a\cos\theta_b + b\cos\theta_a\cos\theta_b - 2b\sin\theta_a\sin\theta_b + c) &= 0 \\
L_{bc}^2 - (d\cos\theta_b + d\cos\theta_c + e\cos\theta_b\cos\theta_c - 2e\sin\theta_b\sin\theta_c + f) &= 0 \\
L_{ac}^2 - (g\cos\theta_a + g\cos\theta_c + h\cos\theta_a\cos\theta_c - 2h\sin\theta_a\sin\theta_c + i) &= 0
\end{aligned}
$$

Watson uses the tangent half-angle formulas to convert this system to polynomial form:

$$
\begin{aligned}
\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_1^2 x_2^2 + \alpha_4 x_1 x_2 + \alpha_5 &= 0 \\
\beta_1 x_2^2 + \beta_2 x_3^2 + \beta_3 x_2^2 x_3^2 + \beta_4 x_2 x_3 + \beta_5 &= 0 \\
\gamma_1 x_3^2 + \gamma_2 x_1^2 + \gamma_3 x_3^2 x_1^2 + \gamma_4 x_3 x_1 + \gamma_5 &= 0
\end{aligned}
$$

For this application, are interested in the real roots, but for illustration, we will try to compute all real and complex roots.)

How many roots are there?

$$
\begin{aligned}
\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_1^2 x_2^2 + \alpha_4 x_1 x_2 + \alpha_5 &= 0 \\
\beta_1 x_2^2 + \beta_2 x_3^2 + \beta_3 x_2^2 x_3^2 + \beta_4 x_2 x_3 + \beta_5 &= 0 \\
\gamma_1 x_3^2 + \gamma_2 x_1^2 + \gamma_3 x_3^2 x_1^2 + \gamma_4 x_3 x_1 + \gamma_5 &= 0
\end{aligned}
$$

The degree of the 1st equation is 4 (because of the $x_1^2 x_2^2$ term).

The degree of each of the other two equations is 4, too.

So Bezout theory tells us that the maximum number of roots is $4^3 = 64$.

But many of these roots are at infinity: for this problem, 48 of them are.

How to solve the system

Idea 1: Choose some different starting guesses and run our favorite nonlinear equation solver on the problem, starting from each of the guesses.

Favorite solver: quasi-Newton.

Initial guesses: 2 per orthant.

Problem: Many different starting guesses may lead to same solution.

Idea 2: Use a homotopy method.

Easy function:

- We want a polynomial with the same structure: each equation should be degree 4 and the highest-order terms should involve the same variables.
- But we need it to be solved easily, for example:

$$\begin{aligned}(x_1^2 - \hat{a})(x_2^2 - \hat{c}) &= 0\\ (x_2^2 - \hat{b})(x_3^2 - \hat{a}) &= 0\\ (x_3^2 - \hat{c})(x_1^2 - \hat{b}) &= 0\end{aligned}$$

  where $\hat{a}$, $\hat{b}$, and $\hat{c}$ are three numbers.

- Note that there are 16 real solutions to our easy problem: $(\pm\sqrt{\hat{a}}, \pm\sqrt{\hat{b}}, \pm\sqrt{\hat{c}})$ and $(\pm\sqrt{\hat{b}}, \pm\sqrt{\hat{c}}, \pm\sqrt{\hat{a}})$.

So if we call our real problem $p(x) = 0$ and our easy problem $q(x) = 0$, then our homotopy becomes

$$\rho(\lambda, x) = \lambda p(x) + (1 - \lambda) q(x)\,.$$

It can be shown that this homotopy, started from each of the 16 solutions to the easy problem, gives a path to a distinct one of the 16 finite real and complex solutions to our original polynomial.

Let's compare this to quasi-Newton.