AMSC/CMSC 661 Quiz 1 , Spring 2005

1. (10) Let $\bar{\Omega} = [0,1]$ and let

$$u(x) = e^{5x} + x^2.$$

Evaluate $\|u\|_{L_2}$, $\|u\|_{\mathcal{C}}$, and $\|u\|_1$.

Answer: (I didn't mean to choose a u for which the integrations would be complicated. Sorry. Because of this, I gave extra credit on Problem 1 for completing the integrations, and on Problem 2 for providing lots of information about u.)

$$\|u\|_{L_2}^2 = \int_0^1 u(x)^2 dx = \int_0^1 (e^{5x} + x^2)^2 dx = \int_0^1 e^{10x} + x^4 + 2x^2 e^{5x} dx$$

We use integration by parts to evaluate the $2x^2e^{5x}$ term:

$$\begin{aligned}
\int_0^1 x^2 e^{5x} dx &= x^2 \frac{e^{5x}}{5}|_0^1 - \int_0^1 2xe^{5x}/5dx \\
&= \frac{e^5}{5} - 2x\frac{e^{5x}}{25}|_0^1 + \int_0^1 2e^{5x}/25dx \\
&= \frac{e^5}{5} - \frac{2e^5}{25} + \frac{2e^{5x}}{125}|_0^1 \\
&= \frac{e^5}{5} - \frac{2e^5}{25} + \frac{2e^5}{125} - \frac{2}{125}.
\end{aligned}$$

so

$$\|u\|_{L_2} = \sqrt{\frac{e^{10}}{10} - \frac{1}{10} + \frac{1}{5} + \left(\frac{2}{5} - \frac{4}{25} + \frac{4}{125}\right) e^5 - \frac{4}{125}}.$$

$\|u\|_{\mathcal{C}} = \max_{x\in[0,1]} |u(x)| = u(1) = e^5 + 1.$

$|u|_1^2 = \int_0^1 u'(x)^2 dx = \int_0^1 (5e^{5x} + 2x)^2 dx$ can also be evaluated using integration by parts.

$\|u\|_1 = \sqrt{\|u\|_{L_2}^2 + |u|_1^2}.$

2. (10) Consider the differential equation

$$-u'' + 8.125\pi \cot((1+x)\pi/8)u' + \pi^2 u = -3\pi^2 \text{ on } \Omega = (0,1)$$

with boundary conditions $u(0) = -2.0761$, $u(1) = -2.2929$. Without using a Green's function or an explicit solution to the problem, tell me about the solution: Does it exist? Is it unique? What are upper and lower bounds on the solution? Justify each of your answers by citing a theorem and verifying its hypotheses. (Hint: One bound can be obtained by comparing the solution to $u(x) = -3$.)

Answer:

- We let $a(x) = 1 > 0$, $b(x) = 8.125\pi \cot((1+x)\pi/8)$, $c(x) = \pi^2 > 0$, and $f(x) = -3\pi^2$. These are all smooth functions on $\bar{\Omega}$.
- Therefore, the Green's Function Theorem tells us that the solution exists (and is equal to the function in the theorem minus $2.0761U_0(x)$ minus $2.2929U_1(x)$).
- Cor 2.2a says that the solution is unique.
- Since $f(x) < 0$, the Maximum Principle tells us that $\max_{x\in\Omega} u(x) \leq \max(-2.0761, -2.2929, 0) = 0$.
- Letting $v(x) = -3$, we see that

$$-v'' + 8.125\pi \cot((1+x)\pi/8)v' + \pi^2 v = -3\pi^2$$

  and $v(0) = v(1) = -3$. Therefore the Monotonicity Theorem (Cor 2.2c in the notes) says that $u(x) \geq v(x)$ for $x \in \bar{\Omega}$.
- Therefore we conclude $-3 \leq u(x) \leq 0$ for $x \in \bar{\Omega}$.

Note on how I constructed the problem: The true solution to the problem is $u(x) = \cos((1+x)\pi/8) - 3$, which does indeed have the properties we proved about it. But we can obtain a lot of information about the solution (as illustrated in this problem) without ever evaluating it!