AMSC/CMSC 661 Quiz 2 , Spring 2005

1a. (3) Write the definition of $|u|_{C^2(\bar{\Omega})}$ when the domain is $\bar{\Omega} = [0, 1]$.

Answer:

From p.6,

$$|u|_{C^2(\bar{\Omega})} = \max_{x\in[0,1]} |u''(x)|.$$

1b. (7) Prove that for $x \in \bar{\Omega}$ and $u \in C^2(\bar{\Omega})$,

$$\left| u'(x) - \frac{u(x+h) - u(x)}{h} \right| \leq Ch|u|_{C^2(\bar{\Omega})}.$$

Answer: Taylor series tells us that there is a point $\xi \in [x, x + h]$ so that

$$u(x + h) = u(x) + hu'(x) + \frac{h^2}{2} u''(\xi).$$

Therefore,

$$\left| u'(x) - \frac{u(x+h) - u(x)}{h} \right| = \frac{h}{2} |u''(\xi)| \leq Ch|u|_{C^2(\bar{\Omega})},$$

at least if $x, x + h \in \bar{\Omega}$.

2. Consider the problem

$$-u'' + \pi u = f \text{ on } \Omega = (0,1)$$

with boundary conditions $u'(0) = u'(1) = 0$.

a. (5) Write the weak formulation. (Use test functions $v \in H^1$.)

Answer: The derivation proceeds as for the problem with the old boundary conditions:

$$\begin{aligned} \int_0^1 (-u'' + \pi u) v dx &= \int_0^1 f v dx \\ \int_0^1 (u'v' + \pi uv) dx - u'v \Big|_0^1 &= \int_0^1 f v dx \end{aligned}$$

but this time the boundary term is zero since $u'(0) = u'(1) = 0$. Therefore, the weak formulation is

$$\int_0^1 (u'v' + \pi uv) dx = \int_0^1 f v dx$$

for all $v \in H^1$.

b. (5) Show that if $u \in C^2(\bar{\Omega})$ and u solves the weak formulation, then u solves the differential equation and satisfies the boundary conditions.

Answer: As before, we reverse the argument: for all $v \in H^1$,

$$\begin{aligned} \int_0^1 f v dx &= \int_0^1 (u'v' + \pi uv) dx \\ &= \int_0^1 (-u'' + \pi u) v dx + u'v \Big|_0^1 \end{aligned}$$

Since $H_0^1 \subset H^1$, we know that for all $v \in H_0^1$,

$$\int_0^1 f v dx = \int_0^1 (-u'' + \pi u) v dx,$$

and, as before, this shows that u satisfies the differential equation.

Once we know that u satisfies the differential equation, then for any $v \in H^1$, we have

$$\int_0^1 (-u'' + \pi u) v dx - \int_0^1 f v dx = 0,$$

so

$$u'v \Big|_0^1 = u'(1)v(1) - u'(0)v(0) = 0$$

for all $v \in H^1$. If we choose a v satisfying $v(1) = 0$ and $v(0) = 1$, we see that $u'(0) = 0$. Similarly choose a v so that $v(1) = 1$ to conclude that $u'(1) = 0$, so u also satisfies the boundary conditions.