AMSC/CMSC 661 Quiz 3 , Spring 2005

1. Consider solving the equation

$$-\bigtriangledown \cdot \bigtriangledown u = f$$

in Ω, with $u = 0$ on the boundary of Ω, using piecewise linear finite elements, which gives us a linear system of equations $\mathbf{AU} = \mathbf{g}$. The domain Ω (see figure) has been divided into 26 triangles.
Let P be the point where triangles 3 and 7 intersect, and let Q be the point where triangles 14 and 20 intersect.
Let ϕ_P be the basis function that is 1 at P, and let ϕ_Q be the basis function that is 1 at Q.

a. (2) What is the dimension of the matrix $\mathbf{A}$?

Answer: 8×8, since there are 8 interior vertices.

b. (2) One of the matrix entries is equal to $a(\phi_P, \phi_Q)$. Which triangles are used in computing this entry?

Answer: 15, 17.

c. (2) How many nonzeros are in the row of the matrix corresponding to the equation $a(u_h, \phi_P) = (f, \phi_P)$?

Answer: 5, since the domain where ϕ_P is nonzero overlaps the domain of 4 other basis functions and we also have the element $a(\phi_P, \phi_P)$.

d. (2) How many triangles are used in computing (f, ϕ_P)?

Answer: 6. (They are 3, 4, 5, 7, 15, and 17.)

2. (10) Consider the differential equation

$$-5u_{xx} - u_{yy} + \pi^2 u = -3\pi^2$$

on the unit square $\Omega = (0,1) \times (0,1)$ with boundary conditions $u(x,y) = x^2 + y^3$ on the boundary. Without using a Green's function or an explicit solution to the problem, tell me about the solution:
(3 pts) Does it exist?
(2 pts) Is it unique?
(5 pts) What are upper and lower bounds on the solution?
Justify each of your answers by citing a theorem and verifying its hypotheses.

Answer: First, notice that none of the theorems in the book apply to this problem, because

- The boundary has corners, so it is not smooth. (The region is a convex polygon, though.)
- The function a in this problem is a matrix, not a scalar:

$$a = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, some of Chapter 3 (including Thm 3.6) assumes $u = 0$ on the boundary.

But we can verify that $c \geq 0$, a is positive definite (the generalization of "nonnegative" for matrices), $f = -3\pi^2$ is smooth, and the boundary condition is smooth.
It is true that the solution is unique, by a generalization of Thm 3.6 or the Green's theorem.
The solution exists, by a generalization of Green's theorem or a generalization of Cor 3.2a (class notes).
A generalization of the Maximum Principle shows that u in Ω is bounded above by its maximum on the boundary, which is 2.
As in Quiz 1, notice that $v = -3$ satisfies the same differential equation, with boundary condition $v = -3$. So let $w = u - v$. Applying the minimum principle, we see that $w \geq \min(w_\Gamma, 0) = 0$, so $u \geq -3$ in Ω.