AMSC/CMSC 661 Quiz 4 , Spring 2005

1. Suppose we use a piecewise linear finite element algorithm to compute an approximate solution u_h to an elliptic partial differential equation

$$\mathcal{A}u = f$$

in a smooth domain Ω, with $u = 0$ on the boundary of Ω. List the 4 most important sources of error that make our computed solution u_h^{comp} different from u. (If you list more than 4 sources, I'll give full credit if the most important 4 are in your list.)

Answer:

- Approximating the true solution by a piecewise linear function. This introduces errors of $O(h^2)$ and is by far the biggest contribution to the error. We design our algorithm so that all other errors are small relative to this one.
- Approximating Ω by a polygon. This only becomes significant if we use piecewise quadratics or higher.
- Approximating the matrix entries $a(\phi_i, \phi_j)$ by a numerical integration formula, where ϕ_i and ϕ_j are hat functions. The barycentric formula makes this error small relative to $O(h^2)$.
- Approximating the right-hand side entries (f, ϕ_j) by a numerical integration formula.
- Rounding error in the algorithm used to solve the linear system and in the function evaluation.
- We also might have errors due to poor meshing (small angles), ill-conditioning in the matrix, discontinuous coefficients, etc.

2a. Let $\mathcal{A}u = -u_{xx} - u_{yy}$, and verify that for any positive integers j and k,

$$v_{jk}(x, y) = \sin(j\pi x)\sin(k\pi y)$$

is an eigenfunction of $\mathcal{A}$. What is the corresponding eigenvalue?

Answer: Let $v = \sin(j\pi x)\sin(k\pi y)$. Then

$$\begin{aligned} -v_{xx} &= (j\pi)^2 \sin(j\pi x)\sin(k\pi y) \\ -v_{yy} &= (k\pi)^2 \sin(j\pi x)\sin(k\pi y) \end{aligned}$$

so

$$\mathcal{A}v = (j^2 + k^2)\pi^2 \sin(j\pi x)\sin(k\pi y).$$

Therefore, v_{jk} is an eigenfunction with eigenvalue $\lambda_{jk} = (j^2 + k^2)\pi^2$.

2b. Note that $v_{jk} = 0$ on the boundary of the unit square $\Omega = (0, 1) \times (0, 1)$. Consider solving the problem

$$-u_{xx} - u_{yy} = f$$

in Ω, with $u = 0$ on the boundary of Ω. Express the solution u in terms of the eigenfunctions of $\mathcal{A}$ and give a computable expression for the coefficients of the eigenfunctions.

Answer: Let

$$u = \sum_{j,k} \alpha_{jk} v_{jk}, \quad f = \sum_{j,k} \beta_{jk} v_{jk}$$

where the sums run from $j, k = 1$ to ∞, and the coefficients α_{jk} and β_{jk} are to be determined. Then

$$\mathcal{A}u = \mathcal{A}\sum_{j,k} \alpha_{jk} v_{jk} = \sum_{j,k} \alpha_{jk} \mathcal{A} v_{jk} = \sum_{j,k} \alpha_{jk} \lambda_{jk} v_{jk},$$

and, since the eigenfunctions are orthogonal, we have by Lemma 6.1a

$$\beta_{jk} = (f, v_{jk})/(v_{jk}, v_{jk}).$$

Equating terms in $\mathcal{A}u = f$ (or, equivalently, taking the inner product of both sides of the equation $\mathcal{A}u = f$ with v_{jk}), we see that

$$\alpha_{jk} = \frac{(f, v_{jk})}{\lambda_{jk}(v_{jk}, v_{jk})}$$

where

$$(u, v) = \int_0^1 \int_0^1 u(x, y)v(x, y)dxdy\,.$$