AMSC/CMSC 661 Quiz 8 , Spring 2005

1. (10) Consider the equation

$$u_t + 2u_x - 2u_y + 6u = 5(x + y + t)^2$$

where the domain Ω is the unit circle, and $u : \Omega \times \mathcal{R}_+ \to \mathcal{R}$. Draw the unit circle, and mark on it the points that define the inflow boundary for this problem. Justify your answer.

Answer: The inflow boundary is the set of points on the unit circle satisfying $a \cdot n < 0$, where n is the exterior normal and $a = [2, -2]^T$. Therefore, we need $2n_1 - 2n_2 < 0$ or $n_1 < n_2$. This is satisfied for points on the unit circle corresponding to $\pi/4 < \theta < 5\pi/4$.

2. (10) Consider the problem

$$u_{tt} - c^2 \Delta u = e^{-i\omega t} f(x)$$

with initial conditions $u(x,0) = u_t(x,0) = 0$ for $x \in \Omega \subset \mathcal{R}^d$.
a. Assume that the solution is of the form $u = e^{-i\omega t} z(x)$. Substitute this solution into the equation to obtain a problem of the form of the Helmholtz equation

$$-\Delta z - \kappa^2 z = g$$

with z given on the boundary of Ω. How are g and κ defined?
b. Let $\kappa = 4$ and let Ω be the square $(-1,1) \times (-1,1)$. Suppose we make a Galerkin finite element approximation to this problem. This gives us a linear system of equations to solve. In Homework 3, you solved a similar system of equations using conjugate gradients. Why can't conjugate gradients be used on our new linear system?

Answer:

a. With this definition of u, we see that $u_{tt} = (-i\omega)^2 e^{-i\omega t} z(x)$, so our equation becomes

$$(-i\omega)^2 e^{-i\omega t} z(x) - c^2 e^{-i\omega t} \Delta z(x) = e^{-i\omega t} f(x),$$

so

$$-\Delta z - \frac{\omega^2}{c^2} z = \frac{1}{c^2} f.$$

Therefore, $\kappa = \omega / c$ and $g(x) = f(x)/c^2$.
b. We know from Homework 3 that the Laplacian on the given domain has eigenvalues less than 16 and bigger than 16. Therefore, the matrix approximation to $\Delta - \kappa^2 I$ with $\kappa^2 = 16$ will have both positive and negative eigenvalues. Conjugate gradients requires a positive definite matrix and therefore cannot be applied.