AMSC/CMSC 661 Scientific Computing II
Spring 2010
Solution of Elliptic PDEs
Part 1
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These notes are based on the 2003 textbook
of Stig Larsson and Vidar Thomée.

Elliptic Partial Differential Equations

The plan:

- The problem and boundary conditions
- An important special case
- A motivating problem
- The Maximum Principle
- The Green's function
- The variational formulation
- Solution and error estimates using finite differences
- Solution and error estimates using finite elements

Note the parallel with ODE-BVP presentation.

The philosophy:

- Emphasize what is different.
- Omit proofs if we have seen the main idea before.
- Concentrate on computational aspects.

Reference: Chapters 3-5.

The problem and boundary conditions (p. 25)

Find the function $u(x)$ that satisfies

$$\mathcal{A}u = -\bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu = f \text{ in } \Omega \subset \mathcal{R}^d$$

where the functions $a(x), b(x), c(x), f(x)$ are given, subject to appropriate boundary conditions on $\Gamma = \bar{\Omega} - \Omega$:

- The Dirichlet problem specifies function values. $u(x) = g(x)$ for $x \in \Gamma$.
- The Neumann problem specifies the normal derivative.

$$\frac{\partial u(x)}{\partial n} = g(x)$$

- Robin's boundary conditions specify some linear combination.

$$a\frac{\partial u(x)}{\partial n} + h(u - g) = 0$$

  for $x \in \Gamma$.
- Mixed boundary conditions specify Dirichlet conditions on part of Γ and Neumann conditions on the rest.

Assumptions:

- The coefficients a, b, and c may depend on x.
- The coefficients a, b, and c are smooth functions and so are f and g; i.e., they have as many continuous derivatives as we need.
- $a(x) \geq a_0 > 0$ for $x \in \bar{\Omega}$. (Why?)
- $c(x) \geq 0$ for $x \in \bar{\Omega}$. (The reason is not as obvious.)

An important special case: origin of harmonic functions

$$\mathcal{A}u = - \bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu = f \text{ in } \Omega$$

Poisson's equation (p. 26) results from setting $a = 1$, $b = 0$, $c = 0$. For $(x, y, z) \in \mathcal{R}^3$, this gives

$$-\Delta u \equiv -u_{xx} - u_{yy} - u_{zz} = f(x, y, z)$$

and if $f = 0$, we call this Laplace's equation and the solutions are harmonic functions.

This problem is well-studied, and analytic solution formulas exist for many domains Ω. In Section 3.3, the formula is derived for $\mathcal{R}^2$ when Γ is a circle, but we will skip this.

Principle of superposition

Suppose we have solved two somewhat simpler problems:

$$\begin{aligned} \mathcal{A}v &= 0 \text{ in } \Omega, \qquad v = g \text{ on } \Gamma \\ \mathcal{A}w &= f \text{ in } \Omega, \qquad w = 0 \text{ on } \Gamma \end{aligned}$$

Then $u = v + w$ solves our original problem.

This trick can be used to simplify analysis and computation.

A motivating problem (Selvadurai, p. 236)

Here is an example of how ODE-BVPs arise in modeling physical problems.

- Suppose we have a piece of steel that is
  - homogeneous (of uniform content).
  - isotropic (with properties independent of direction of measurement).
- We know that steel conducts heat: it feels cold to the touch, because it conducts heat away from our finger.
- Fourier (1768-1830) derived a good model of this heat conduction, relating the amount of heat entering or escaping from a small piece of the metal to the rate at which the temperature T changes normal to the surface.

We no longer need the wild assumptions we used for the ODE:

- The steel is infinite in y and z (or at least so large that it doesn't matter), but extends from $x = 0$ to $x = 1$,
- and any external source of heat is applied at $(0, y, z)$ for all values of y and z, so the only direction left to study is x.

Now, the steel is a finite domain in 3 dimensions.

Let's see what happens.

According to Fourier's model, the amount of heat entering a volume V of steel is

$$\int_V \bigtriangledown \cdot (a \bigtriangledown T) dV$$

where a is the proportionality between temperature and heat. (This is known as the thermal conductivity of the steel.)

If there is a heat source f within that volume, then it generates an amount of heat equal to

$$\int_V f dV.$$

The heat contained in V is

$$\int_V \rho c \frac{\partial T}{\partial t} dV$$

where ρ and c are two constants depending on the material: ρ is the mass-density of the steel and c is its specific heat.

To balance things out, we must have

$$\int_V \left(\bigtriangledown \cdot (a \bigtriangledown T) + f - \rho c \frac{\partial T}{\partial t} \right) dV = 0,$$

and taking limits over small volumes yields

$$\bigtriangledown \cdot (a \bigtriangledown T) + f = \rho c \frac{\partial T}{\partial t}.$$

Finally, if we assume steady state, in which T is unchanging, we obtain the equation

$$\bigtriangledown \cdot (a \bigtriangledown T) + f = 0,$$

and we can solve this for values of T in the interior of the steel once we know what is happening at the boundary.

With such physical problems in mind, we return to the study of the theory of elliptic PDEs.

The Maximum Principle (p. 26)

$$\mathcal{A}u = - \bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu = f \text{ in } \Omega$$

Theorem 3.1a (p. 26): Assume

- $u \in \mathcal{C}^2(\bar{\Omega})$;
- $\mathcal{A}u \leq 0$ in Ω.

Then

- If $c = 0$, then
$$\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \Gamma} u(x)$$
- If $c(x) \geq 0$ for $x \in \Omega$, then
$$\max_{x \in \bar{\Omega}} u(x) \leq \max(\max_{x \in \Gamma} u(x), 0).$$

Compare: p.16

The Minimum Principle

$$\mathcal{A}u = - \bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu = f \text{ in } \Omega$$

Theorem 3.1b (p. 26): Assume

- $u \in \mathcal{C}^2(\bar{\Omega})$;
- $\mathcal{A}u \geq 0$ in Ω.

Then

- If $c = 0$, then
$$\min_{x \in \bar{\Omega}} u(x) = \min_{x \in \Gamma} u(x)$$
- If $c(x) \geq 0$ for $x \in \Omega$, then
$$\min_{x \in \bar{\Omega}} u(x) \geq \min(\min_{x \in \Gamma} u(x), 0).$$

Compare: p.16

Uses of the Maximum Principle

- Bounding the solution in terms of the data.
- Proving uniqueness of solutions.
- Proving stability of solutions.
- Monotonicity properties.

Bounding the solution in terms of the data

Theorem 3.2 (p. 27): If $u \in \mathcal{C}^2$, then

$$\|u\|_{\mathcal{C}(\bar{\Omega})} \leq \|u\|_{\mathcal{C}(\Gamma)} + C\|\mathcal{A}u\|_{\mathcal{C}(\bar{\Omega})}$$

where the constant C depends on a, b, and c.

Compare: p.17

Usefulness: Even if $f(x)$ is not always ≥ 0 in Ω, we have an upper and lower bound on the solution.

Proving uniqueness

Corollary 3.2a: Our problem has a unique solution.

Proof: (as before) Suppose we have two solutions u and v, and let $w = u - v$. Then

$$\begin{aligned} \mathcal{A}w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \Gamma \end{aligned}$$

Therefore, Theorem 3.2 tells us that $w(x) = 0$ for $x \in \Omega$, so $u = v$. []

Proving stability

Corollary 3.2b: Our problem is stable: small changes in the data make small changes in the solution.

Compare: p.17

Proof: (as before) Suppose that

$$\mathcal{A}u = f_1 \text{ in } \Omega, \;\; u = g_1 \text{ on } \Gamma,$$

$$\mathcal{A}v = f_2 \text{ in } \Omega \;\; v = g_2 \text{ on } \Gamma.$$

Then, letting $w = u - v$, we see that

$$\begin{aligned} \mathcal{A}w &= f_1 - f_2 \quad \text{in } \Omega, \\ w &= g_1 - g_2 \quad \text{on } \Gamma. \end{aligned}$$

Now apply the stability estimate Theorem 3.2 to w:

$$\|w\|_{\mathcal{C}(\bar{\Omega})} \leq \|g_1 - g_2\|_{\mathcal{C}(\Gamma)} + C\|f_1 - f_2\|_{\mathcal{C}(\bar{\Omega})},$$

and stability is established. []

The Green's function (p. 30)

$$\mathcal{A}u = -\bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu = f \text{ in } \Omega$$

For convenience, again we work with a special case: $b = 0$.

Recall from ODE-BVP that the Green's function gives us a formula for the solution to our problem in terms of simpler problems:

- First handle the function f.
- Then consider the boundary conditions.

The derivation for PDEs is a bit more complicated, and uses the weak formulation of the problem

$$(\mathcal{A}u, \phi) = (f, \phi)$$

for all $\phi \in \mathcal{C}_0^\infty(\mathcal{R}^d)$.

Note: We define the adjoint operator $\mathcal{A}^*$ as the operator that satisfies

$$(\mathcal{A}u, \phi) = (u, \mathcal{A}^* \phi)$$

for all u, ϕ. Since $b = 0$, it can be shown that $\mathcal{A}^* = \mathcal{A}$: i.e., $\mathcal{A}$ is self-adjoint.

Our goals: To prove that a solution exists, and to express the solution in terms of the solution to simpler problems.

We express our solution in terms of the fundamental solution U that satisfies

$$\mathcal{A}U = \delta$$

where δ is the Dirac delta-function (p. 241), defined to be 0 when $x \neq 0$ and to have an integral of 1. (A mathematician would not like this definition, but it will do.)

Note that this means that

$$(U, \mathcal{A}\phi) = \phi(0).$$

We call U the Green's function.

Theorem 3.4 (Green's Function Theorem) (p. 30) If $f \in \mathcal{C}_0^1(\mathcal{R}^d)$, then the solution to the problem $\mathcal{A}u(x) = f(x)$ for $x \in \mathcal{R}^d$ is

$$u(x) = \int_{\mathcal{R}^d} U(x-y) f(y) dy \,.$$

Notes:

- For some problems (e.g., Poisson's equation, p. 31), the function U is known explicitly, so the solution to the differential equation on the infinite domain with f arbitrary can be computed by integration. Easy!
- For finite domains, we need to impose boundary conditions. This leads to a numerical technique called the boundary integral method (older terminology: capacitance matrix techniques), but we won't study it in this course. See Section 14.4 if you are interested.

Proof: (Rather different from the ODE-BVP techniques).

Let $z = x - y$. Then

$$\int_{\mathcal{R}^d} U(x-y)\mathcal{A}\phi(x)dx = \int_{\mathcal{R}^d} U(z)\mathcal{A}\phi(z+y)dz = \phi(y).$$

Therefore,

$$\begin{array}{rcll} (u, \mathcal{A}\phi) = & \int_{\mathcal{R}^d}\int_{\mathcal{R}^d} U(x-y)f(y)dy\mathcal{A}\phi(x)dx & \\ = & \int_{\mathcal{R}^d}\int_{\mathcal{R}^d} U(x-y)\mathcal{A}\phi(x)dxf(y)dy & \text{interchanging order of integration} \\ = & \int_{\mathcal{R}^d} \phi(y)f(y)dy & \text{from our previous equation} \\ = & (f, \phi). & \end{array}$$

We have assumed enough smoothness to use integration by parts, so we get

$$(u, \mathcal{A}\phi) = (\mathcal{A}u, \phi) = (f, \phi)$$

for all $\phi \in \mathcal{C}_0^\infty(\mathcal{R}^d)$, so we have a solution to the problem $\mathcal{A}u = f$, as desired. []

The variational formulation (p. 32)

We have already hinted at the variational formulation, a powerful tool for solving our pde.

A weak formulation of our problem

$$\mathcal{A}u = -\bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu = f \text{ in } \Omega$$

$$u = 0 \text{ on } \Gamma.$$

A change in assumptions:

- a, b, c are smooth functions,
- $a(x) \geq a_0 > 0 \; x \in \Omega$,
- $c(x) - \bigtriangledown \cdot b(x)/2 \geq 0$ for $x \in \Omega$.

Now choose an arbitrary function $v \in C_0^1$, and notice that

$$(\mathcal{A}u, v) = \int_\Omega (-\bigtriangledown \cdot (a \bigtriangledown u) + b \cdot \bigtriangledown u + cu) v dx = \int_\Omega f v dx = (f, v).$$

Now use integration by parts on the first term:

$$(\mathcal{A}u, v) = \int_\Omega (a \bigtriangledown u \cdot \bigtriangledown v + b \cdot \bigtriangledown uv + cuv) dx = \int_\Omega f v dx.$$

Technicalities:

- $\mathcal{C}_0^1$ is dense in H_0^1, so we can take $v \in H_0^1$.
- The solution u lives in H_0^1, so this is good.

So we have shown that if u solves our PDE, then u satisfies the weak formulation:

Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) \equiv (\mathcal{A}u, v) = \int_\Omega (a \bigtriangledown u \cdot \bigtriangledown v + b \cdot \bigtriangledown uv + cuv) dx = \int_\Omega f v dx \equiv (f, v)$$

for all $v \in H_0^1(\Omega)$.

The converse is not quite true; we say that u is a weak solution of our problem if $u \in H_0^1$ satisfies the variational form of the problem, but it must be in C^2 (in fact, $H^2 \cap H_0^1$) to solve the strong (original) form of the problem.

In weakness there is strength

The weak formulation has two important uses:

- It provides a set of numerical methods, called Galerkin methods. These come from enforcing $a(u, v) = (f, v)$ over a subspace of H_0^1. We'll follow up on this when we discuss finite element methods.
- It provides an alternative existence proof for the solution.

Existence and uniqueness for the solution to the weak problem

Theorem 3.6 (p. 33): Under our assumptions $a(x) \geq a_0 > 0$ and $c(x) - \bigtriangledown \cdot b(x)/2 \geq 0$ for $x \in \Omega$, if $f \in L_2$, then there exists a unique solution of $a(u, v) = (f, v)$ for all $v \in H_0^2$, with $\|u\|_1 \leq C\|f\|$, and this solution solves $\mathcal{A}u = f$ in Ω with $u = 0$ on Γ.

Compare with p. 22.

Another important tool: minimization of energy

If $b = 0$, then $a(u, v) = a(v, u)$, so a is both self-adjoint (symmetric) and positive definite. In this case, we can find the solution by minimizing

$$F(u) \equiv \frac{1}{2}a(u, u) - (f, u)$$

for $u \in H_0^1$.

This principle, Dirichlet's principle (Theorem 3.7) is an important computational tool.

Physical interpretation: Suppose we are modeling an elastic membrane attached at its boundary. Then

- $F(u)$ is the potential energy of the membrane, where u is the deflection.
- $a(u, u)$ is the internal elastic energy.
- (f, u) is the load potential.
- In physics, this is sometimes called the principle of minimizing potential energy or minimizing virtual work.

Other boundary conditions

We derived the weak formulation for the homogeneous Dirichlet boundary condition. What about other cases?

Nonhomogeneous Dirichlet: $u = g$ on Γ.

Find $u \in H^1$ such that

$$a(u, v) = (f, v)$$

for all $v \in H_0^1$, with $\gamma u = g$.

Homogeneous Neumann: $\partial u / \partial n = 0$ on Γ. (Assume $c(x) \geq c_0 > 0$ on Ω.)

Find $u \in H^1$ such that

$$a(u, v) = (f, v)$$

for all $v \in H^1$.

(The boundary condition comes into the integration-by-parts formula; it is not enforced explicitly but instead follows naturally from the formulation.)

Existence, uniqueness, and stability can be established (pp. 34-37).

Regularity (p. 37)

Theorem 2.6 (p.37; Compare with p.23) Assume

- smooth coefficients,
- $f \in L_2$.
- Γ smooth, or Γ a convex polygon.

Then

$$\|u\|_2 \leq C\|f\|.$$

where C is independent of f.

This is a regularity result; it shows that u and its 1st and second derivatives can be bounded in terms of the data f, a rather remarkable fact.

Note: The result does not hold for regions in which Γ has an interior corner; for example, for an L-shaped domain.