AMSC/CMSC 661 Scientific Computing II

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Transforms and Wavelets: Supplement

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Important identities for wavelets:

$$\begin{aligned}\phi(\hat{x}/2) &= \phi(\hat{x}) + \phi(\hat{x}-1)\\ \psi(\hat{x}/2) &= \phi(\hat{x}) - \phi(\hat{x}-1)\end{aligned}$$

or, from adding and subtracting these identities,

$$\begin{aligned}\phi(\hat{x}) &= \frac{1}{2}\phi(\hat{x}/2) + \frac{1}{2}\psi(\hat{x}/2)\\ \phi(\hat{x}-1) &= \frac{1}{2}\phi(\hat{x}/2) - \frac{1}{2}\psi(\hat{x}/2)\end{aligned}$$

Approximation to $f(x)$ using intervals of length 2^{-m} (e.g., step function, if we use Haar wavelets):

$$f_m(x) = \sum_{k=-\infty}^{\infty} a_m[k]\phi(2^m x - k)$$

Now double-up the terms

$$\begin{aligned}f_m(x) &= \sum_{k=-\infty}^{\infty} a_m[2k]\phi(2^m x - 2k) + a_m[2k+1]\phi(2^m x - 2k - 1)\\ &= \sum_{k=-\infty}^{\infty} a_m[2k]\phi(2(2^{m-1}x - k)) + a_m[2k+1]\phi(2(2^{m-1}x - k) - 1)\end{aligned}$$

Now use the two identities above with $\hat{x} = 2(2^{m-1}x - k)$.

$$\begin{aligned}f_m(x) &= \sum_{k=-\infty}^{\infty} a_{m-1}[k]\phi(2^{m-1}x - k) + b_{m-1}[k]\psi(2^{m-1}x - k)\\ &\equiv f_{m-1}(x) + d_{m-1}(x),\end{aligned}$$

where

$$\begin{aligned}a_{m-1}[k] &= \frac{1}{2}a_m[2k] + \frac{1}{2}a_m[2k+1]),\\ b_{m-1}[k] &= \frac{1}{2}a_m[2k] - \frac{1}{2}a_m[2k+1]).\end{aligned}$$

$f_{m-1}(x)$ is called the frame and $d_{m-1}(x)$ is called the detail.

Formula for the frame coefficients:

$$a_{m-1}[k] = 2^{m-1}\int_{-\infty}^{\infty} f(x)\phi(2^{m-1}x - k)\,dx.$$