AMSC/CMSC 661 Scientific Computing II
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Solution of Ordinary Differential Equations, Boundary Value Problems
Part 2

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These notes are based on the 2003 textbook of Stig Larsson and Vidar Thomée.

Solution and error estimates using finite differences (p. 43)

Note: A small part of this material is covered in 660, too.

Notation (a slight change):

$$\mathcal{A}u = -au'' + bu' + cu = f$$

with a, b, c smooth and $a(x) > 0$, $c(x) \geq 0$ in $\bar{\Omega}$.

We would like to write down an approximation to this equation that would permit us to solve for values of u at selected points in $[0, 1]$.

Unquiz 2: Suppose u has 4 continuous derivatives. Prove that **the central difference approximations** satisfy

$$\begin{aligned} u'(x) &= \frac{\mathbf{u(x+h) - u(x-h)}}{\mathbf{2h}} + O(h^2)\,, \\ u''(x) &= \frac{\mathbf{u(x-h) - 2u(x) + u(x+h)}}{\mathbf{h^2}} + O(h^2) \end{aligned}$$

for small values of h. []

More formally,

$$\left| u''(x) - \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} \right| \leq Ch^2 |u|_{\mathcal{C}^4}$$

and similarly for $u'(x)$, where

$$|u|_{\mathcal{C}^4} = \max_{x \in \bar{\Omega}} |u''''(x)|$$

So the **finite difference approach** is to choose **mesh points** $x_j = jh$, where $h = 1/M$ for some large integer M, and solve for $u_j \approx u(x_j)$ for $j = 0, 1, \ldots, M$.

Unquiz 3: Consider the equation

$$\mathcal{A}u = -u'' + bu' + u = f$$

where $b(x) = x$. Let $M = 5$, and write the 4 finite difference equations for u at $x = .2, .4, .6$, and $.8$. []

Properties of the finite difference formulation:

- We obtain a system of linear equations $AU = g$, where g is determined by the function f and the boundary conditions.
- A is $(M-1)\times(M-1)$ and **tridiagonal**. In the jth row, the main diagonal element is $2a_j/h^2 + c_j$ and the off-diagonal elements are $-(a_j/h^2 \pm b_j/(2h))$. **(The book's A is h^2 times ours.)**
- For small enough h, the matrix A is **row diagonally dominant**: the main diagonal element is at least as big as the sum of the absolute values of the off-diagonal elements. This ensures that the matrix **has no zero eigenvalues** and therefore a unique solution U exists.

Now we need an error estimate, which we obtain from

- a discrete maximum principle.
- a stability estimate.

A discrete maximum principle

Lemma 4.1 (p. 44): Assume h is small enough that $a_j \pm \frac{1}{2}hb_j \geq 0$ and that $AU \leq 0$.

- (i) If $c = 0$, then

$$\max_j U_j = \max(U_0, U_M).$$

- (ii) If $c \geq 0$ then

$$\max_j U_j \leq \max(U_0, U_M, 0).$$

Proof of (i): The jth equation $(1 \leq j \leq M-1)$:

$$2a_jU_j/h^2 - (a_j + hb_j/2)U_{j-1}/h^2 - (a_j - hb_j/2)U_{j+1}/h^2 = g_j \leq 0$$

so

$$\begin{aligned} U_j &= \mathbf{\frac{h^2}{2a_j}} g_j + \frac{a_j - hb_j/2}{2a_j}U_{j+1} + \frac{a_j + hb_j/2}{2a_j}U_{j-1} \\ &\leq \frac{a_j - hb_j/2}{2a_j}U_{j+1} + \frac{a_j + hb_j/2}{2a_j}U_{j-1}. \end{aligned}$$

Suppose U_j is the maximum. Then $U_j = U_{j-1} = U_{j+1}$ because the coefficients on the right add to 1. Continuing this reasoning, we see that U is constant, so the result holds. Therefore, either U is constant or the max occurs at an endpoint. []

A stability estimate

We use the ∞-norm of the vector U:

$$\|U\|_\infty = \max_j |U_j|.$$

Lemma 4.2 (p. 45): If $b = 0$, then

$$\|U\|_\infty \leq \max(|U_0|, |U_M|) + C\|AU\|_\infty,$$

where C depends on $\mathcal{A}$ but not h or U.

Proof: Let $w(x) = x - x^2$, $W_j = w(x_j)$, and

$$\alpha = \min_{x \in \bar{\Omega}} a(x).$$

Then

$$\begin{aligned}
(AW)_j &= (2a_j + h^2 c_j)W_j/h^2 - a_j W_{j-1}/h^2 - a_j W_{j+1}/h^2 \\
&= c_j W_j + \frac{a_j(\mathbf{2x_j} - 2x_j^2 - (\mathbf{x_j} - \mathbf{h}) + (x_j - h)^2 - (\mathbf{x_j} + \mathbf{h}) + (x_j + h)^2)}{h^2} \\
&= c_j W_j + 2a_j \\
&\geq 2\alpha.
\end{aligned}$$

Now let

$$V_j^{\pm} = \pm U_j - (2\alpha)^{-1}\|AU\|_\infty W_j,$$

so that

$$(AV)_j^{\pm} = \pm(AU)_j - (2\alpha)^{-1}\|AU\|_\infty (AW)_j \leq 0.$$

Since $W_0 = W_M = 0$, we conclude from Lemma 4.1 that

$$V_j^{\pm} = \pm U_j - (2\alpha)^{-1}\|AU\|_\infty W_j \leq \max(|U_0|, |U_M|)$$

and therefore

$$|U_j| \leq \max(|U_0|, |U_M|) + (2\alpha)^{-1}\|AU\|_\infty |W_j|,$$

and since

$$\max_j |W_j| = \max_j x_j - x_j^2 = \max_j 1/4 - (x_j - 1/2)^2 = 1/4$$

the result follows with $C = 1/(8\alpha)$. []

The error in the finite difference solution

Theorem 4.1 (p. 45): If $b = 0$, then

$$\max_j |U_j - u(x_j)| \leq Ch^2 \|u\|_{C^4}.$$

Proof: Let $e_j = U_j - u(x_j)$. Then by Unquiz 2,

$$|(Ae)_j| \leq Ch^2 \|u\|_{C^4},$$

so the result follows from Lemma 4.2, noting that $e_0 = e_M = 0$. []

Summary

- The finite difference approximation to our problem leads to a system of linear equations to be solved.
- The approximation is $O(h^2) = O(M^{-2})$, so the more accuracy we need in the solution, the larger the system.
- To get approximations to the solution at points between mesh points, we could use **interpolation**; see van Loan's text for details.

Solution and error estimates using finite elements (p. 51)

Notation:

$$\mathcal{A}u = -(au')' + cu = f \text{ in } \Omega = (0,1)$$

with $u(0) = u(1) = 0$.

Assumptions:

- $a(x)$ and $c(x)$ smooth functions.
- $a(x) \geq \alpha > 0$, $c(x) \geq 0$ in $\bar{\Omega}$.
- $f \in L_2(\Omega)$.

Recall the **variational formulation**

$$a(u,v) = (f,v), \quad v \in H_0^1$$

where

$$\begin{aligned} a(u,v) &= \int_{\Omega} (au'v' + cuv)dx \\ (f,v) &= \int_{\Omega} fv dx \end{aligned}$$

As in finite differences, we choose a **mesh** $0 = x_0 < x_1 < \ldots < x_M = 1$.

$$\begin{aligned} h_j &= x_j - x_{j-1}, \\ K_j &= [x_{j-1}, x_j], \\ h &= \max_j h_j. \end{aligned}$$

But rather than solve for u at the mesh points, we seek an approximate solution of a particular form:

- continuous,
- satisfying the boundary conditions,
- and **piecewise linear** in each of the subintervals K_j.

We call the space of such functions S_h and note that it is a subset of H_0^1, the space where the solution lives.

A convenient basis

We can construct our solution using **any** basis for S_h, but one basis is particularly convenient: the set of **hat functions** ϕ_i, $i = 1, \ldots, M-1$, where

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x - x_{i+1}}{x_i - x_{i+1}} & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

These are designed to satisfy $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ if $i \neq j$.

Picture.

Any function $v \in S_h$ can be written as

$$v(x) = \sum_{i=1}^{M-1} v_i \phi_i(x)$$

where $v_i = v(x_i)$.

The resulting equations

Our original problem: Find $u \in H_0^1$ satisfying

$$a(u, v) = (f, v)$$

for all $v \in H_0^1$.

Our new problem: Find $u_h \in S_h$ satisfying

$$a(u_h, v) = (f, v)$$

for all $v \in S_h$.

Because the ϕ_i form a basis, our new problem becomes: Find

$$u_h = \sum_{i=1}^{M-1} u_i \phi_i(x)$$

satisfying

$$a(u_h, \phi_j) = (f, \phi_j)$$

for $j = 1, \ldots, M-1$.

Unquiz 4: Write the resulting system of equations $AU = g$ and compare with the answer to Unquiz 3. []

Some properties

- This method of constructing the discrete equations is called **Galerkin's method** and is characterized by seeking u_h in some subspace of the space H_0^1 that contains the solution, and making the residual $a(u_h, v) - (f, v)$ zero on that subspace.
- A is called the **stiffness matrix** and g is called the **load vector**.
- A is symmetric (because $a(\phi_i, \phi_j) = a(\phi_j, \phi_i)$) and $V^T A V = a(v, v) > 0$ when

  $$v = \sum_{i=1}^{M-1} v_i \phi_i(x) \neq 0.$$

  Therefore, the solution exists and is unique.
- A is tridiagonal.

Error analysis

The error analysis of the finite element method proceeds in two steps:

- **Step 1:** Show that for every function $u \in H_0^1$, there is a function $\hat{u}_h \in S_h$ that is close to it.
- **Step 2:** Show that the system of equations yields a solution close to $\hat{u}_h$.

Step 1: Approximability

For any $u \in H_0^1$, let $\hat{u}_h \in S_h$ be defined by

$$I_h u \equiv \hat{u}_h = \sum_{i=1}^{M-1} u(x_i) \phi_i(x).$$

(This is the piecewise linear **interpolating function**.)

A standard result in approximation theory tells us that over the interval K_j we have

$$\begin{aligned} \|I_h u - u\|_{K_j} &\leq Ch_j^2 |u|_{2,K_j}, \\ \|(I_h u)' - u'\|_{K_j} &\leq Ch_j |u|_{2,K_j}. \end{aligned}$$

(The proof follows from Taylor series expansions.)

(Remember notation: $|u|_2 = L_2$ norm of u''.)

So

$$\begin{aligned} \|I_h u - u\| &= \left(\sum_{j=1}^{M-1} \|I_h u - u\|_{K_j}^2 \right)^{1/2} \\ &\leq \left(\sum_{j=1}^{M-1} C^2 h_j^4 |u|_{2,K_j}^2 \right)^{1/2} \\ &\leq Ch^2 \|u\|_2, \end{aligned}$$

and similarly

$$\|(\mathbf{I_h u})' - \mathbf{u}'\| \leq Ch\|u\|_2.$$

Step 2: u_h is close to $I_h u$

We use the **energy norm**

$$\|v\|_a = a(v,v)^{\mathbf{1/2}}.$$

Theorem 5.1a (p. 54):

$$(**) \; \|u_h - u\|_a = \min_{v \in S_h} \|v - u\|_a$$

A note: Let $e = u - u_h$. We know that $a(u,v) = (f,v)$ and $a(u_h, v) = (f,v)$ for all $v \in S_h$, so

$$(***) \; a(e,v) = 0$$

for all $v \in S_h$. This means that the **error is orthogonal to S_h**, or, in other words, u_h is the **orthogonal projection** (with respect to the inner product a) of u onto S_h, and therefore $(**)$ holds, as we now prove in detail.

Proof: Using (***), we see that for any $v \in S_h$,

$$\|e\|_a^2 = a(e,e) = a(e, u - u_h - v) \equiv a(e, u - \hat{v}) \leq \|e\|_a \|u - \hat{v}\|_a,$$

where $\hat{v} = v + u_h \in S_h$. Therefore, $\|e\|_a \leq \|u - \hat{v}\|_a$ for all $\hat{v} \in S_h$. []

Theorem 5.1b (p. 54):

$$\|\mathbf{u_h'} - \mathbf{u'}\| \leq Ch\|u\|_2.$$

Proof: (With more detail added.)
Notice that if $v \in H_0^1$, then

$$\begin{aligned}\|v\|_a^2 &= \int_0^1 a(x)(v'(x))^2 + c(x)v(x)^2 dx \\ &\geq \min_{x\in[0,1]} a(x) \int_0^1 (v'(x))^2 dx \\ &\geq \alpha\|v'\|^2\end{aligned}$$

and

$$\begin{aligned}\|v\|_a^2 &= \int_0^1 a(x)(v'(x))^2 + c(x)v(x)^2 dx \\ &\leq \max_{x\in[0,1]} a(x) \int_0^1 (v'(x))^2 dx + \max_{x\in[0,1]} c(x) \int_0^1 (v(x))^2 dx \\ &\leq C\|v'\|^2 + C\|v\|^2 \\ &\leq C\|v'\|^2,\end{aligned}$$

where the last step follows from equation (2.17). Thus,

$$(*)\ \sqrt{\alpha}\|v'\| \leq \|v\|_a \leq C\|v'\|$$

for $v \in H_0^1$. **Now, Theorem 5.1a implies**

$$\|e\|_a \leq \|z - u\|_a$$

for all $z \in S_h$, and since $z - u \in H_0^1$, by (*) we conclude that

$$\|e\|_a \leq \|z - u\|_a \leq C\|z' - u'\|$$

Therefore,

$$\|e\|_a \leq C \min_{z\in S_h} \|z' - u'\|$$

so, using (*) again,

$$\|e'\| \leq C\|e\|_a \leq C \min_{z\in S_h} \|z' - \mathbf{u'}\|.$$

Now let $z = I_h u$ and use the interpolation bound

$$\|(\mathbf{I_h u})' - \mathbf{u'}\| \leq Ch\|\mathbf{u}\|_\mathbf{2}.[]$$

This is nice, but it gives us a result on the **energy norm**, not the L_2 norm, so we need to work a little more.

Theorem 5.2 (p. 55):

$$\|e\| \leq Ch^2 \|u\|_2.$$

Proof: We use a **duality argument**.

Original problem: Find $u \in H_0^1$ such that $a(u, \phi) = (f, \phi)$ for all $\phi \in H_0^1$.

Dual problem: Find $\phi \in H_0^1$ such that $a(w, \phi) = (w, e)$ for all $w \in H_0^1$.

We proved that

$$\|\phi\|_1 \leq C\|e\|,$$

but it is also true (see (2.22)) that

$$\|\phi\|_2 \leq C\|e\|.$$

Now

$$\begin{array}{rcll} (e,e) = & & \|e\|^2 & \text{(definition)} \\ & = & a(e,\phi) & (a(w,\phi) = (w,e)) \\ & = & a(e, \phi - I_h\phi) & \text{(orthogonality)} \\ & \leq & \|e\|_a \|(\phi - I_h\phi)\|_a & \text{Cauchy-Schwarz} \\ & \leq & C\|e'\| \|(\phi - I_h\phi)'\| & \text{(*)} \\ & \leq & Ch\|e'\| \|\phi\|_2 & \text{(approximability)} \\ & \leq & Ch\|e'\| \|e\|, & \text{(previous equation)} \end{array}$$

so $\|e\| \leq Ch\|e'\|$, and by Theorem 5.1b, this is bounded by $Ch^2\|u\|_2$. []

Higher order approximation

We derived our finite element equation using the space of **piecewise linear functions** (i.e., piecewise polynomials of degree 1) with a convenient basis, the basis of **hat functions**.

We could also use **higher order polynomials**: quadratics, cubics, etc. The basis we choose consists of our old hat functions plus **quadratic** or **cubic** hat functions that vanish at all mesh points.

Picture: p. 57.

Because the approximability properties are better, we get higher order estimates for the error: if we use **piecewise polynomials of degree $r-1$**, then

$$\begin{array}{rcl} \|u - u_h\| & \leq & Ch^r \|u\|_r, \\ \|u' - u_h'\| & \leq & Ch^{r-1} \|u\|_r, \end{array}$$

when $u \in H^r$.

h-p methods

Result: If we want a better approximation, we have two choices:

- decrease h.
- increase r.

The parameter r is often called p in the literature, so the resulting adaptive methods are called **h-p methods**.

Summary

- We have shown existence, uniqueness, and stability of the solution to our ODE-BVP.
- We have introduced several tools for analysis, including
  - the maximum principle,
  - Green's functions,
  - approximability,
  - duality,
  - the energy norm,
  - regularity.
- We have defined a finite difference approximation to the ODE-BVP, reducing the problem to solving a linear system of equations.
- We showed existence and uniqueness of the finite difference approximation, as well as an error bound.
- Omitted: We could also have applied **shooting methods** to solve our ODE-BVP (660).
- We have defined a finite element approximation.
- We showed existence and uniqueness of the finite element approximation, as well as an error bound.