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CMSC661

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Transforms and Wavelets

Introduction:

We've seen transform methods used to solve time-dependent PDEs and also in a discrete fashion to solve discretized PDE in homework 4. We now re-examine these transform methods and investigate new methods.

The idea of the transform methods is that given a function or set of values (n-dimensional vector of values), we put it through some transform "black box" to get a new function or vector of values, respectively.

We will be looking at three particular types of transforms and their properties:
- Fourier transform
- Discrete Fourier transforms
- Wavelet transforms

Some useful transforms we will look at: Fourier transform, discrete Fourier transform, discrete sine transform, Haar transform, and discrete Haar transform. We don't have time to examine other transforms such as the Laplace transforms, Z-transforms, or Daubechies Wavelet transform, but we will develop a good framework so that we can easily understand new transforms encountered in the future.

Most of the material in the slides is from the text "A First Course in Fourier Analysis", by David W. Kammler, which includes many examples.

Setup:

The following are needed for a transform method:

- Domain Ω:
    - Could be $(-\infty,\infty)$, unit square, discrete set of values, etc.
- Inner product (u,v) on Ω
    - Depends on the variable x.
    - We've looked at vector inner products and L2 inner products
- Set of basis functions $z(s,x)$:

  - Contains twice as many variables as inner product.
  - Depends on variables *x* and *s*.
  - Usually orthogonal.
- Function *f(x)* of interest
  - Must be defined over some certain, proper space.

We define the transform of *f* (using the basis functions and inner product) to be:

$$F(s) = (f(x),z(s,x))$$

We define the inverse transform to be the function that maps the transform function *F(s)* back to original *f(x)*.

Example 1a: The Fourier Transform (as before, but 1-dimensional space):

This definition is used by the Larsson and Thomee book.

- Domain $\Omega = (-\infty,\infty)$

- Inner product:

$$(u,v) = \int_{-\infty}^{\infty} u(x)v(x)dx$$

- Basis functions are the exponentials, depends on 2 sets of variables (s and x):

$$z(s,x) = e^{-isx}$$

- Definition of the transform:

$F(s) = \int_{-\infty}^{\infty} f(x)e^{-ixs}dx$, results in a transform function that is a function of s.

- Definition of the inverse transform:

$f(s) = \frac{1}{(2\pi)}\int_{-\infty}^{\infty} F(s)e^{+ixs}ds$, where $\frac{1}{(2\pi)}$ is the normalization constant.

Example 1b: The Fourier Transform (an alternative definition):

There are several ways to define the Fourier transform. The following is another definition that is used in the Kammler book. Whatever you can do with the previous definition of the Fourier transform you can do with this one.

- Domain $\Omega = (-\infty,\infty)$ [no change]

- Inner product [no change]

$$(u,v) = \int_{-\infty}^{\infty} u(x)v(x)dx$$

- Basis functions is an exponential [- sign replaced by 2π] :

  $z(s,x) = e^{2\pi isx}$

- Definition of the transform [- sign replaced by 2π] :

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{2\pi ixs} dx$$

- Def of the inverse transform [$\frac{1}{(2\pi)}$ normalization constant gone, - sign replaced by 2π] :

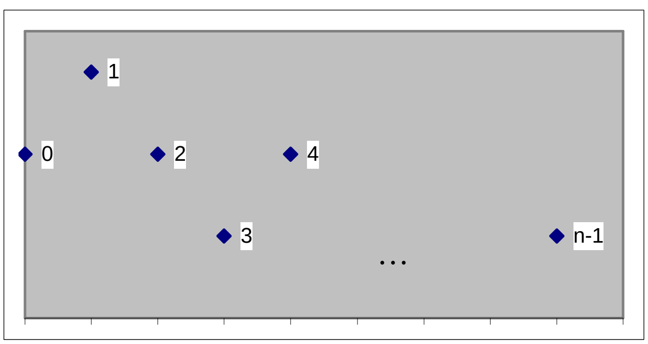
$$f(s) = \int_{-\infty}^{\infty} F(s) e^{-2\pi ixs} ds$$

The advantage of this definition is that it's symmetric; there is no normalization constant.

Example 2: The Discrete Fourier Transform:

- Domain is a set of discrete numbers, Ω = {0,1,…,n-1}

  So we get a function that is justified at *n* points, and we use the function values at these discrete points:



- Inner product:

$$(u, v) = \sum_{x=0}^{n-1} u(x)v(x)$$, *u(x)* and *v(x)* each has *n* different values

- Basis functions is an exponential:

  $z(s,x) = e^{2\pi isx/n}$, same as before except normalized so that x/n ranges from 0…1

- Definition of the transform:

$$F(s) = \sum_{x=0}^{n-1} f(x) e^{2\pi isx/n}$$, x varies while s stays constant.

- Definition of the inverse transform:

$$f(s) = \sum_{s=0}^{n-1} F(s) e^{-2\pi isx/n}$$

Example 3: The Discrete Sine Transform:

Used in homework 4, part 1.

- Domain: Ω = {1,2,…,n}

- Inner product:

$$(u, v) = \sum_{x=1}^{n} u(x)v(x)$$

- Basis functions:

$$z(s,x) = sin(\frac{sx\pi}{n+1}) ,$$

We have *x* start from 1 instead of 0 because we don't want *x*=0, because sin(0) = 0 doesn't do anything for us. Note that as we increase *s* in *z(s,x)*, the resulting sine function will oscillate faster and faster.

- Definition of the transform:

$F(s) = \alpha_s \sum_{x=1}^{n} f(x) \sin(\frac{sx\pi}{n+1})$, normalization factor α_s varies from source to source.

- Definition of the inverse transform:

$f(s) = \alpha_x \sum_{s=1}^{n} F(s) \sin(\frac{sx\pi}{n+1})$, (n+1) term keeps us off the situation of sin(xπ) = 0.

Example 4: The Haar Transform:

- Domain: Ω = [0,1]

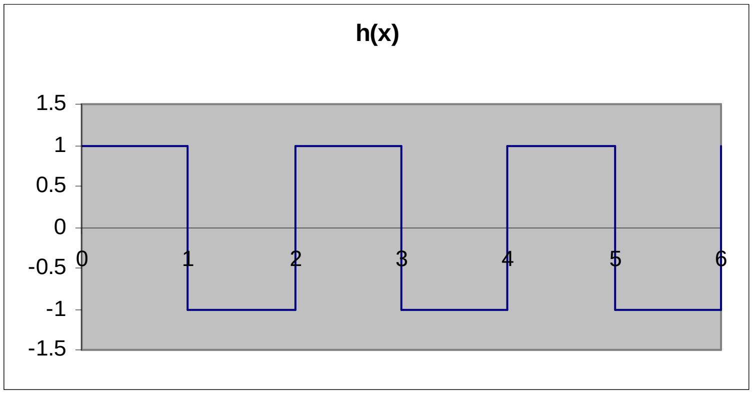
- Inner product:

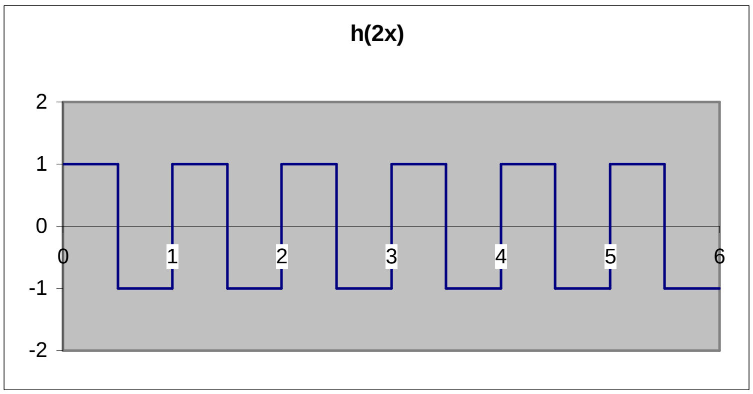
$$(u, v) = \int_0^1 u(x)v(x)dx$$

- Basis functions:

z(s,x) = *h(sx)* , where *h(sx)* is the Haar function

$h(x) = \begin{cases} +1 & if \ 2k \le x < 2k+1 \\ -1 & if \ 2k+1 \le x < 2(k+1) \end{cases}$, where k is an integer





As *s* gets bigger in *z(s,x)*, *h(sx)* is moving from 0 to 1 faster. Thus *h(2x)* oscillates 2 times as fast as *h(x)*, *h(3x)* will oscillate 3 times as fast, etc.

- Definition of the transform:

$$F(s) = \alpha_s \int_0^1 f(x) z(s,x) dx$$, α_s is a normalization factor.

- Definition of the inverse transform:

$$f(s) = \alpha_s \int_0^1 F(s) z(s,x) ds$$

Example 5: The Discrete Haar Transform:

This breaks function into components of various frequencies.

- Domain: Ω = $\{0, \frac{1}{n}, \ldots, \frac{n-1}{2}\}$, where n = 2^k to get a nice boundary.

- Inner product:

$$(u,v) = \sum_{j=0}^{n-1} u(j/n) v(j/n)$$

- Basis functions:

  $z(s,x) = h(sx)$,

- Definition of the transform:

  $$F(s) = \alpha_s \sum_{j=0}^{n-1} f(j/n) z(s, j/n)$$, α_s is a normalization factor.

- Definition of the inverse transform:

  $$f(s) = \alpha_x \sum_{k=0}^{n-1} F(k/n) z(x, k/n)$$

This transform has an advantage over Fourier and sine in that it is easier to compute the integrals.

Review: Important properties of the Fourier Transform

In the Kammler notation:

- Fourier inversion formula. If we inverse Fourier transform something that we Fourier transformed, we get the original function back: $\mathcal{F}^{-1}(\mathcal{F}v)=b$

- Parseval's formula:

  $$\int_{-\infty}^{\infty} f(x)\overline{g}(x)dx = \int_{-\infty}^{\infty} F(s)\overline{G}(s)ds$$, where $\overline{g}(x)$ and $\overline{G}(s)$ are complex conjugates.

- A norm relation: $\|v\| = \|V\| \rightarrow$ Fourier doesn't change norm.

- A translation relation: if $w(x) = v(x-y)$, y is fixed constant, then the Fourier transform of $v(x-y)$ is same as the Fourier transform of $v(x)$ but multiplied by an exponential:

  $w(x) = v(x-y) \rightarrow W(s) = e^{-2\pi iys}V(s)$

- A scaling relation: if $w(x) = v(ax)$, $a > 0$ is a fixed scalar, then the Fourier transform of $v(ax)$ is the Fourier transform of $v(x)$ with s replaced by s/a and scaled by a^{-1}:

  $w(x) = v(ax) \rightarrow W(s) = a^{-1}V(a^{-1}s)$

- A convolution relation:

  If we define $(v * w)(x) = \int_R v(y)w(x - y)dy$

  Then $\mathcal{F}(v*w)(s) = V(s)W(s)$.

  Fourier transforms are useful because they break apart convolutions nicely.

- A differentiation formula that holds as long as v and its derivatives go to zero for large $|x|$: the Fourier transform of $v'(x)$ is the Fourier transform of $v(x)$ multiply by $2\pi is$:

  $\mathcal{F}v'(s) = 2\pi isV(s)$