Lecture Notes on Transforms and Wavelets I

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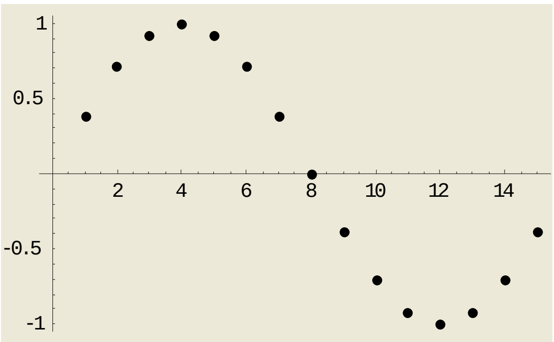
The integral transform is an integral with parameters which establishes a one-to-one mapping between two functions in different domains. For instance, a function $f(t)$ in time domain $(-\infty,\infty)$ represents a signal can be mapped to a function $\hat{f}(\omega)$ in frequency domain which characterizes the spectral property of the signal by Fourier transform. The domains are usually interpreted from physical perspective, and can help to understand the properties of the transform better.

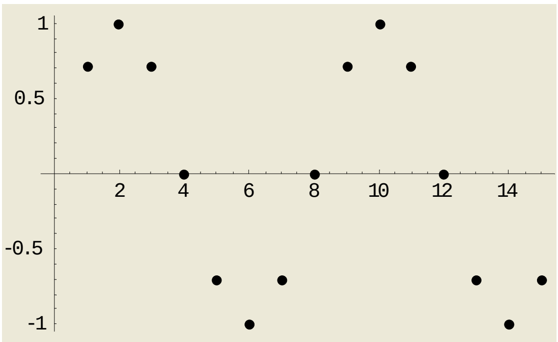
An integral transform needs 4 ingredients: a domain Ω on which both the original function and its image function are defined, $(-\infty,\infty)$ for Fourier transform for example; an inner product which defines the operation of the original (or image) function and the basis function (kernel); the basic function is a function with two sets of variables, one of which is the variables of the original function and the other is of the image function (therefore the basis function has twice as many variables as the original (or image) function; the inverse transform defines the mapping from the image function back to the original one. The definition of inner product usually requires some math property of the original (or image) function. For example, Fourier transform deals with function that is integrable on $(-\infty,\infty)$, that is $f(t)\in L(R)$ and $\hat{f}(\omega)\in L(R)$.

Discrete transform shares the same essence with integral transform, only replacing the integral by sum of the integrand values on discrete (and usually uniform) points. The discrete Sine transform is defined as $F(s)=\alpha_s\sum_{x=1}^{n} f(x)\sin(sx\pi/(n+1))$, where $s\in\{1,2,\ldots n\}$ to make each term in the summation nonzero if $f(x)$ is nonzero.

The basis function is usually a set of orthogonal functions with respect to both domains, under the definition of inner product mentioned above. For example, for $s_1,s_2\in\{1,2,\ldots n\}$, $s_1\neq s_2$, $\alpha_s\sum_{x=1}^{n}\sin(s_1x\pi/(n+1))\sin(s_2x\pi/(n+1))=0$. Therefore the function $f(x)=\sum_{s=1}^{n} c_s\sin(sx\pi/(n+1))$ is determined by $c_s=F(s)=\alpha_s\sum_{x=1}^{n} f(x)\sin(sx\pi/(n+1))$, by the orthogonality of the basis function.

The parameter in the basis function often stands for frequency. For the discrete Sine transform, $\sin((2s)x\pi/(n+1))$ is a function oscillating twice as fast as $\sin((s)x\pi/(n+1))$. The list plot of $\sin(2x\pi/(15+1))$ and $\sin(4x\pi/(15+1))$ are as follows:





Ingrid Daubechies is one of the most famous pioneers in wavelet research. In 1988, she proposed smooth orthogonal wavelet basis with compact support known as Daubechies wavelet basis. Her “Ten lectures on wavelets” was the beginning of an upsurge of research in this intriguing area. Please refer to http://www.pacm.princeton.edu/~ingrid/ if you are interested.