AMSC/CMSC 661 Quiz 5 , Spring 2010

1. (6) The following is the CG algorithm for solving the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. As usual, $\boldsymbol{A}$ is a symmetric positive definite matrix with nz nonzero elements and $\boldsymbol{b}$ is a vector of length n. How much storage does the algorithm use? How many multiplications and divisions does it perform? (Express your answers in terms of the parameters n, K, and nz.)

Let $\boldsymbol{r} = \boldsymbol{b}$, $\boldsymbol{x} = \boldsymbol{0}$, and let $\gamma = \boldsymbol{r}^T \boldsymbol{r}$, $\boldsymbol{p} = \boldsymbol{r}$.
for $k = 1, \ldots, K$

$$\boldsymbol{q} = \boldsymbol{A}\boldsymbol{p}$$

$$\alpha = \gamma / (\boldsymbol{p}^T \boldsymbol{q})$$

$$\boldsymbol{x} = \boldsymbol{x} + \alpha \boldsymbol{p}$$

$$\boldsymbol{r} = \boldsymbol{r} - \alpha \boldsymbol{q}$$

$$\hat{\gamma} = \boldsymbol{r}^T \boldsymbol{r}$$

$$\beta = \hat{\gamma}/\gamma, \ \gamma = \hat{\gamma}$$

$$\boldsymbol{p} = \boldsymbol{r} + \beta \boldsymbol{p}$$

end (for k)

Answer: We need storage for matrix $\boldsymbol{A}$ and vectors $\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{q}$, for a total of $nz+5n$ floating-point numbers plus a few scalars, and (in MATLAB) $2nz$ integers.

In each iteration, we need one matrix-vector product, 2 inner products (for α and $\hat{\gamma}$), and 3 vector updates ($\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{p}$) for a total of $K(nz + 5n) + n$ multiplications plus $2K$ divisions.

2a. (6) Suppose we use CG to solve the linear system $\widehat{\boldsymbol{G}}\boldsymbol{x}^* = \boldsymbol{c}$, and we know that all of the eigenvalues of $\widehat{\boldsymbol{G}}$ are in the interval $[1, 10^4]$. Give an upper bound on the number of iterations it will take to reduce the error

$$\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*\|_{\widehat{\boldsymbol{G}}}$$

by a factor of 10^5.

Answer: Let $e_k = \|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*\|_{\widehat{\boldsymbol{G}}}$. Let κ be the ratio of the largest eigenvalue of $\widehat{\boldsymbol{G}}$ to the smallest. The convergence theorem for CG says

$$e_k \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k e_0.$$

We know that $\kappa \le 10^4$. Therefore,

$$e_k \le 2\left(\frac{100-1}{100+1}\right)^k e_0.$$

We need

$$2\left(\frac{100-1}{100+1}\right)^k \le 10^5,$$

or

$$\left(\frac{100-1}{100+1}\right)^k \le \frac{10^5}{2}.$$

Taking logs of both sides (base e or base 10 would be fine for a calculator) gives

$$k(\log 99 - \log 101) \le \log(10^5/2).$$

Since k must be an integer, we set

$$k \ge -\lceil \frac{\log(10^5/2)}{(\log 99 - \log 101)} \rceil,$$

and, with a calculator we could find that we need at most $k = 541$ iterations. (This only makes sense if $n \ge 541$; otherwise, without rounding error, the iteration will stop after n iterations with the exact solution.)

2b. (8) Apply one step of SOR ($\omega = 1.5$) to the linear system

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

with a starting guess of $x_1^{(0)} = x_2^{(0)} = 1$. What is $\boldsymbol{x}^{(1)}$? Will the iteration converge to the true solution? Justify your answer.

Answer: For Gauss-Seidel,

$$\begin{aligned} x_1^{(1)}{}_{GS} &= (4+1)/2 = 2.5 \\ x_2^{(1)}{}_{GS} &= (1+2.5)/2 = 1.75 \end{aligned}$$

Therefore, the SOR iterate is

$$\boldsymbol{x}^{(1)} = (1-\omega)\boldsymbol{x}^{(0)} + \omega \boldsymbol{x}_{GS}^{(1)} = -.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1.5 \begin{bmatrix} 2.50 \\ 1.75 \end{bmatrix} = \begin{bmatrix} 3.250 \\ 2.125 \end{bmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are roots of

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0.$$

Therefore,

$$(2-\lambda)^2 - 1 = 0$$

or

$$2 - \lambda = \pm 1,$$

so

$$\lambda = 1, 3.$$

Therefore matrix $\boldsymbol{A}$ is symmetric and positive definite. (Gershgorin's theorem would also tell us this.)

The "Convergence of SIMs" handout (http://www.cs.umd.edu/users/oleary/c661/simnotes.pdf) says that SOR is convergent for this ω when $\boldsymbol{A}$ is symmetric and positive definite.