AMSC/CMSC 661 Quiz 6 , Spring 2010

1. (10) Let $\Omega = \{\boldsymbol{x} : -1 \leq x_1 \leq 1, \ \ -1 \leq x_2 \leq 1\}$, and let $\Gamma(\Omega)$ be the boundary of Ω. Let

$$q(\boldsymbol{x}, t) = 3(x_1 - 1)(x_1 + 1) - 2t(x_2 - 1)(x_2 + 1) - 6t + 2t^2.$$

Consider the problem

$$\begin{aligned} \frac{\partial u(\boldsymbol{x}, t)}{\partial t} - \Delta u(\boldsymbol{x}, t) = & \quad q(\boldsymbol{x}, t) && \text{for } \boldsymbol{x} \in \Omega \subset \mathcal{R}^2, t \in \mathcal{R}_+ \\ u(\boldsymbol{x}, 0) = & \quad 0 && \text{for } \boldsymbol{x} \in \Omega \\ u(\boldsymbol{x}, t) = & \quad 0 && \text{for } \boldsymbol{x} \in \Gamma(\Omega), t \in \mathcal{R}_+ \end{aligned}$$

Give a bound on

$$\max_{0 \leq t \leq 5} \max_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x}, t)|.$$

Answer:

We could apply the maximum principle (Thm 8.6 of textbook, or p.7 of parabolic theory notes) since $q(\boldsymbol{x}, t) \leq 0$, but the conclusion is that $u(\boldsymbol{x}, t) \leq 0$, which doesn't give us any bound on $|u(\boldsymbol{x}, t)|$.

Instead, we need to apply the stability estimate (Thm 8.7, or p.7 of the notes). In its notation, $g = 0$, $v = 0$, and we can bound the four terms in q to get

$$|f(\boldsymbol{x}, t)| = |q(\boldsymbol{x}, t)| \leq 3 + 10 + 30 + 50.$$

The domain Ω is contained in the circle of radius $\sqrt{2}$ centered at the origin, and the dimension is $d = 2$. Therefore,

$$\max_{0 \leq t \leq 5} \max_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x}, t)| \leq \frac{93}{2}.$$

2a. (5) Show that $z_j(x) = \sin(2\pi jx)$ (j a positive integer) satisfies the equation $-z_{xx}(x) = \lambda z(x)$, and $z(0) = z(1) = 0$. What is λ?

Answer: We verify that z_j satisfies the boundary conditions:

$$z_j(0) = \sin(0) = 0, \quad z_j(1) = \sin(2\pi j) = 0.$$

We compute

$$-(z_j)_{xx} = (2\pi j)^2 \sin(2\pi jx) = (2\pi j)^2 z_j(x),$$

so $\lambda = (2\pi j)^2$ and z_j is an eigenfunction of the differential equation $-z_{xx}(x) = \lambda z(x)$ with boundary conditions $z(0) = z(1) = 0$ and eigenvalue $(2\pi j)^2$.

2b. (5) Consider the problem

$$\begin{array}{rcl} u_t - u_{xx} & = 0 & \text{for } x \in [0,1], t \geq 0, \\ u(x,0) & = x(1-x) & \text{for } x \in \Omega \\ u(x,t) & = 0 & \text{for } x \in \Gamma(\Omega), t \in \mathcal{R}_+ \end{array}$$

Suppose we express the solution as

$$u(x,t) = \sum_{j=1}^{\infty} w_j(t) z_j(x)$$

Give a formula for $w_1(t)$.

Answer: We follow the procedure on p. 5 of the parabolic theory notes (Section 8.2 of the text).

Let

$$w_j(0) = \int_0^1 x(1-x)\sin(2\pi jx)dx.$$

Then

$$w_j(t) = e^{-\lambda_j t} w_j(0),$$

and for $j = 1$, $\lambda_j = 4\pi^2$.