On Bounds for Scaled Projections and Pseudoinverses

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ABSTRACT

Let $X$ be a matrix of full column rank, and let $D$ be a positive definite diagonal matrix. In a recent paper, Stewart considered the weighted pseudoinverse $X_D^\dagger = (X^TDX)^{-1}X^TD$ and the associated oblique projection $P_D = XX_D^\dagger$, and gave bounds, independent of $D$, for the norms of these matrices. In this note, we answer a question he raised by showing that the bounds are computable.

Let $X$ be a matrix of full column rank, and let $D$ be a positive definite diagonal matrix. In a recent paper, Stewart [1] considered the weighted pseudoinverse $X_D^\dagger = (X^TDX)^{-1}X^TD$ and the associated oblique projection $P_D = XX_D^\dagger$. He proved two results. The first is that the spectral norms of these matrices are bounded independently of $D$ as

$$
\sup_{D \in \mathcal{D}_+} \|P_D\| \leq \rho^{-1}
$$

and

$$
\sup_{D \in \mathcal{D}_+} \|X_D^\dagger\| \leq \rho^{-1}\|X^\dagger\|,
$$

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where
\[
\rho = \inf_{\substack{y \in \mathcal{Y} \\
X \in \mathcal{X}} \forall x \in \mathcal{X}} \|y - x\| > 0,
\]

with
\[
\mathcal{X} = \{x \in \mathcal{R}(X) : \|x\| = 1\}, \tag{2}
\]
\[
\mathcal{Y} = \{y : \exists D \in \mathcal{R}_+ \text{ such that } X^TDy = 0\}. \tag{3}
\]

His second result is that if the columns of \( U \) form an orthonormal basis for \( \mathcal{R}(X) \), then
\[
\rho \leq \min \inf_+ (U_i), \tag{4}
\]
where \( U_i \) denotes any submatrix formed from a nonempty set of rows of \( U \).

In this note, we answer a question he raised by showing that
\[
\rho = \min \inf_+ (U_i).
\]

Since \( \mathcal{X} \) and \( \mathcal{Y} \) depend only on the range of \( X \) and not on its entries, we can replace \( X \) in (2) and (3) by \( U \). Thus,
\[
\mathcal{X} = \{U\alpha : \|\alpha\| = 1\}.
\]

Let the sign of a scalar \( t \) be defined by
\[
\text{sg}(t) = \begin{cases} 
1 & \text{if } t > 0, \\
0 & \text{if } t = 0, \\
-1 & \text{if } t < 0,
\end{cases}
\]
and let the sign of a vector \( z \) be denoted by \( \text{sg}(z) \) and defined component-wise. Then \( \mathcal{Y} \) has the property that for any vector \( \hat{y} \in \mathcal{Y} \), every vector \( y \) with \( \text{sg}(y) = \text{sg}(\hat{y}) \) is also an element of \( \mathcal{Y} \). This is verified by letting \( D \) be the nonnegative diagonal matrix such that \( U^TD\hat{y} = 0 \). Then \( U^TDSy = 0 \), where \( S \) is the diagonal matrix with
\[
s_{ii} = \begin{cases} 
\hat{y}_i / y_i & \text{if } y_i \neq 0, \\
1 & \text{if } y_i = 0.
\end{cases}
\]
Now,

\[ \rho = \inf_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \| y - x \| \]

\[ = \inf_{\tilde{y} \in \mathcal{Y}} \inf_{\lambda \in \mathcal{X}} \| y - \lambda \|
\]

\[ = \inf_{\tilde{y} \in \mathcal{Y}} \inf_{\lambda \in \mathcal{X}} \| y - U\alpha \|, \]

In the inner infimum, for every choice of \( \alpha \) there is a set of rows of \( U\alpha \) that agree in sign with \( \tilde{y} \) and a set that disagree. The set of rows that disagree in sign must be nonempty; otherwise \( y = U\alpha \in \mathcal{Y} \), and the infimum would be zero, which contradicts (1). Let the set of those that disagree be denoted by the subscript \( I \). For this choice of \( \alpha \), the best \( y \) equals \( U\alpha \) in all rows that agree in sign and has elements zero or arbitrarily close to zero in the other rows. The resulting value of \( \| y - U\alpha \| \) is no less than \( \|(U\alpha)_I\| = \|U_I\alpha\| \), and this value is bounded below by the smallest singular value of \( U_I \). Thus we have shown that

\[ \rho \geq \min \inf_+ (U_I), \]

and combining this with Stewart's result (4) establishes the equality.

REFERENCES


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