JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 6, November–December 1992

Design of Reduced-Order Observers with Precise Loop Transfer Recovery

Moghen M. Monahemi,* Jewel B. Barlow,† and Dianne P. O'Leary‡
University of Maryland, College Park, Maryland 20742

This paper concerns the design of reduced-order observers for systems in which the number of measurements is more than the number of controls. We develop an algorithm that applies to regular systems that have no transmission zeros. The algorithm uses eigenstructure assignment whereas other approaches use Kalman filter methods. The advantages of this approach are the following: 1) precise loop transfer recovery rather than approximate loop transfer recovery, 2) finite observer gain rather than asymptotic observer gain, and 3) modest computational tools and operations counts. Case studies are presented illustrating these features.

Introduction

THE problem of designing an observer that can achieve loop transfer recovery (LTR) has received continuous attention since Doyle[1] presented an example with a Kalman-filter-based observer [linear quadratic Gaussian (LQG)] design lacking robustness even though the full-state feedback [linear quadratic regulator (LQR)] controller had impressive robustness properties, namely gain margins of -6 dB to $+\infty$ dB and phase margins of ± 60 deg.[2] To alleviate this problem, Doyle and Stein[3] developed a robustness recovery procedure in which fictitious process noise is added to the input in the design model. The LQR robustness properties are preserved with the loop open at the input since the loop transfer function is recovered asymptotically as the intensity of the fictitious process noise is increased. Stein and Athans[4] call this procedure LQG/LTR. There have been further developments and applications of LQG/LTR by a number of workers. Madiwale and Williams[5] extended the theory to reduced-order observer-based LQG designs for nonsquare, minimum phase, and left invertible plants. Calise and Prasad[6] developed an approach for designing a fixed-order compensator and obtained an approximate LTR for nonsquare, minimum phase systems. It is similar to the full-order compensator design of ordinary LQG/LTR. Fu[7] developed the necessary and sufficient condition for exact LTR employing a general feedback structure for model matching control that includes the observer-based state feedback control as a special case. He formulated the problem of model matching by assigning certain stable matrices such that the desired loop transfer function became equal to the associated loop transfer function of the closed-loop input-output transfer function. He claimed that exact LTR is achievable under the conditions for strong stabilization of systems. Tsui[8,9] introduced a theoretical analysis of an entirely new approach to the problem of loop transfer recovery. His approach was to minimize the observer gain to the system input by observer pole selection. Furthermore, he claimed that this new approach aimed directly at achieving the necessary and sufficient condition of LTR.

The purpose of this paper is to present a computational algorithm for the solution of the constrained matrix Sylvester equation that arises in Tsui's approach for designing robust reduced-order Luenberger observers.[10] Precise LTR is achieved with finite observer gain for regular nonsquare plants with no transmission zeros, for which the number of sensors is greater than the number of controls. Other approaches have been unable to attain these results. The method developed here provides freedom to select eigenvalues and eigenvectors for the observer.

Formulation of the Loop Transfer Recovery Problem for Reduced-Order Observers

Consider a nonsquare controllable and observable (linear time invariant) state-space model of a system,

$$\dot{x} = Ax + Bu, \quad x \in R^{n\times 1}, \quad u \in R^{p\times 1}, \quad y \in R^{m\times 1} \tag{1}$$

$$y = Cx, \qquad p < m < n \tag{2}$$

and its full-state feedback control law

$$u = -K_c x \tag{3}$$

This K_c can be separately designed to provide both stability robustness and performance robustness of the corresponding state feedback system using standard state-space control techniques (e.g., LQR, pole placement, direct eigenspace assignment/full-state feedback).

Because the state x is ordinarily not completely measurable, we usually need an estimator. We design an observer z of fixed order $n-m$ satisfying

$$\dot{z} = Fz + (TB)u + K_f y \tag{4}$$

$$u = -K_c\hat{x} = Nz + My \tag{5}$$

The observer-based feedback system (1–4) is shown in Fig. 1. Equation (4) is the general form of a Luenberger observer[10] that takes the system input u and output y as its inputs, and that estimates $K_c x$. The familiar Kalman filter, in which the parameters (F, T, N, M) are fixed to be $(A-K_f C, I, K_c, O)$ is one example of a Luenberger observer. The generalization to $M \neq 0$ has also been made,[10] but up until now the generalization to $T \neq I$ and $N \neq K_c$ has not been successful.

The loop transfer function at the break point x in Fig. 1 is

$$(LTF)_u = \left[I + N(sI-F)^{-1}TB\right]^{-1}\left[M + N(sI-F)^{-1}K_f\right]P(s) \tag{6}$$

The loop transfer function of the direct-state feedback system, the regulator loop transfer function $RLTF$, is

$$RLTF = K_c(sI-A)^{-1}B \tag{7}$$

Received Feb. 14, 1991; revision received Dec. 24, 1991; accepted for publication Jan. 2, 1992.

*Graduate Student, Aerospace Engineering Department. Student Member AIAA.

†Associate Professor, Aerospace Engineering Department. Senior Member AIAA.

‡Professor, Computer Science Department and Institute for Advanced Computer Studies.

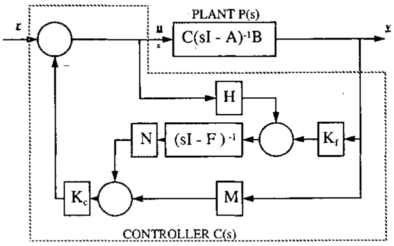

Fig. 1 Observer-based feedback control system.

The objective of precise LTR is to make $(LTF)_u = RLTF$.

Recently, Tsui[8,9] has proven that a necessary and sufficient condition for LTR at all frequencies is

$$N(sI-F)^{-1}TB = 0 \tag{8}$$

for every s, where N satisfies the equation

$$K_c = NT + MC \tag{9}$$

and T satisfies the Sylvester matrix equation

$$TA - FT = K_f C \tag{10}$$

Condition (8) can be achieved if we find $T \in R^{(n-m)\times n}$ satisfying

$$TB = 0 \tag{11}$$

Thus, Eqs. (9-11) are sufficient conditions for LTR. It is clear that Eq. (11) cannot be achieved by the Kalman filtering algorithms that have the restriction $T=I$. These algorithms aim for approximate LTR by using conditions less restrictive than Eq. (11).[3,11] Commonly, the Doyle-Stein identity is employed, an identity that can be only approximately achieved in the limit as system input noise is increased. Each Kalman filter pole is required to approach a system zero or negative infinity. Therefore, this approach produces a K_f with infinite gain. This approach breaks down when nonminimum-phase plants are present, since the Kalman filter poles would approach the unstable region of the s plane and produce an unstable compensator. This situation is very undesirable from a robustness point of view.

In the next section we construct a more general observer with $T \neq I$ that does achieve precise LTR with finite gain for regular nonsquare plants having $m > p$ and having no transmission zeros, either minimum-phase or nonminimum-phase. This last constraint is not very restrictive, since nonsquare plants rarely exhibit transmission zeros.[12] Generically, nonsquare transfer functions have no zeros.[13]

Designing an Observer

Given A, B, C, K_c in Eqs. (1-3) and the observer dynamic matrix F, we wish to calculate the observer gain matrix K_f of the observer (4) and the associated matrices M and N. The observer dynamic matrix F may be chosen with wide latitude; however, its eigenvalues must not overlap those of A.[14] The flexibility in selection of observer poles can be used to meet other performance requirements.

The following algorithm uses readily available software modules to compute K_f, M, and N: matrix multiplication routines, computation of QR factors of a matrix (orthogonal matrix times a triangular one), triangular system solvers (see, e.g., Linpack[15]), and a Sylvester equation solver (see, e.g., Ref. 16). The algorithm is quite economical, requiring a constant times n^3 operations. Following the algorithm, we discuss conditions under which the Luenberger observer K_f achieves precise loop transfer recovery.

Algorithm: 1) Perform a QR factorization of B: $[W,S] = qr(B)$:

$$W = n\overset{\begin{array}{cc} p & n-p \end{array}}{[W_1 \quad W_2]}, \qquad S = \begin{array}{c} p \\ n-p \end{array}\overset{p}{\begin{bmatrix} S_1 \\ 0 \end{bmatrix}} \tag{12}$$

2) Let

$$C_1 = CW_1, \qquad A_1 = W_2^T A W_1, \qquad A_2 = W_2^T A W_2$$

3) Perform QR factorization, $[Q,R] = qr(C_1)$:

$$Q = m\overset{\begin{array}{cc} p & m-p \end{array}}{[Q_1 \quad Q_2]}, \qquad R = \begin{array}{c} p \\ m-p \end{array}\overset{p}{\begin{bmatrix} R_1 \\ 0 \end{bmatrix}} \tag{13}$$

4) Let

$$E = Q^T C W_2 = \begin{array}{c} p \\ m-p \end{array}\overset{n-p}{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}}$$

5) Solve the Sylvester equation:

$$Z(A_2 - A_1 R_1^{-1} E_1) - FZ = L_2 E_2 \tag{14}$$

where the elements of L_2 are chosen at random (see Remark).

6) Set

$$K_f = [ZA_1 R_1^{-1} \quad L_2]Q^T \qquad \text{and} \qquad T = ZW_2^T$$

7) Solve

$$[N \quad M] = K_c \begin{bmatrix} T \\ C \end{bmatrix}^{-1}$$

Recall that a *transmission zero* of the system (A, B, C, 0) is a value of s for which the *transmission matrix*

$$\begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} \tag{15}$$

has less than full rank. The success of the algorithm is related to the existence of transmission zeros.

Theorem 1: If $m > p$ and if the system $[A, B, C, 0]$ is regular (i.e., CB has full rank p), then the algorithm produces a solution to Eqs. (10) and (11) if the reduced system $[A_2 - A_1 R_1^{-1} E_1, E_2]$ is observable and the eigenvalues of $A_2 - A_1 R_1^{-1} E_1$ are distinct from those of F.

Theorem 2: If the system $[A, B, C, 0]$ is regular (i.e., CB has full rank p), then the system has no transmission zeros if and only if the reduced system $[A_2 - A_1 R_1^{-1} E_1, E_2]$ is observable; i.e., the only vector y satisfying $[A_2 - A_1 R_1^{-1} E_1]y = \mu y$ and $E_2 y = 0$ must be the vector $y = 0$.

The proofs of these results are given in the Appendix.

Remark: The matrix K_f produced by the algorithm will be a Luenberger observer that achieves precise loop transfer recovery provided that the matrix

$$\mathfrak{I} \equiv \begin{bmatrix} T \\ C \end{bmatrix} \tag{16}$$

has full rank, so that Eq. (9) can be satisfied. We are interested only in regular systems, since we have proven in Ref. 14 that regularity is a necessary condition for $\mathfrak{I}$ to be full rank. We believe that under the assumptions of the theorem, $\mathfrak{I}$ will have full rank for almost every choice of F and L_2. Although our numerical experience supports this belief, we have not proven it. The suggestion to use a random choice of L_2 in step 5 of the algorithm avoids deficiencies in rank caused by certain unfortunate but natural choices, such as a matrix of all ones that is rank deficient.

Recently Chen, Saberi, and Sannuti[17] gave conditions similar to those of theorem 1, guaranteeing that reduced-order observers can achieve precise LTR.

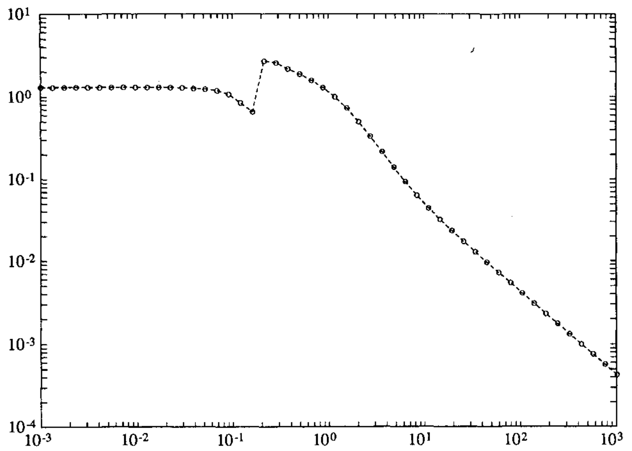

Fig. 2 Target loop transfer function (dashed line) and control system loop transfer function (open circles).

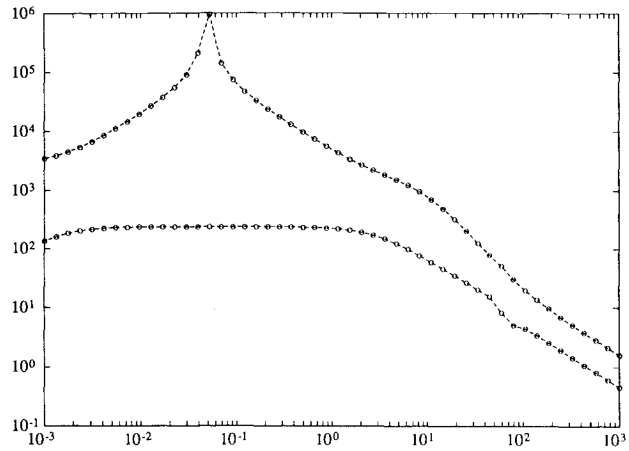

Fig. 3 Target loop transfer function (dashed line) and control system loop transfer function (open circles).

Control System Development

We shall now focus on the design of the compensator using the separation principle.[18] The closed-loop system, comprising the process under control together with the compensator, combines the dynamics of the closed-loop system designed for full-state feedback with that of the reduced-order observer. The poles of the overall system are those for the system with full-state feedback plus those for the reduced-order observer. Each is designed independently. Friedland[19] showed that the poles of the closed-loop system, when a reduced-order observer is used in the compensator, are the eigenvalues of $A - BK_c$ and those of F. Thus there can be as many as $2n - m$ poles, the roots of the equation

$$\det[sI - A + BK_c]\,\det[sI - F] = 0 \tag{17}$$

$$A = \begin{bmatrix} -4.9320e-001 & 1.2900e-004 & 1.4168e+000 & 0 \\ 0 & -5.0770e-002 & -3.8610e+000 & -3.2170e+001 \\ 1.0000e+000 & -1.1700e-003 & -5.1640e-001 & 0 \\ 1.0000e+000 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

On the basis of the separation principle, the control law when the compensator is based on a reduced-order observer is given by Eq. (5) as $u = K_c\hat{x} = Nz + My$. The transfer function of the compensator is obtained by the use of Eq. (5) with $TB = 0$ and Eq. (1). Thus

$$\dot{z} = Fz + K_f y \tag{18}$$

or

$$z(s) = [sI - F]^{-1} K_f y(s) \tag{19}$$

In implementation, there is certainly a basic limit to the performance and robustness of the control system. It appears that, whereas performance at low frequency can be satisfactorily attained, there might exist problems at high frequencies. To overcome these problems one can make the bandwidth of the compensator larger than that of the plant by a factor of 10; however, the actuator and the actuation constraints will be affected by this large bandwidth and may prove to be inadequate. Whether this is acceptable or not depends on the specific application and the importance of robustness. We simply want to point out that the tradeoff exists and should be considered when selecting the operating frequencies of the control system.

Case Studies

In this section, the use of the algorithm will be demonstrated via two case studies: two aircraft flight dynamic problems taken from Refs. 20 and 21, respectively. The computational results were obtained using *MATLAB*.[22]

Case Study 1[20]

The example aircraft is a linearized approximation to the AFTI/F-16 on landing approach with $V = 139$ kt. The objective is to design feedback configurations using both an angle-of-attack sensor and an attitude gyro. The state space matrices corresponding to the small perturbation longitudinal-vertical equations of motion (1) are given by

$$B = \begin{bmatrix} -1.6450e+000 \\ 0 \\ -7.1700e-002 \\ 0 \end{bmatrix} \tag{21}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

where $x = [q, u, \alpha, \theta]'$ and $u = \delta_E$. The states are the perturbations in pitch rate, speed, angle of attack, and pitch attitude, respectively. The control is elevator deflection.

The plant is typical of a statically unstable aircraft and, for the given arrangements of sensors, does not have any transmission zeros. Satisfactory and acceptable flying qualities for this aircraft would result if the airplane were augmented to produce the following short period and phugoid mode characteristics:

$$\omega_{\mathrm{SP}} = 2.5 \text{ rad/s} \qquad \xi_{\mathrm{SP}} = 0.5 \tag{23}$$

$$\omega_{\mathrm{PH}} = 0.1 \text{ rad/s} \qquad \xi_{\mathrm{PH}} = 0.1 \tag{24}$$

where ω is natural frequency and ξ is the damping ratio. The full-state feedback gain matrix, using the robust pole placement technique[23] is

$$K_c = [3.3853e-001 \quad -1.4634e-002 \quad -1.7655e+000 \quad -4.2847e-002]$$

For the observer dynamics matrix

$$F = \begin{bmatrix} -7 & 0 \\ 0 & -9 \end{bmatrix} \tag{25}$$

and an arbitrary L_2 such as

$$L_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{26}$$

the T matrix, the observer gain matrix K_f, and the corresponding observer matrices N, M are computed by the algorithm as

$$T = \begin{bmatrix} 8.6823e-03 & -3.3698e-05 & -1.9920e-01 & 1.4270e-01 \\ 7.6911e-03 & -2.3180e-05 & -1.7645e-01 & 1.1103e-01 \end{bmatrix} \tag{27}$$

$$K_f = \begin{bmatrix} -1.2791e+00 & 1.0000e+00 \\ -1.4860e+00 & 1.0000e+00 \end{bmatrix} \tag{28}$$

$$N = \begin{bmatrix} 5.8840e+01 & -2.2407e+01 \end{bmatrix} \tag{29}$$

$$M = \begin{bmatrix} 6.0014e+00 & -5.9516e+00 \end{bmatrix} \tag{30}$$

Figure 2 is a plot of singular values of the control system loop transfer function $C(s)P(s)$. The target loop, $K_c[sI-A]^{-1}B$, is precisely recaptured. The multivariable stability margins of the observer-based system are the same as those of the regulator, and they exhibit excellent (optimal)[24] gain and phase margins:

$$-6.0206 \leq GM \leq \infty \text{ dB} \tag{31}$$

$$-60 \leq PM \leq 60 \text{ deg} \tag{32}$$

Case Study 2[21]

This case uses a model of a generic, forward-swept wing aircraft. The generic aircraft is roughly the same size as the X29. The wings are swept forward at a 30 deg angle. The operating point used in this study is level flight at a velocity of 1000 ft/s at sea level, which is about Mach 0.9. This corresponds to a dynamic pressure of 1189 lb/ft^2. Three models were given in Ref. 21 corresponding to three different center-of-gravity locations. The model with the center-of-mass location at 0.30 ft ahead of the wing root elastic axis was used in this study. The aircraft data, and the structural mode data developed in Ref. 21 are all the information needed to obtain the mathematical model of the FSW aircraft configuation under consideration. The model is in linear state variable form where the system matrices A and B are as follows:

Columns 1–4

$$A = \left[\begin{matrix} 5.2660e-004 & 9.2764e-002 & -5.6200e-001 & -2.5360e-001 \\ -3.6887e-003 & -2.8810e+000 & -4.6720e-004 & 1.0060e+000 \\ 0 & 0 & 0 & 1.0000e+000 \\ 1.1648e-004 & 7.9560e+001 & 1.4750e-005 & -8.3110e-001 \\ 0 & 0 & 0 & 0 \\ -9.4390e-001 & -5.4384e+002 & -1.1832e-006 & 1.1589e+000 \\ 0 & 0 & 0 & 0 \\ 3.3630e-003 & 1.3106e+000 & -2.7297e-007 & -1.1222e-002 \end{matrix}\right.$$

Columns 5–8

$$\left.\begin{matrix} -1.4050e-001 & 1.5070e-003 & 2.7430e+000 & 1.9840e-005 \\ 4.3699e+000 & -4.6879e-002 & -8.5313e+001 & -6.1708e-004 \\ 0 & 0 & 0 & 0 \\ -6.0447e+001 & 1.0096e+000 & 1.4330e+003 & 8.3537e-002 \\ 0 & 0 & 0 & 0 \\ -3.6240e+003 & -2.0640e+001 & -2.8050e+004 & 3.8550e-002 \\ 0 & 0 & 0 & 0 \\ -7.6250e-002 & -8.1300e-004 & -4.5240e+004 & -3.6000e-002 \end{matrix}\right] \tag{33}$$

$$B = \begin{bmatrix} 1.2296e+000 & 4.9255e-001 \\ -5.5524e+000 & -1.5324e+001 \\ 0 & 0 \\ -2.3328e+002 & 1.8399e+003 \\ 0 & 0 \\ -1.2985e+003 & 1.4750e+002 \\ 0 & 0 \\ 2.8023e-001 & 3.4029e+001 \end{bmatrix} \tag{34}$$

For the airplane configuration under consideration, the state vector x is

$$x = [u, \alpha, \theta, q, n_1, \dot{n}_1, n_2, \dot{n}_2]' \tag{35}$$

The first four states are the usual rigid body variables, as in case study 1.

The remaining four states represent flexible degrees of freedom. The first flexible mode represents the wing bending: n_1 is the wing tip deflection in feet, $\dot{n}_1$ is its rate of deflection in feet/second. The second flexible mode represents wing torsion, n_2 is the wing rotation about the elastic axis in radians, $\dot{n}_2$ is the rate of deflection in radians/second.

The control vector u is

$$u = [\delta_F, \delta_C]' \tag{36}$$

where $\delta_F(t)$ is the perturbation deflection from trim of the full-span flaperon in radians, and $\delta_C(t)$ is the perturbation deflection from trim of the canard in radians. The aircraft under consideration has an unstable pole corresponding to the short period mode ($\lambda_{SP} = 7.308,\ -11.918$ rad/s).

A full-state feedback regulator is designed to stabilize the pitch rate and control the wing tip bending rate in the face of a wind gust.

$$K_c = \begin{bmatrix} 4.9393e-04 & -7.6933e+01 & -6.3529e-01 & -4.8729e-01 & -5.5159e-01 & 5.1513e-03 & 0 & 0 \\ 5.6533e-05 & -6.3450e+01 & 7.7227e+01 & 2.9781e-01 & -1.3466e-01 & -2.2119e-04 & 0 & 0 \end{bmatrix} \tag{37}$$

Given this matrix, the multivariable phase and gain margins are found to be[21]

$$\inf_{\omega} \sigma\left[I + K_c(sI - A)^{-1}B\right] = \alpha = 0.59 \tag{38}$$

$$-4 \leq GM \leq 7.72 \text{ dB} \tag{39}$$

$$-34.26 \text{ deg} \leq PM \leq 34.26 \text{ deg} \tag{40}$$

when the full-state feedback is included in the flight control system.

For the given configuration of sensors, the measurements are

$$y = [u, \alpha, \theta, n_1]' \tag{41}$$

For the observer dynamics matrix

$$F = \begin{bmatrix} -7 & -4 & 0 & 0 \\ 4 & -7 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 5 & -9 \end{bmatrix} \tag{42}$$

and matrix

$$L_2 = \begin{bmatrix} 6.7886e-01 & 5.1942e-01 \\ 6.7930e-01 & 8.3097e-01 \\ 9.3469e-01 & 3.4572e-02 \\ 3.8350e-01 & 5.3462e-02 \end{bmatrix} \tag{43}$$

the algorithm produces T and the observer gain K_f, with associated N, M matrices as follows:

Columns 1–4

$$T = \begin{bmatrix} 1.1288e+00 & -1.4952e-01 & -1.4277e-01 & -9.2849e-04 \\ 3.0572e+00 & 4.8747e-01 & 2.6094e-01 & 3.2375e-03 \\ 4.3738e-01 & 6.4453e-02 & 3.3707e-02 & 4.1684e-04 \\ 1.8269e+00 & 2.7351e-01 & 1.7543e-01 & 1.7788e-03 \end{bmatrix}$$

Columns 5–8

$$\begin{bmatrix} -8.8686e-03 & -2.6270e-04 & 6.8523e-04 & 3.4769e-04 \\ 1.7045e-02 & 2.2871e-04 & 6.7291e-03 & -7.7218e-04 \\ 1.4784e-03 & 6.3663e-05 & 3.5590e-03 & -1.2016e-04 \\ 1.1394e-02 & 2.4082e-04 & 3.8344e-03 & -4.9760e-04 \end{bmatrix} \tag{44}$$

$$K_f = \begin{bmatrix} 4.3271e+00 & 1.2987e+00 & 6.7886e-01 & 5.1942e-01 \\ 2.5915e+01 & 3.0218e+00 & 6.7930e-01 & 8.3097e-01 \\ 1.3071e+01 & 1.8009e+00 & 9.3469e-01 & 3.4572e-02 \\ 1.4255e+01 & 1.5307e+00 & 3.8350e-01 & 5.3462e-02 \end{bmatrix} \tag{45}$$

$$N = \begin{bmatrix} 2.0750e+05 & -1.4364e+05 & -2.2264e+05 & 4.2166e+05 \\ -1.3105e+05 & 9.0701e+04 & 1.4062e+05 & -2.6628e+05 \end{bmatrix} \tag{46}$$

$$M = \begin{bmatrix} 4.1057e+02 & -5.2786e+00 & 6.3758e+02 & -1.8704e+02 \\ -2.5469e+02 & -1.0739e+02 & -3.2603e+02 & 1.1764e+02 \end{bmatrix} \tag{47}$$

The minimum return difference matrix of control system, loop transfer function $\inf \sigma[I + C(s)P(s)] = \beta = 0.59$. Therefore, the multivariable stability margins,[24] gain margin and phase margin, are precisely the same as those of the regulator when the observer is included in the flight control system. Figure 3 shows the target loop and control system loop transfer function.

Conclusions

We have given an algorithm that achieves precise loop transfer recovery and provides freedom of eigenstructure assignment for nonsquare plants in a situation in which the number of sensors exceeds the number of controls. It is important to note that the algorithm yields finite observer gain, a critical requirement for pragmatic design. This approach is computationally simple, requiring on the order of n^3 operations, and works for regular systems with no transmission zeros.

The flexibility in selection of observer eigenvalues can be used to meet other performance requirements. In particular, in the case of flight control problems this flexibility can be used to meet handling and flying quality requirements. Much work can be done in the area of exploring the selection of observer poles needed to achieve certain desired handling qualities and better performance in general.

The situation in which the number of actuators exceeds the number of sensors is dual to this case, and corresponds to loop transfer recovery at the output. This dual version of loop transfer recovery has been explained by a number of researchers.[4,8,11] Tsui[8] states that the necessary and sufficient conditions for dual loop transfer recovery are $CT = 0$ and $AT - TF = BK_c$. In this case, the state feedback system does not have a reduced-order version. There exists no full rank matrix T to exactly satisfy these dual conditions.[14] One can conclude that, in this case, precise loop transfer recovery is not possible.

Appendix

Proof of Theorem 1

The assumption on the rank of CB is sufficient to guarantee the existence of the QR factors in steps 1 and 3 and the invertibility of R_1. The distinct eigenvalue hypothesis guarantees that the Sylvester equation in step 5 has a unique solution. Thus, all of the indicated computations in steps 1–6 can be performed. We now verify that we have satisfied Eqs. (11) and (10). In step 6, we set $T = ZW_2^T$, so that $TB = (ZW_2^T)(W_1 S_1) = 0$, since $W_2^T W_1 = 0$, and so Eq. (11) is satisfied. Now, the matrix T satisfies the Sylvester equation (10) if and only if

$$ZW_2^T A - FZW_2^T = K_f C \tag{A1}$$

or, when we multiply by the nonsingular matrix W,

$$ZW_2^T AW - FZW_2^T W = K_f CW \tag{A2}$$

Equation (A2) can be rewritten as

$$Z[A_1 \quad A_2] - FZ[0 \quad I] = K_f[C_1 \quad CW_2] \tag{A3}$$

which is equivalent to the two conditions

$$ZA_1 = K_f C_1 \tag{A4}$$

and

$$ZA_2 - FZ = K_f CW_2 \tag{A5}$$

We now verify that the matrices Z and K_f determined by the algorithm satisfy these relations. By step 6,

$$K_f C_1 = [ZA_1 R_1^{-1} \quad L_2] Q^T C_1 = [ZA_1 R_1^{-1} \quad L_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = ZA_1 \tag{A6}$$

and by steps 5, 4, and 6,

$$ZA_2 - FZ = ZA_1 R_1^{-1} E_1 + L_2 E_2 \tag{A7}$$

$$= [ZA_1 R_1^{-1} \quad L_2] \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{A8}$$

$$= [ZA_1 R_1^{-1} \quad L_2] Q^T CW_2 \tag{A9}$$

$$= K_f CW_2 \tag{A10}$$

as desired. □

The second theorem concerns the meaning of transmission zeros. We will use several facts about the rank of a matrix. A matrix Y with at least as many rows as columns has full rank if and only if there exists no nonzero vector h such that $Yh = 0$. The rank of a matrix is unchanged if 1) the matrix is multiplied by a square, nonsingular matrix, 2) row operations are performed, adding a multiple of one row to another, and 3) rows or columns are reordered.

Finally, if

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ 0 & Y_3 \end{bmatrix} \tag{A11}$$

where Y_1 is square, then Y is full rank if and only if Y_1 and Y_3 are full rank.

Proof of Theorem 2

We multiply the transmission matrix by square nonsingular matrices to produce the following product:

$$\begin{array}{cc} & \begin{array}{ccc} p & n-p & m \end{array} \\ \begin{array}{c} p \\ n-p \\ m \end{array} & \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^T \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} p \\ n-p \\ m \end{array} & \begin{bmatrix} W_1^T & 0 \\ W_2^T & 0 \\ 0 & I \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{cc} n & p \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} A-sI & B \\ C & 0 \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccc} p & n-p & p \end{array} \\ \begin{array}{c} n \\ p \end{array} & \begin{bmatrix} W_1 & W_2 & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{ccc} p & n-p & m \end{array} \\ \begin{array}{c} p \\ n-p \\ m \end{array} & \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^T \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccc} p & n-p & p \end{array} \\ \begin{array}{c} p \\ n-p \\ m \end{array} & \begin{bmatrix} W_1^TAW_1-sI & W_1^TAW_2 & W_1^TB \\ W_2^TAW_1 & W_2^TAW_2-sI & W_2^TB \\ CW_1 & CW_2 & 0 \end{bmatrix} \end{array} \tag{A12}$$

$$= \begin{array}{cc} & \begin{array}{ccc} p & n-p & m \end{array} \\ \begin{array}{c} p \\ n-p \\ m \end{array} & \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^T \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccc} p & n-p & p \end{array} \\ \begin{array}{c} p \\ n-p \\ m \end{array} & \begin{bmatrix} W_1^TAW_1-sI & W_1^TAW_2 & S_1 \\ A_1 & A_2-sI & 0 \\ C_1 & C_2 & 0 \end{bmatrix} \end{array} \tag{A13}$$

$$= \begin{array}{cc} & \begin{array}{ccc} p & n-p & p \end{array} \\ \begin{array}{c} p \\ n-p \\ p \\ m-p \end{array} & \begin{bmatrix} W_1^TAW_1-sI & W_1^TAW_2 & S_1 \\ A_1 & A_2-sI & 0 \\ R_1 & E_1 & 0 \\ 0 & E_2 & 0 \end{bmatrix} \end{array} \tag{A14}$$

After permutation, and subtraction of $A_1R_1^{-1}$ times the third block of rows from the second block, we obtain the following matrix, whose rank is the same as that of the transmission matrix:

$$\begin{array}{cc} & \begin{array}{ccc} p & p & n-p \end{array} \\ \begin{array}{c} p \\ p \\ n-p \\ m-p \end{array} & \begin{bmatrix} S_1 & W_1^TAW_1-sI & W_1^TAW_2 \\ 0 & R_1 & E_1 \\ 0 & 0 & A_2-A_1R_1^{-1}E_1-sI \\ 0 & 0 & E_2 \end{bmatrix} \end{array} \tag{A15}$$

Since, by regularity, R_1 and S_1 are full rank, we see that there are no transmission zeros if and only if the reduced system is observable. □

Acknowledgments

The work of the third author was supported by Air Force Office of Scientific Research Grant AFOSR-87-0158. The authors would like to express their sincere appreciation to Chia-Chi Tsui of the City University of New York, Staten Island College, for many discussions with the first author. His interest is gratefully acknowledged. We are also grateful to Siva Banda and the referees for their careful reading and very helpful comments.

References

[1]Doyle, J. C., "Guaranteed Margins for LQG Regulators," *IEEE Transactions on Automatic Control,* Vol. AC-23, Aug. 1978, pp. 756–757.

[2]Safonov, M. G., and Athans, M., "Gains and Phase Margin of Multiloop LQG Regulators," *IEEE Transactions on Automatic Control,* Vol. AC-22, April 1977, pp. 173–179.

[3]Doyle, J. C., and Stein, G., "Robustness with Observers," *IEEE Transactions on Automatic Control,* Vol. AC-24, Aug. 1979, pp. 607–611.

[4]Stein, G., and Athans, M., "The LQS/LTR Procedure for Multivariable Feedback Control Design," *IEEE Transactions on Automatic Control,* Vol. AC-32, Feb. 1987, pp. 105–114.

[5]Madiwale, A. N., and Williams, D. E., "Some Extensions of Loop Transfer Recovery," *Proceedings of the IEEE American Control Conference* (Boston, MA), Inst. of Electrical and Electronics Engineers, New York, 1985, pp. 790–795.

[6]Calise, A. J., and Prasad, J. V. R., "An Approximate Loop Transfer Recovery Method for Designing Fixed-Order Compensators," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 2, 1990, pp. 297–302.

[7]Fu, M., "Exact, Optimal, and Partial Loop Transfer Recovery," *Proceedings of the IEEE Decision and Control Conference* (Honolulu, HI), Inst. of Electrical and Electronics Engineers, New York, 1991, pp. 1841–1846.

[8]Tsui, C. C., "On the Loop Transfer Recovery," *Proceedings of the IEEE American Control Conference* (Pittsburgh, PA), Inst. of Electrical and Electronics Engineers, New York, 1989, pp. 2184–2189.

[9]Tsui, C. C., "New Approach to Robust Observer Design," *International Journal of Control,* Vol. 47, 1988, pp. 745–751.

[10]Luenberger, D. G., "Introduction to Observers," *IEEE Transactions on Automatic Control,* Vol. AC-16, Dec. 1971, pp. 596–602.

[11]Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for Classical/Modern Synthesis," *IEEE Transactions on Automatic Control,* Vol. AC-26, Feb. 1981, pp. 4–16.

[12]Ridgely, B. D., and Banda, S. S., "Introduction to Robust Multivariable Control," Air Force Wright Aeronautical Lab., AFWAL-TR-85-3102, Wright-Patterson AFB, OH, Feb. 1986, pp. 2–34.

[13]Maciejowski, J. M., *Multivariable Feedback Design,* Addison-Wesley, Reading, MA, 1989, pp. 52–55.

[14]Barlow, J. B., Monahemi, M. M., and O'Leary, D. P., "Constrained Matrix Sylvester Equations," *SIAM Journal on Matrix Analysis and Applications,* Vol. 13, Jan. 1992, pp. 1–9.

[15]Dongarra, J. J., Moler, C. B., Bunch, J. R., and Stewart, G. W., *LINPACK User's Guide,* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1979.

[16]Bartels, R. H., and Stewart, G. W., "Algorithm 432, Solution of a Matrix Equation $AX+XB=C$," *Communications of the ACM,* Vol. 15, Sept. 1972, pp. 820–826.

[17]Chen, B. M., Saberi, A., and Ly, U-L., "Closed-Loop Transfer Recovery with Observer-Based Controllers—Part 1: Analysis," *Proceedings of the AIAA Guidance, Control, and Navigation Conference* (New Orleans, LA), AIAA, Washington, DC, 1991, pp. 1095–1109.

[18]Joseph, P. D., and Tou, J., "On Linear Control Theory," *Transactions of the AIEE,* Pt. II, Vol. 90, Sept. 1961, pp. 193–196.

[19]Friedland, B., *Control System Design,* McGraw-Hill, New York, 1986, pp. 298–301.

[20]Rynaski, E. G., "Flight Control System Design Using Robust Output Observers," AGARD-CP 321, 1982, pp. 14-1–14-8.

[21]Gilbert, M., "Dynamic Modeling and Active Control of Aeroelastic Aircraft," M.S. Thesis, Dept. of Aeronautics and Astronautics, Purdue Univ., West Lafayette, IN, May 1982.

[22]*MATLAB User's Guide,* MathWorks, Inc., South Natick, MA, 1991.

[23]Kautsky, J., Nichols, N. K., and Van Dooren, P., "Robust Pole Assignment in Linear State Feedback," *International Journal of Control,* Vol. 45, No. 5, 1985, pp. 1129–1155.

[24]Lehtomaki, N. A., "Practical Robustness Measures in Multivariable Control System Analysis," Ph.D. Dissertation, Dept. of Electrical Engineering and Computer Science, Massachusetts Inst. of Technology, Cambridge, MA, May 1981.