$$K_f = \begin{bmatrix} 4.3271e+00 & 1.2987e+00 & 6.7886e-01 & 5.1942e-01 \\ 2.5915e+01 & 3.0218e+00 & 6.7930e-01 & 8.3097e-01 \\ 1.3071e+01 & 1.8009e+00 & 9.3469e-01 & 3.4572e-02 \\ 1.4255e+01 & 1.5307e+00 & 3.8350e-01 & 5.3462e-02 \end{bmatrix} \tag{45}$$

$$N = \begin{bmatrix} 2.0750e+05 & -1.4364e+05 & -2.2264e+05 & 4.2166e+05 \\ -1.3105e+05 & 9.0701e+04 & 1.4062e+05 & -2.6628e+05 \end{bmatrix} \tag{46}$$

$$M = \begin{bmatrix} 4.1057e+02 & -5.2786e+00 & 6.3758e+02 & -1.8704e+02 \\ -2.5469e+02 & -1.0739e+02 & -3.2603e+02 & 1.1764e+02 \end{bmatrix} \tag{47}$$

The minimum return difference matrix of control system, loop transfer function $\inf \sigma[I + C(s)P(s)] = \beta = 0.59$. Therefore, the multivariable stability margins,[24] gain margin and phase margin, are precisely the same as those of the regulator when the observer is included in the flight control system. Figure 3 shows the target loop and control system loop transfer function.

Conclusions

We have given an algorithm that achieves precise loop transfer recovery and provides freedom of eigenstructure assignment for nonsquare plants in a situation in which the number of sensors exceeds the number of controls. It is important to note that the algorithm yields finite observer gain, a critical requirement for pragmatic design. This approach is computationally simple, requiring on the order of n^3 operations, and works for regular systems with no transmission zeros.

The flexibility in selection of observer eigenvalues can be used to meet other performance requirements. In particular, in the case of flight control problems this flexibility can be used to meet handling and flying quality requirements. Much work can be done in the area of exploring the selection of observer poles needed to achieve certain desired handling qualities and better performance in general.

The situation in which the number of actuators exceeds the number of sensors is dual to this case, and corresponds to loop transfer recovery at the output. This dual version of loop transfer recovery has been explained by a number of researchers.[4,8,11] Tsui[8] states that the necessary and sufficient conditions for dual loop transfer recovery are $CT = 0$ and $AT - TF = BK_c$. In this case, the state feedback system does not have a reduced-order version. There exists no full rank matrix T to exactly satisfy these dual conditions.[14] One can conclude that, in this case, precise loop transfer recovery is not possible.

Appendix

Proof of Theorem 1

The assumption on the rank of CB is sufficient to guarantee the existence of the QR factors in steps 1 and 3 and the invertibility of R_1. The distinct eigenvalue hypothesis guarantees that the Sylvester equation in step 5 has a unique solution. Thus, all of the indicated computations in steps 1–6 can be performed. We now verify that we have satisfied Eqs. (11) and (10). In step 6, we set $T = ZW_2^T$, so that $TB = (ZW_2^T)(W_1S_1) = 0$, since $W_2^TW_1 = 0$, and so Eq. (11) is satisfied. Now, the matrix T satisfies the Sylvester equation (10) if and only if

$$ZW_2^TA - FZW_2^T = K_fC \tag{A1}$$

or, when we multiply by the nonsingular matrix W,

$$ZW_2^TAW - FZW_2^TW = K_fCW \tag{A2}$$

Equation (A2) can be rewritten as

$$Z[A_1 \quad A_2] - FZ[0 \quad I] = K_f[C_1 \quad CW_2] \tag{A3}$$

which is equivalent to the two conditions

$$ZA_1 = K_fC_1 \tag{A4}$$

and

$$ZA_2 - FZ = K_fCW_2 \tag{A5}$$

We now verify that the matrices Z and K_f determined by the algorithm satisfy these relations. By step 6,

$$K_fC_1 = [ZA_1R_1^{-1} \quad L_2]Q^TC_1 = [ZA_1R_1^{-1} \quad L_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = ZA_1 \tag{A6}$$

and by steps 5, 4, and 6,

$$ZA_2 - FZ = ZA_1R_1^{-1}E_1 + L_2E_2 \tag{A7}$$

$$= [ZA_1R_1^{-1} \quad L_2]\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{A8}$$

$$= [ZA_1R_1^{-1} \quad L_2]Q^TCW_2 \tag{A9}$$

$$= K_fCW_2 \tag{A10}$$

as desired. □

The second theorem concerns the meaning of transmission zeros. We will use several facts about the rank of a matrix. A matrix Y with at least as many rows as columns has full rank if and only if there exists no nonzero vector h such that $Yh = 0$. The rank of a matrix is unchanged if 1) the matrix is multiplied by a square, nonsingular matrix, 2) row operations are performed, adding a multiple of one row to another, and 3) rows or columns are reordered.

Finally, if

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ 0 & Y_3 \end{bmatrix} \tag{A11}$$

where Y_1 is square, then Y is full rank if and only if Y_1 and Y_3 are full rank.

Proof of Theorem 2

We multiply the transmission matrix by square nonsingular matrices to produce the following product:

$$\begin{matrix} p \\ n-p \\ m \end{matrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^T \end{bmatrix} \begin{matrix} p \\ n-p \\ m \end{matrix}\begin{bmatrix} W_1^T & 0 \\ W_2^T & 0 \\ 0 & I \end{bmatrix} \begin{matrix} n \\ m \end{matrix}\begin{bmatrix} A-sI & B \\ C & 0 \end{bmatrix} \begin{matrix} n \\ p \end{matrix}\begin{bmatrix} W_1 & W_2 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{matrix} p \\ n-p \\ m \end{matrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^T \end{bmatrix} \begin{matrix} p \\ n-p \\ m \end{matrix}\begin{bmatrix} W_1^TAW_1-sI & W_1^TAW_2 & W_1^TB \\ W_2^TAW_1 & W_2^TAW_2-sI & W_2^TB \\ CW_1 & CW_2 & 0 \end{bmatrix} \tag{A12}$$

$$= \begin{matrix} p \\ n-p \\ m \end{matrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^T \end{bmatrix} \begin{matrix} p \\ n-p \\ m \end{matrix}\begin{bmatrix} W_1^TAW_1-sI & W_1^TAW_2 & S_1 \\ A_1 & A_2-sI & 0 \\ C_1 & C_2 & 0 \end{bmatrix} \tag{A13}$$

$$= \begin{matrix} p \\ n-p \\ p \\ m-p \end{matrix}\begin{bmatrix} W_1^TAW_1-sI & W_1^TAW_2 & S_1 \\ A_1 & A_2-sI & 0 \\ R_1 & E_1 & 0 \\ 0 & E_2 & 0 \end{bmatrix} \tag{A14}$$

After permutation, and subtraction of $A_1R_1^{-1}$ times the third block of rows from the second block, we obtain the following matrix, whose rank is the same as that of the transmission matrix:

$$\begin{matrix} p \\ p \\ n-p \\ m-p \end{matrix}\begin{bmatrix} S_1 & W_1^TAW_1-sI & W_1^TAW_2 \\ 0 & R_1 & E_1 \\ 0 & 0 & A_2-A_1R_1^{-1}E_1-sI \\ 0 & 0 & E_2 \end{bmatrix} \tag{A15}$$

Since, by regularity, R_1 and S_1 are full rank, we see that there are no transmission zeros if and only if the reduced system is observable. □

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