CMSC 858S: Randomized Algorithms

Fall 2001

Final Exam

Instructions:

1. You have 3 hours. There are four questions, arranged approximately in increasing order of difficulty.
2. There are 40 points in total. Questions 3 and 4 give you an option of "buying a hint". If you choose to buy a hint (you can choose to do so for one, both, or none of these two problems–it is up to you) for a problem, you will get a hint, and your final score for that problem will be half of the score you get for that problem.
3. If you think you have some ideas that deserve partial credit, please *itemize them concisely*. However, if it is clear that several irrelevant observations have been made, such partial solutions will be ignored and *not* given credit; so, please only include partial ideas if you see them as relevant. Also, if you are unable to solve a problem but can do so by making some assumptions, please clearly state these assumptions and then proceed.

Good luck!

1. (**5 points**) Consider a multiple-access channel with some n users to start with, where each user has one packet to send to the channel. Suppose time is discrete (counted as $0, 1, 2, \ldots$ etc.). If at a time step, more than one user tries to send a packet to the channel, then no user succeeds in that time step; a user succeeds in a time step iff it was the *unique* user to send a packet to the channel at that time step.

Consider the distributed scheme where each user independently sends its packet to the channel with probability $1/n$ at each time step until it is successful; once a user succeeds in a time step, it leaves the system (i.e., no longer transmits any packets). Suppose we are given that there are at least $n/2$ users still in the system at some time step t. Show that there is a constant $a > 0$ such that the probability of some user succeeding at time step t, is at least a.

2. (**10 points**) Suppose we are given a hypergraph $H = (V, E)$: V is some set of n elements, and E is a collection of subsets of V. We are also given an integer k, where $1 \leq k \leq n$. If each element v of V is given a weight $w(v)$, the weight of a set $S \subseteq V$ is defined as usual to be $\sum_{v \in S} w(v)$.

Given an assignment of weights w to the elements of V, let us define some terms. Suppose the sets in E get some ℓ distinct weights, ordered as $x_1 < x_2 < \cdots < x_\ell$. Then, a set $S \in E$ is called *pth smallest set in E* iff $w(S) = x_p$. For instance, suppose $V = \{1, 2, 3\}$, and $E = \{\{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$; suppose also that $w(1) = 2$, $w(2) = 3$, and $w(3) = 5$. Then, $x_1 = 2$, $x_2 = 5$, $x_3 = 8$, and $x_4 = 10$. Also,

- the set $\{1\}$ is the first smallest set in E;
- the sets $\{3\}$ and $\{1, 2\}$ are second smallest sets in E;
- the set $\{2, 3\}$ is the third smallest set in E; and
- the set $\{1, 2, 3\}$ is the fourth smallest set in E.

Note that more than one set can be a pth smallest set in E; also, it is possible no set is a pth smallest set (e.g., if $p \geq 5$ in the above example).

Recall that we are given $H = (V, E)$ and k, where $|V| = n$ and $1 \leq k \leq n$. Show that there is a value s upper-bounded by a polynomial in n, such that the following holds. If we choose a weight uniformly at random from $\{1, 2, \ldots, s\}$ and independently for each $v \in V$, then the probability of getting more than one kth smallest set in E, is at most $1/2$.

3. **(10 points; you can buy a hint for this problem if you wish)** Suppose you are given an undirected graph $G = (V, E)$ where the maximum degree of any vertex is some $\Delta \geq 2$. (Recall that the degree of a vertex is the number of edges incident upon it.) Let $d(u)$ be the degree of vertex u; given a partition of V into two sets A and B, let $d_A(u)$ and $d_B(u)$ denote the number of neighbors of u in A and the number of neighbors of u in B, respectively. Show that there is a constant $C > 0$ and a partition of V into two sets A and B such that for all u,

$$d_A(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta} \text{ and } d_B(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta}.$$

(One approach will be to start with a random partition.)

4. **(You can buy a hint for this problem if you wish)** We are given integers n and ℓ (with $\ell \leq n$); for this problem, we will only consider hypergraphs $H = (V, E)$ for which $|V| = n$, and where each hyperedge (i.e., element of E) has exactly ℓ elements.

(a) **(5 points)** Show that if $|E| \leq 2^{\ell-1}$, then any such $H = (V, E)$ is 2-colorable. (Note that $|E|$ is not necessarily smaller than $2^{\ell-1}$; the case $|E| = 2^{\ell-1}$ is possible.)

(b) **(10 points)** If n is even, then a 2-coloring of H (if one exists) is called *equitable* if, in addition to being a valid 2-coloring, say with colors Red and Blue, we also have that exactly $n/2$ vertices are colored red, and exactly $n/2$ are colored Blue. We want to show that there is some function $f(\ell)$ [which we want to keep as small as possible] such that the following holds for every even integer ℓ: there exists some hypergraph $H = (V, E)$ with $n = \ell^2$, $|E| = f(\ell)$, and with each hyperedge having ℓ elements, such that H has *no equitable 2-coloring.*

To achieve this goal, we will employ the probabilistic method as briefly sketched now (you have to fill in the details). Suppose we start with a set V of $n = \ell^2$ elements, and construct a random hypergraph R as follows: choose $f(\ell)$ edges independently, each edge being chosen uniformly at random from the set of $\binom{n}{\ell}$ subsets of V that contain ℓ elements. Use this random process and the probabilistic method to show the existence of a hypergraph $H = (V, E)$ which has $|V| = n = \ell^2$, $|E| = f(\ell)$, and with each hyperedge having ℓ elements, such that H has no equitable 2-coloring. In the process, you need to find as small a value for $f(\ell)$ as you can.

Here is a useful starting point. Suppose we *fix* an arbitrary Red-Blue coloring χ of V, which colors exactly $n/2$ elements of V with the color Red (with the remaining $n/2$ elements being colored Blue). What is the probability that χ is a valid 2-coloring for the above random hypergraph R? Also, you may find the following inequality (which you do not need to prove) useful:

$$\text{if } r \leq b \leq a, \text{ then } \frac{\binom{b}{r}}{\binom{a}{r}} \geq \left(\frac{b-r+1}{a-r+1}\right)^r.$$

End of Exam