Economical Delone Sets for Approximating Convex Bodies

Ahmed Abdelkader and David Mount

Department of Computer Science
University of Maryland, College Park

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Curse of dimensionality

Upper-Bound Theorem [McMullen, 1970], also [Seidel, 1995]

Let $P \subset \mathbb{R}^d$ be a convex polytope.

- If P has n vertices, then the number of faces is $O(n^{\lfloor d/2 \rfloor})$.
- If P has n facets, then the number of vertices is $O(n^{\lfloor d/2 \rfloor})$.

./figs/images/mom1.png

./figs/images/mom2.png

Moment curve: $t \mapsto (t, t^2, \ldots, t^d)$

Figure from [George Papazafeiropoulos]

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Workarounds

Settle for an approximation

- Explicit: find an approximating polytope
- Implicit: approximate membership testing

Polytope Membership Queries

.. implementing a membership oracle

Polytope membership queries

Given a polytope K in $\mathbb{R}^d$, preprocess K to answer membership queries:

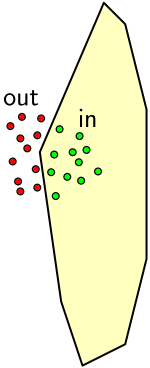
Given a query point $q \in \mathbb{R}^d$, is $q \in K$?

Assumptions

- Dimension d is a constant
- K is given as the intersection of n halfspaces

Dual: Halfspace emptiness queries [Matoušek'92, Erickson'98]

- $d \leq 3 \;\Rightarrow$ Query time: $O(\log n)$ with $O(n)$ storage
- $d \geq 4 \;\Rightarrow$ (roughly) $O(n^{1-1/\lfloor d/2 \rfloor})$ time with $O(n)$ storage
- Restriction $\Rightarrow \tilde{\Omega}(n^{1-1/d})$ time if storage is $\tilde{O}(n)$

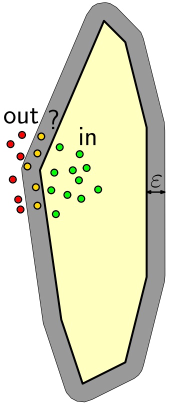


Approximate Polytope Membership Queries

How about an *approximate* membership oracle?

ε-APM queries

- Specify $\varepsilon > 0$ at preprocessing time
- Assume the polytope scaled to unit diameter
- If the query point's distance from K:
  - $0 \Rightarrow$ Inside
  - $> \varepsilon \Rightarrow$ Outside
  - Otherwise: Either answer is acceptable



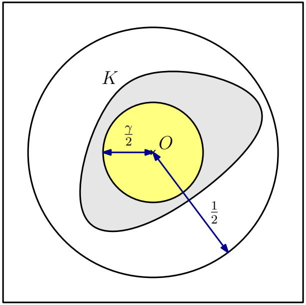
Goal

Query time: $O(\log \frac{1}{\varepsilon})$ ← Logarithmic
Storage: $O(1/\varepsilon^{O(d)})$ ← Efficient?

Canonical Form

Easier to work with fat bodies

In $O(n)$, find an invertible affine transform T s.t. $T(K)$ is in canonical form.



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If q is a point at distance greater than $\varepsilon \cdot diam(K)$ from K, then $T(q)$ is at distance greater than ε/d from $T(K)$.

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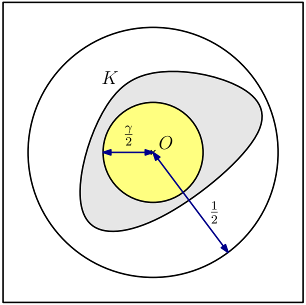

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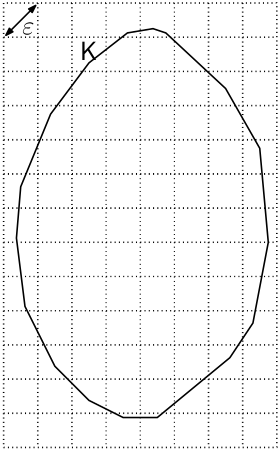
Time-efficient Solution [Bentley et al. (1982)]

- Create a grid with cells of diameter ε
- For each column, store the topmost and bottommost cells intersecting K
- Query processing:
    - Locate the column that contains q
    - Compare q with the two extreme values



Time-efficient:

- $O(1/\varepsilon^{d-1})$ columns
- Query time: $O(\log \frac{1}{\varepsilon})$ ← optimal
- Storage: $O(1/\varepsilon^{d-1})$ ← high!

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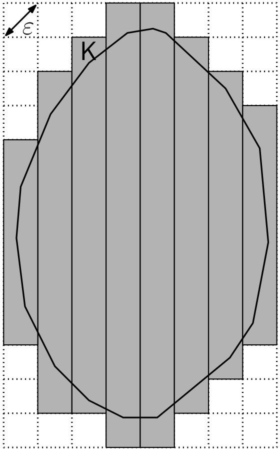

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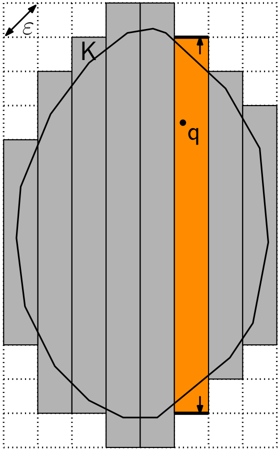

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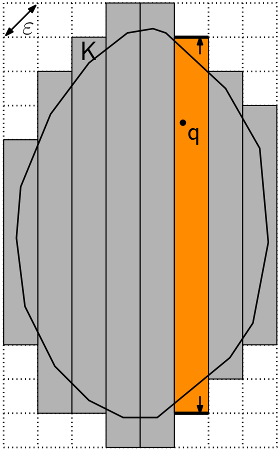

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Space-efficient Solution [Arya et al. (2011, 2012)]

- $Q \leftarrow$ unit hypercube; $t \leftarrow \widetilde{O}(1/\varepsilon^{(d-1)/8})$
- Find an ε-approximation of $Q \cap P$
- If $\leq t$ facets suffice, Q stores them
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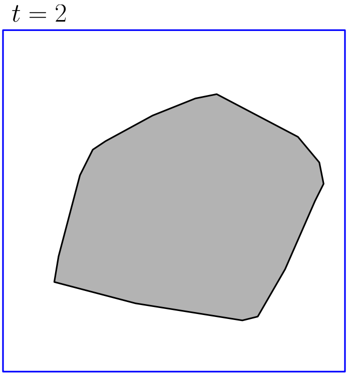

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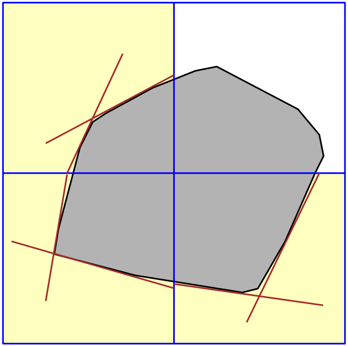

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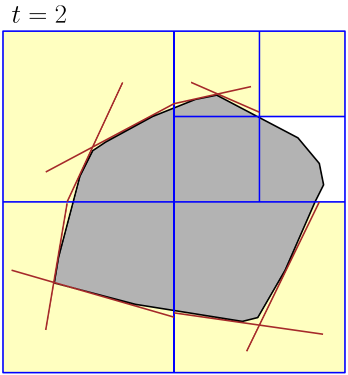

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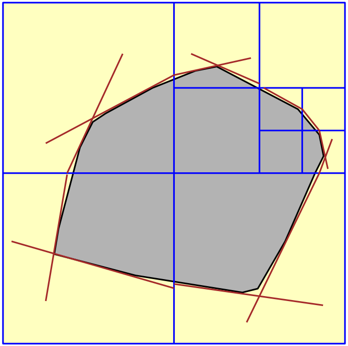

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Digression?

Polytope approximation

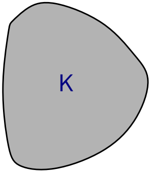
Let $K \subset \mathbb{R}^d$ be a convex body (polytope) of unit diameter. Find P s.t.

- The Hausdorff distance between K and P is at most $\varepsilon > 0$,
- P is a convex polytope having few faces.

Combinatorial Complexity of Approximating Polytopes

Dudley (1974) or Bronshteyn-Ivanov (1976)

Easy construction of P with $O(1/\varepsilon^{(d-1)/2})$ vertices (or facets).



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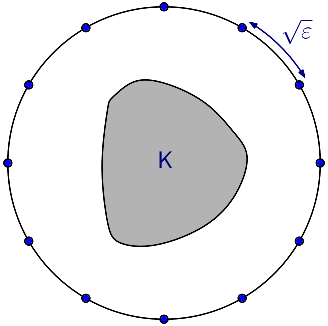

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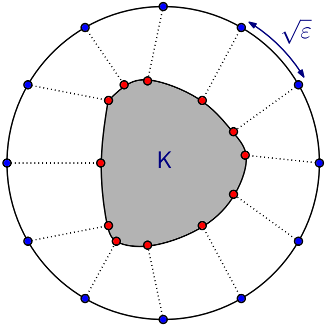

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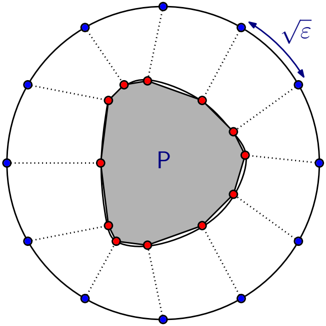

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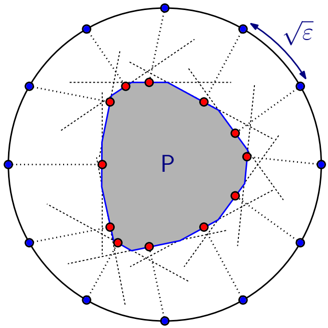

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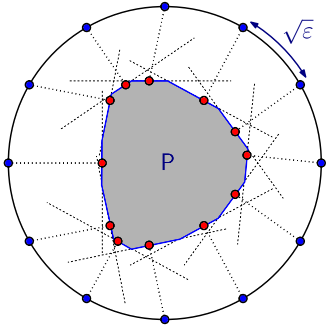

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- Create a hierarchy of eroded bodies
- For each erosion, cover the boundary with a ring of ellipsoids
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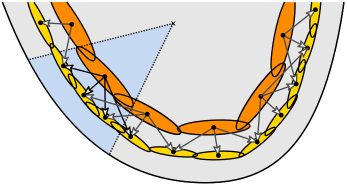

Asymptotically optimal

Query time: $O(\log \frac{1}{\varepsilon})$ ← optimal
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- Lower-bound realized by $\mathbb{S}^{d-1}$
- Ray shooting vs. point location

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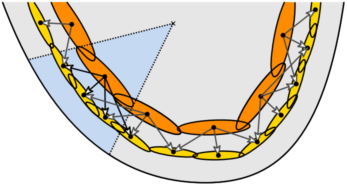

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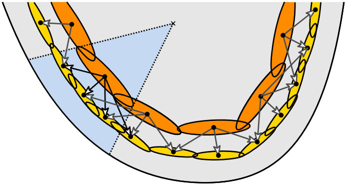

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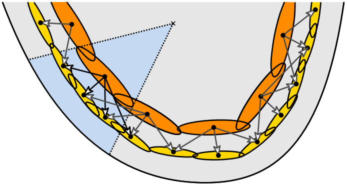

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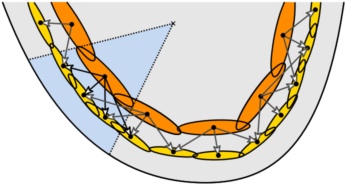

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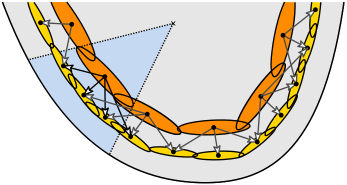

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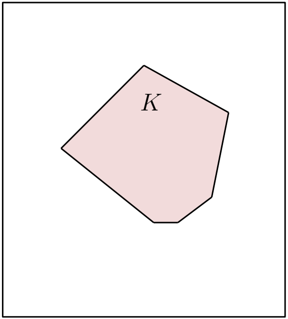
Representational Technicality

A convenient alternative to $K \oplus B(\delta)$

Assuming K is represented as $K = \cap_i H_i$ where each H_i is of the form

$$H_i = \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq a_i\}, \text{ where } c_1\delta \leq \|v_i\| \leq c_2,$$

we work with $K_\delta = \cap_i H_{i,\delta}$, where $H_{i,\delta} = \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq a_i + \delta\}$.



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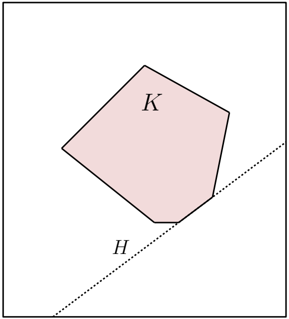

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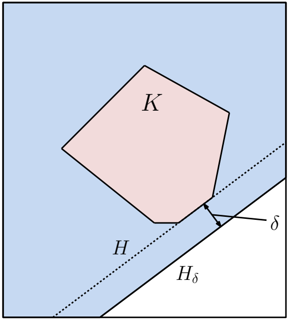


APM by Quadtree Search

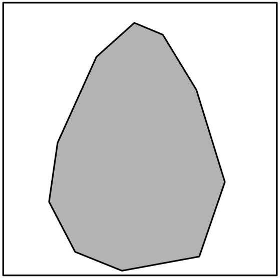
Quadtree-based approach

- Preprocessing: Build a quadtree, subdividing each node that cannot be resolved as being inside or outside
- Stop at diameter ε
- Query: Find the leaf node containing q and return its label

Analysis

Query time: $O(\log \frac{1}{\varepsilon})$ (Quadtree descent)
Storage: $O(1/\varepsilon^{d-1})$ (Number of leaves)

What does it take to beat $O(1/\varepsilon^{d-1})$?



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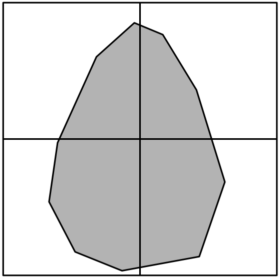

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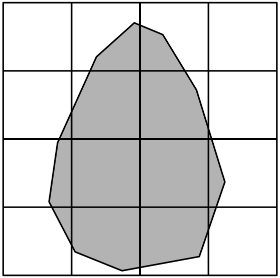

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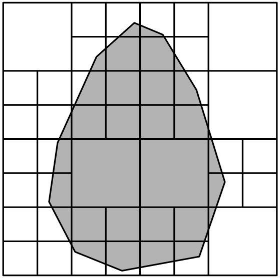

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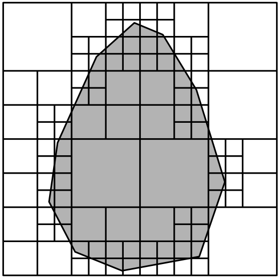
- Preprocessing: Build a quadtree, subdividing each node that cannot be resolved as being inside or outside
- Stop at diameter ε
- Query: Find the leaf node containing q and return its label

Analysis

Query time: $O(\log \frac{1}{\varepsilon})$ (Quadtree descent)
Storage: $O(1/\varepsilon^{d-1})$ (Number of leaves)

What does it take to beat $O(1/\varepsilon^{d-1})$?



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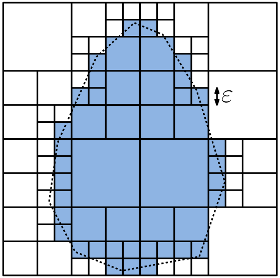

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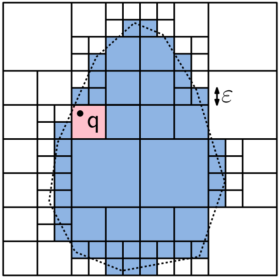

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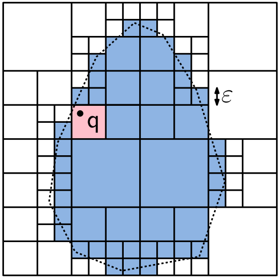

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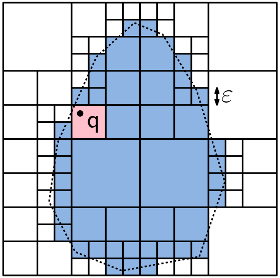


Macbeath Regions

Want cells that conform to K's shape

Macbeath Region [Macbeath (1952)]

Given convex body K, $x \in K$, and $\lambda > 0$:

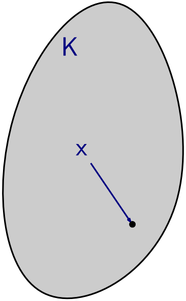
- $M_K(x) = x + (K - x) \cap (x - K)$
  - Intersection of K and its reflection about x
- $M_K^\lambda(x)$: Scaling of $M_K(x)$ by factor λ

Will omit K when clear

John ellipsoid [John (1948)]

May use ellipsoids instead

$$E^\lambda(x) \subseteq M^\lambda(x) \subseteq E^{\lambda\sqrt{d}}(x)$$



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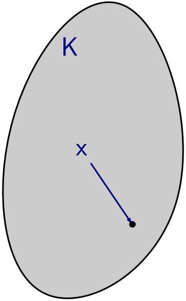

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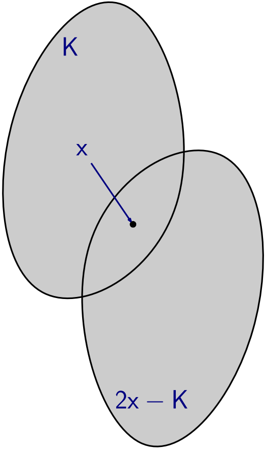

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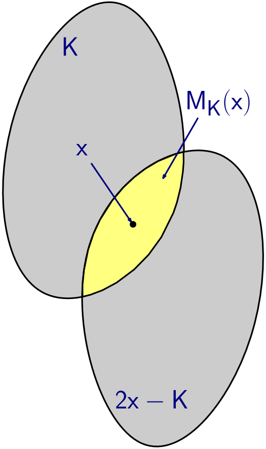

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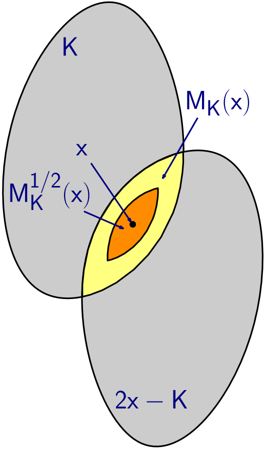

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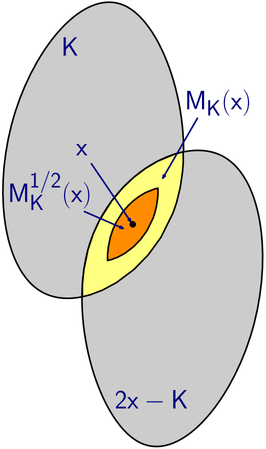

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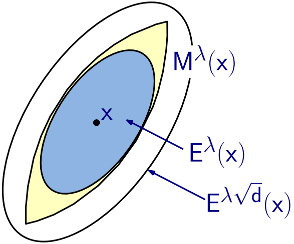


Properties of Macbeath Regions

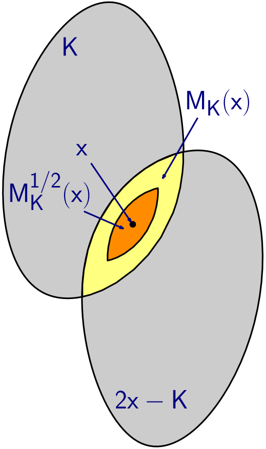
Properties

- $M^{\lambda}(x)$ is convex and symmetric about x
- Expansion-Containment: [Ewald et al (1970)]
  If $M^{\lambda}(x)$ intersects $M^{\lambda}(y)$, with $\lambda < 1$, then

$$M^{\lambda}(y) \subseteq M^{c\lambda}(x), \text{ where } c = \frac{3+\lambda}{1-\lambda}.$$

Good approximation primitives

- Macbeath regions act like metric balls
- But, with respect to which metric?



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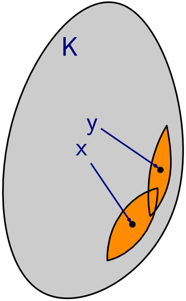

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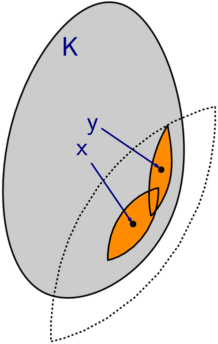

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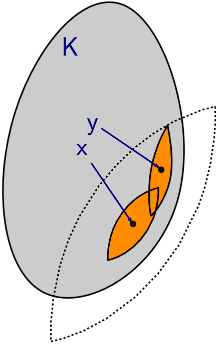

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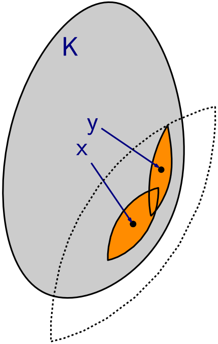

Macbeath Regions and the Hilbert Geometry

From Hilbert geometry [Hilbert (1895)]

- Hilbert Metric: Given $x, y \in K$, define

$$f_K(x,y) = \frac{1}{2} \ln \left(\frac{\|x' - y\|}{\|x' - x\|} \frac{\|x - y'\|}{\|y - y'\|} \right)$$

- Hilbert Ball: $B_H(x,\delta) = \{y \in K : f_K(x,y) \leq \delta\}$
- Locally-sensitive to the shape of the convex body

.. to Macbeath regions [Vernicos and Walsh (2016)]

For all $x \in K$ and $0 \leq \lambda < 1$:

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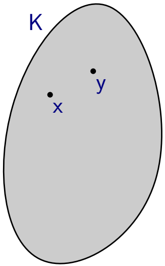

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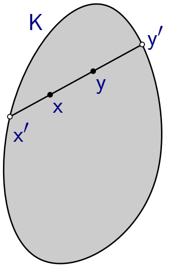

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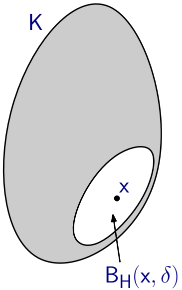

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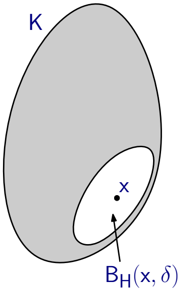

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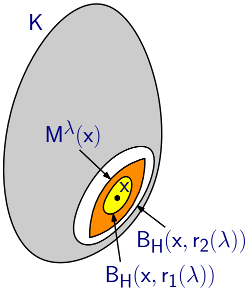
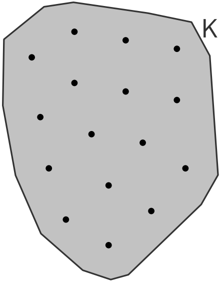


More from Metric Spaces



Where to put the Macbeath regions?

Point set layouts in a metric space $\mathbb{X}$

- ε-packing: If the balls of radius $\varepsilon/2$ centered at every point of X are disjoint
- ε-covering: If every point of $\mathbb{X}$ is within distance ε of some point of X
- $(\varepsilon_p, \varepsilon_c)$-Delone Set: If X is an ε_p-packing and an ε_c-covering
- ε-Net: If X is an $(\varepsilon, \varepsilon)$-Delone set

We seek economical Delone sets for K, that fit within an expansion of K by $\delta \in \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \varepsilon\}$

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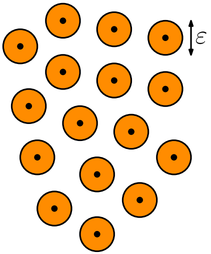

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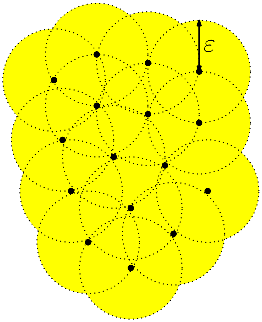

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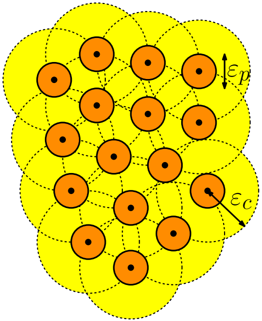

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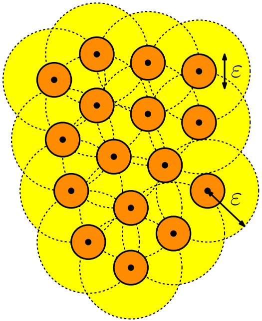

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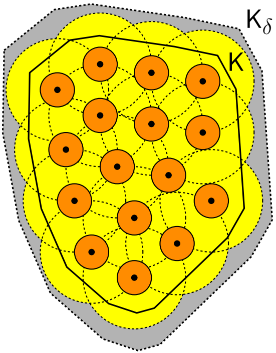

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Macbeath Ellipsoids and Delone Sets

Delone sets from Macbeath ellipsoids

- Let K_δ be an expansion of K by $\delta > 0$
- Let $X_\delta \subset K$ be a maximal set of points such that $E^{\lambda_0}(x)$ are disjoint for all $x \in X_\delta$

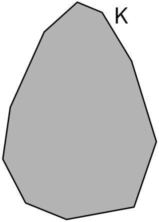
Lemma

$$K \subseteq \bigcup_{x \in X_\delta} E^{\frac{1}{2}}(x) \subseteq K_\delta$$

Macbeath-Based Delone Set

X_δ is essentially a $(\frac{1}{2}, 2\lambda_0)$-Delone set for K

λ_0 is a small constant $1/(4\sqrt{d}+1)$



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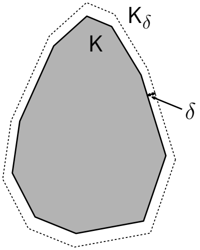

Macbeath Ellipsoids and Delone Sets

Delone sets from Macbeath ellipsoids

- Let K_δ be an expansion of K by $\delta > 0$
- Let $X_\delta \subset K$ be a maximal set of points such that $E^{\lambda_0}(x)$ are disjoint for all $x \in X_\delta$

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$$K \subseteq \bigcup_{x \in X_\delta} E^{\frac{1}{2}}(x) \subseteq K_\delta$$

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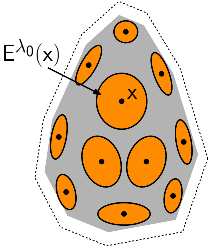

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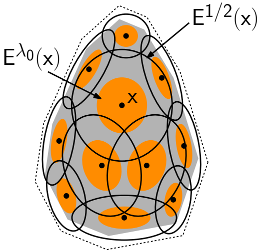

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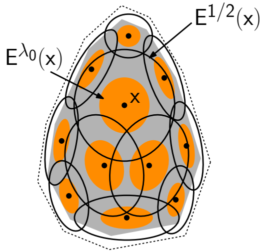

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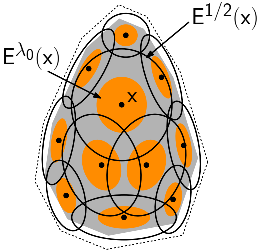

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Expansion-Containment over δ

Lemma

For a convex body $K \subset \mathbb{R}^d$ and $\delta \geq 0$, for all $x \in K$

$$M_\delta(x) \subseteq M_{2\delta}(x) \subseteq M_\delta^2(x).$$

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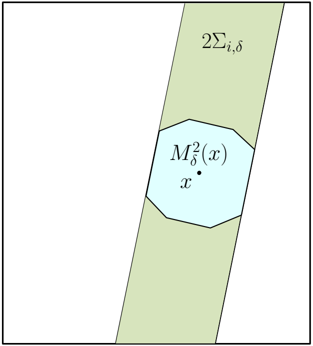

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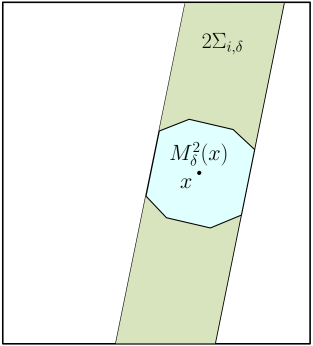

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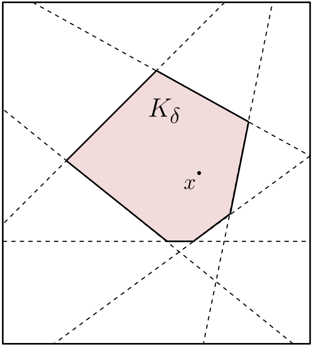

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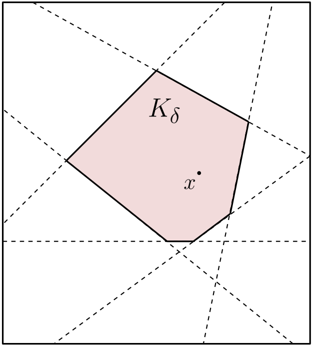

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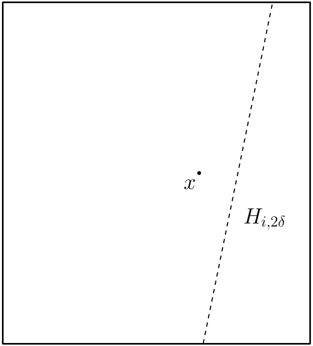

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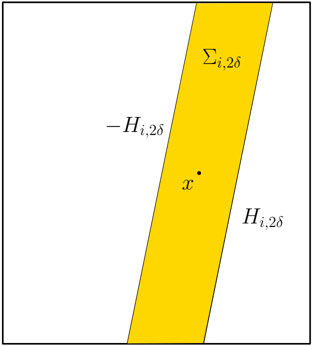

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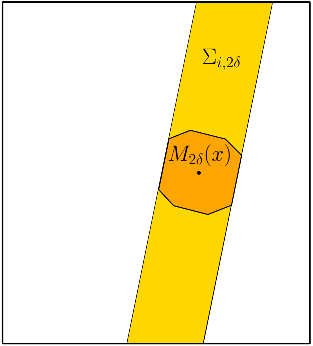

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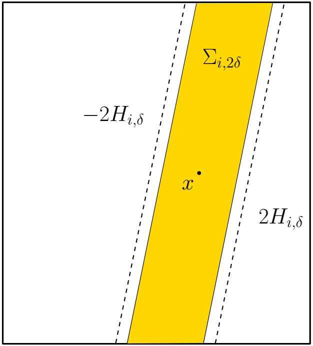

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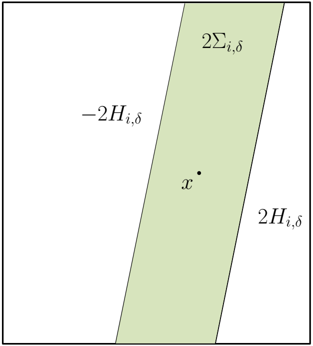

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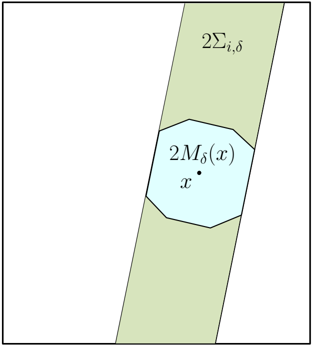

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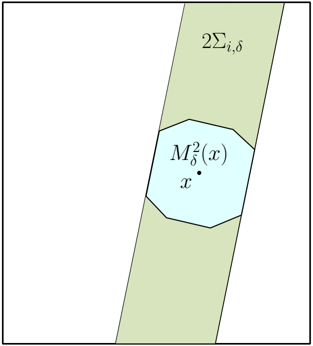

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Preprocessing

- Input: K and $\varepsilon > 0$
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- Stop when $|E_\ell| = 1$ (at $\delta_\ell = O(1)$)

Query Processing

- Descend the DAG from root (level ℓ) until:
  - $q \notin \frac{1}{2}$-scaled child ellipsoids $\Rightarrow$ "outside"
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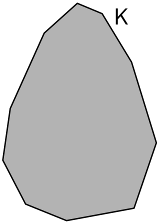

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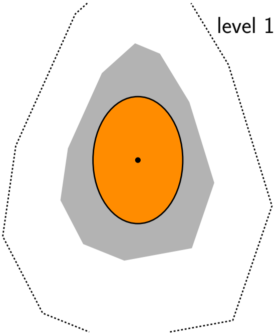

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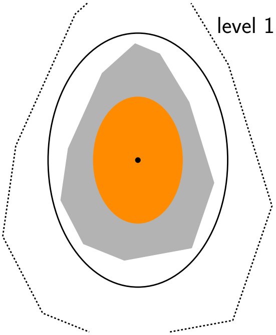

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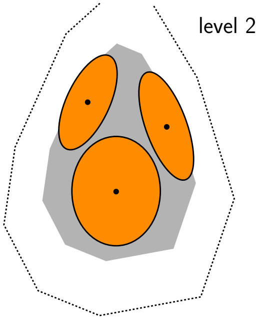

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- Descend the DAG from root (level ℓ) until:
  - $q \notin \frac{1}{2}$-scaled child ellipsoids $\Rightarrow$ "outside"
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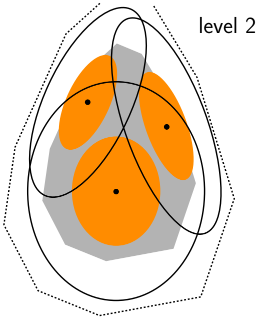

APM Data Structure

Preprocessing

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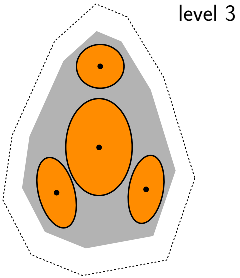

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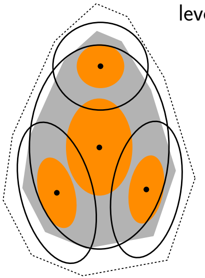

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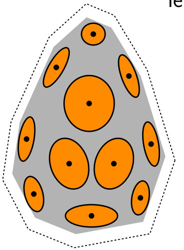
level 4



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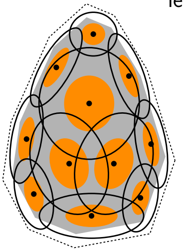

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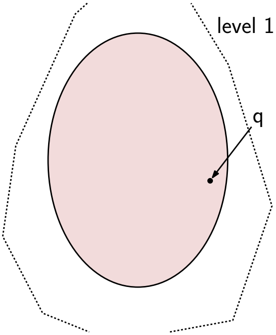

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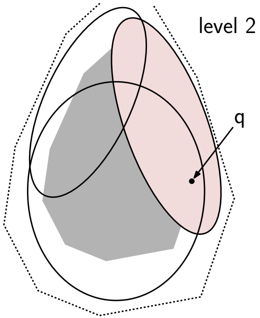

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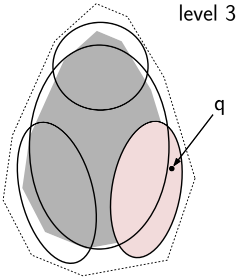

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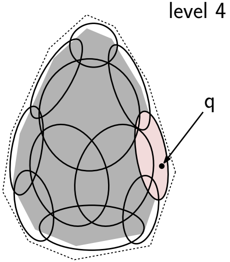
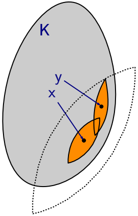


Analysis

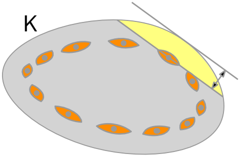
Query time

- Out-degree: $O(1)$ (exp-containment + packing)
- Query time per level: $O(1)$
- Number of levels: $O(\log \frac{1}{\varepsilon})$ (ε to $O(1)$)
- Total query time: $O(\log 1/\varepsilon)$



Storage

- #M-regions to cover K_{δ_i}: $O(1/\delta_i^{(d-1)/2})$ (ECC)
- Storage for bottom level: $O(1/\varepsilon^{(d-1)/2})$
- Leaf level dominates (geometric progression)
- Total storage: $O(1/\varepsilon^{(d-1)/2})$



Economical Cap Cover

[Barany and Larman (1989), Arya, de Fonseca, Mount (2016)]

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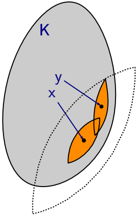

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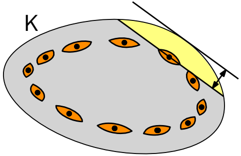

Economical Cap Cover

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Summary

Delone sets for convex bodies

- Intuitive and space-efficient cover
- $O(1/\delta^{(d-1)/2})$ ellipsoids for δ-approximation
- Simple DAG hierarchy for time-efficient look-ups
    - Asymptotically optimal

Thanks for listening

Questions?
akader@cs.umd.edu

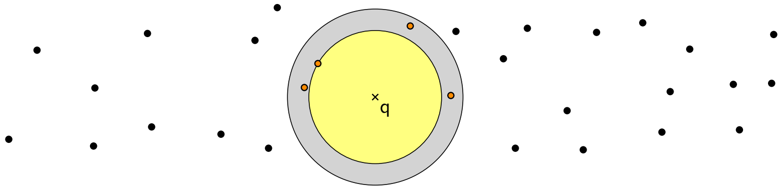
Proximity Search

Approximate Nearest-Neighbor Searching (ANN)

Given a set of n points $P \subset \mathbb{R}^d$, preprocess P into a data structure to such that, given a query point $q \in \mathbb{R}^d$ one can efficiently find a $p \in P$ where

$$\|p - q\| \leq \|nn(q) - q\|,$$

where $nn(q)$ is the *nearest-neighbor* of q in P.



ANN Searching and Polytope Approximation

Lifting and Voronoi Diagrams

Lift the points of P to Ψ, take the upper envelope of the tangent hyperplanes, and project the skeleton back onto the plane. The result is the Voronoi diagram of P.

Intuition: Improved representations of polytopes lead to improvements for ANN