Sampling Conditions for Conforming Voronoi Meshing by the VoroCrust Algorithm

Ahmed Abdelkader[1], Chandrajit Bajaj[2], Mohamed Ebeida[3], Ahmed Mahmoud[4], Scott Mitchell[3], John Owens[4] and Ahmad Rushdi[4]

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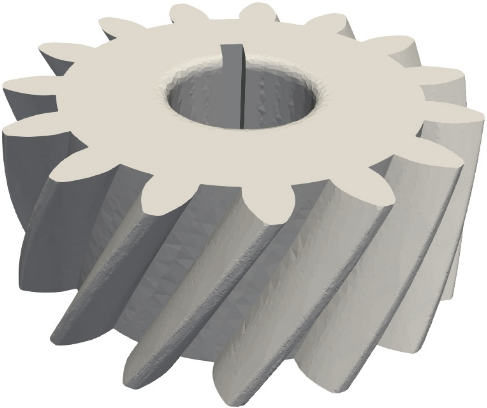
[3]Sandia National Laboratories

[4]University of California, Davis

Motivation

Meshing

Partition into "simple" elements



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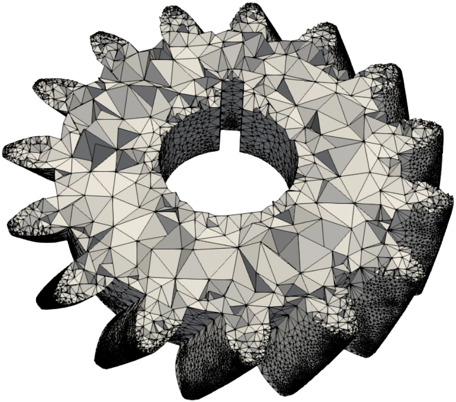
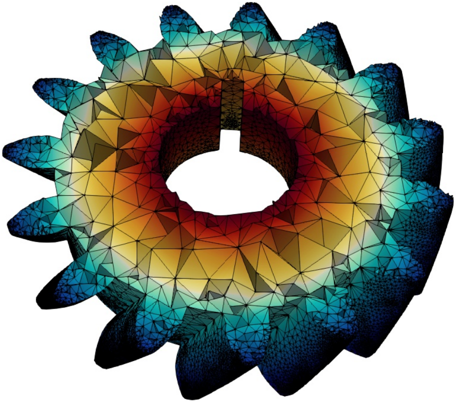


Tetrahedral mesh

Motivation

Finite Element Method (FEM)

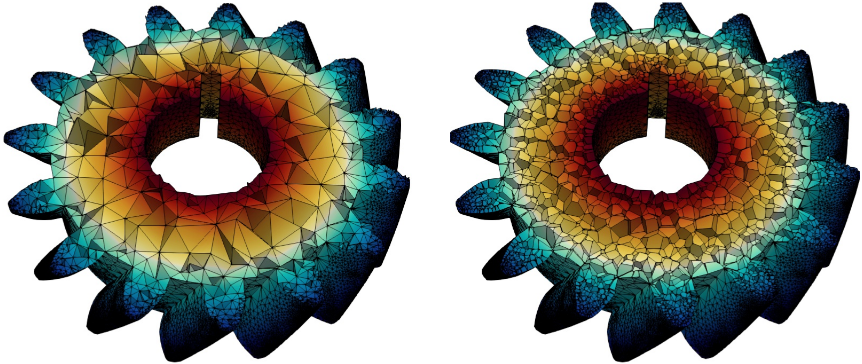
PDE $\xrightarrow{\Delta}$ Mesh $\rightarrow$ Algebra $\rightarrow$ Discrete Approximation $\rightarrow$ Interpolation



Motivation

What type of element to use?

No silver bullet ..



Tetrahedral mesh vs. Voronoi mesh

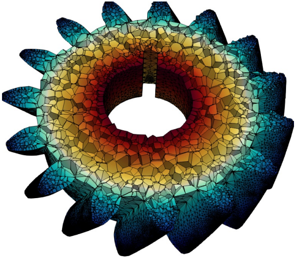
Motivation

Why polyhedral meshing?

- Less sensitive to stretching
  - Efficient meshing of complicated domains
- Higher node degree, even at boundaries
  - Better approximations of gradients

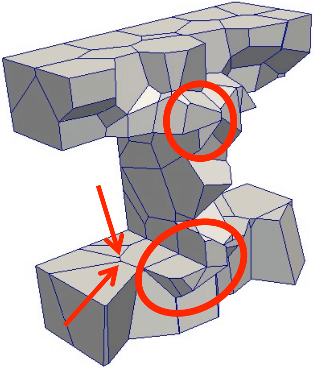
Why Voronoi meshing?

- Convex elements
- Positive Jacobians
- Orthogonal dual: a Delaunay mesh



Voronoi meshing by "clipping"

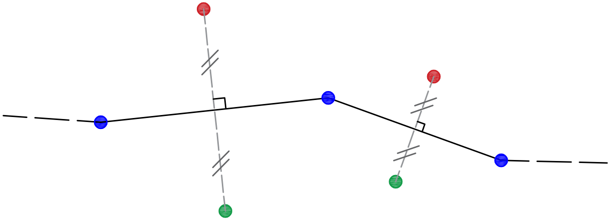
Initial Voronoi mesh → Truncate cells by bounding surface → Defects



Related Work

Voronoi meshing by “mirroring”

Pair seeds naïvely across surface



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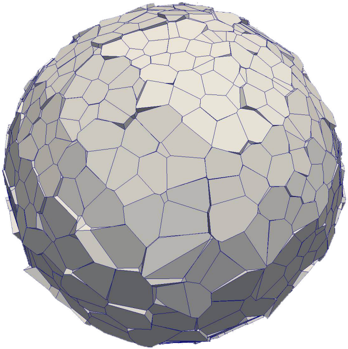
Pair seeds naïvely across surface → Bad surface normals



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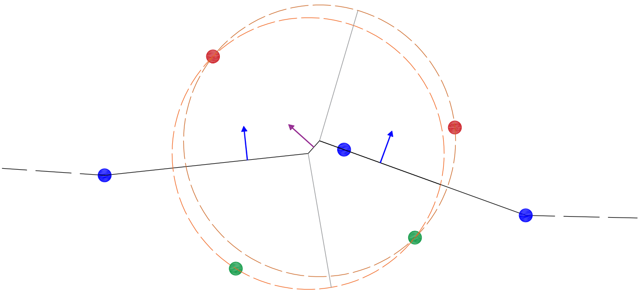
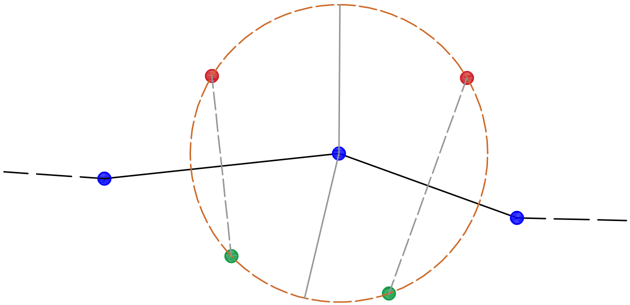


Related Work

Voronoi meshing

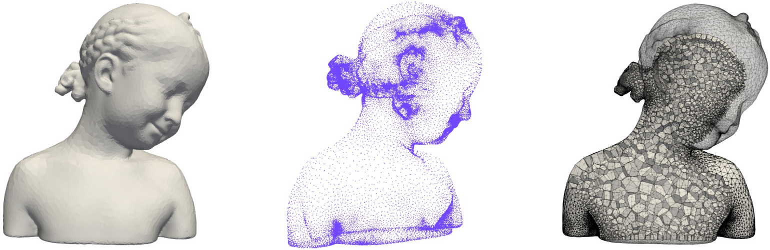
VoroCrust is a principled approach to mirroring



Preliminaries

Notation

- $\mathcal{O}$: bounded open set in $\mathbb{R}^3$; the volume to be meshed
- $\mathcal{M}$: boundary of $\mathcal{O}$; a smooth surface
- $\mathcal{P}$: input sample from $\mathcal{M}$
- $\hat{\mathcal{M}}, \hat{\mathcal{O}}$: surface and volume meshes

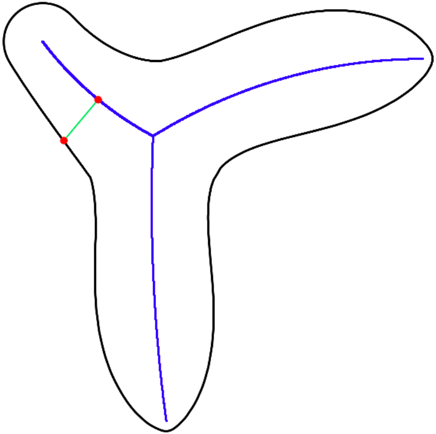


Fillette aux tourterelles [Luigi Pampaloni]

Preliminaries

Local features size (lfs)

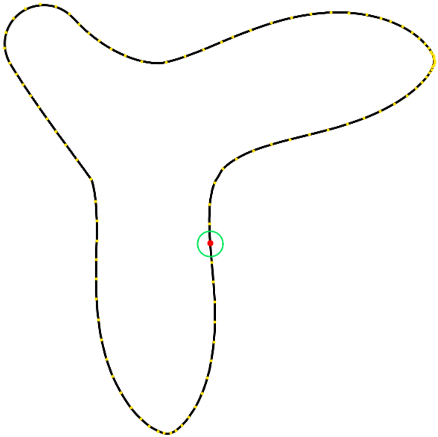
At any $x \in \mathcal{M}$, $lfs(x)$ is the distance from x to the medial axis of $\mathcal{M}$.



Preliminaries

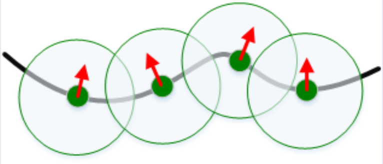
ϵ-sample

A set of points P on $\mathcal{M}$ such that $\forall x \in \mathcal{M}\ \exists p \in P$ s.t. $||px|| \leq \epsilon \cdot \mathit{lfs}(x)$.



VoroCrust Intuition - A 2D Example

(1) Weighted samples (balls)



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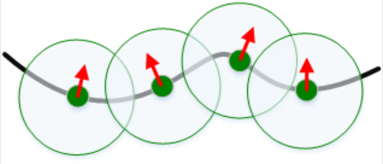
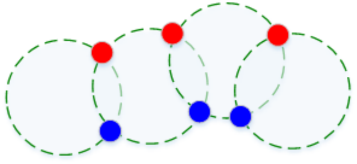


(2) Collect intersection points



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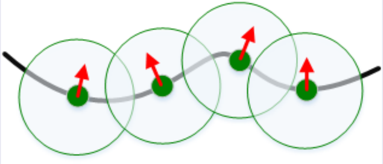

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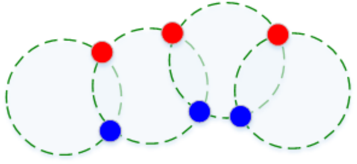
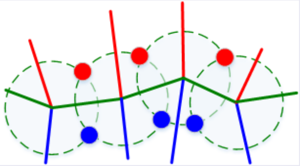


(3) Compute Voronoi diagram



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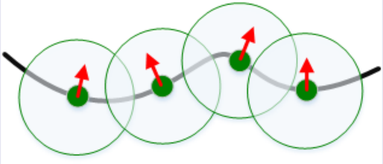

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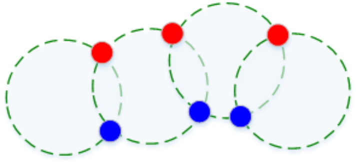

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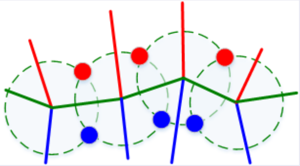
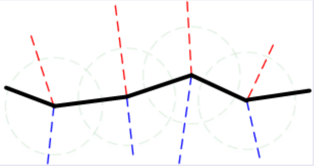


(4) Keep the separating facets



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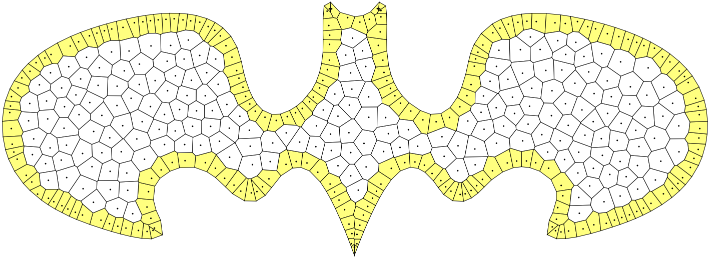

Tools - Non-Uniform Approximation

From balls to surfaces [Chazal, Lieutier (2006)]

Let $\mathcal{P}$ be an ϵ-sample of $\mathcal{M}$ and define b_p as the ball centered at $p \in \mathcal{P}$ with radius $\delta(\epsilon) \cdot \mathit{lfs}(p)$. Then, $\mathcal{M}$ is a deformation retract of $\cup b_p$.

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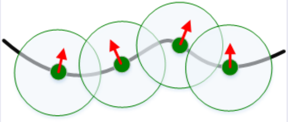

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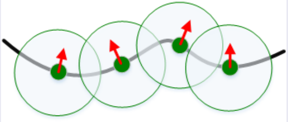

Corollary

For sufficiently small ϵ, we may set $\delta(\epsilon) = c$; we take $c = 2$.

Tools - Union of Balls

Power distance and cell

For a ball $b \in \mathcal{B}$ centered at c with radius r, $\pi(b, x) = \|cx\|^2 - r^2$.
$V_b = \{x \in \mathbb{R}^d \mid \pi(b, x) \leq \pi(b', x) \, \forall b' \in \mathcal{B}\}$.

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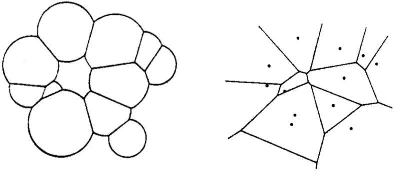

Figures from [Edelsbrunner]

Tools - Union of Balls

Weighted α-complex and α-shape [Edelsbrunner (1992-1995)]

Define $\mathcal{K} = \mathit{Nerve}(\{V_b \cap b \mid b \in \mathcal{B}\})$ and $\mathcal{S}$ as the underlying space $|\mathcal{K}|$.

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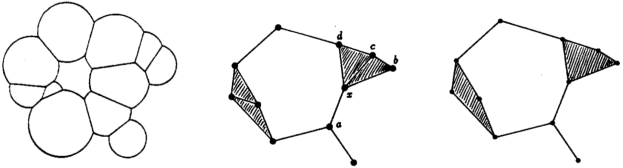

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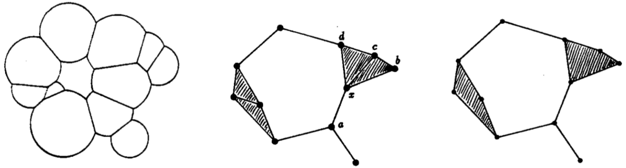

Figures from [Edelsbrunner]

Homotopy-equivalence

The nerve theorem implies $\mathcal{S} = |\mathcal{K}|$ has the same homotopy-type as $\cup\mathcal{B}$.

Tools - Voronoi-based Surface Reconstruction

Medial Axis of a Union of Balls [Amenta, Kolluri (2001)]

Let $\mathcal{U}$ be a union of balls in $\mathbb{R}^d$, let V be the vertices of $\partial\mathcal{U}$ and let $\mathcal{S}$ be the α-shape of $\mathcal{U}$. The medial axis of $\mathcal{U}$ consists of:

1. the singular faces of $\mathcal{S}$ and
2. the subset of $Vor(V)$ which intersects the regular components of $\mathcal{S}$.

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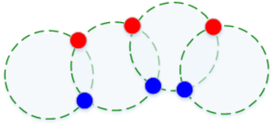
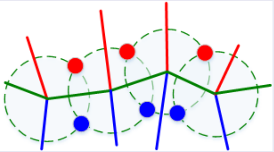
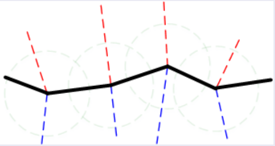
VoroCurst steps (2-4)







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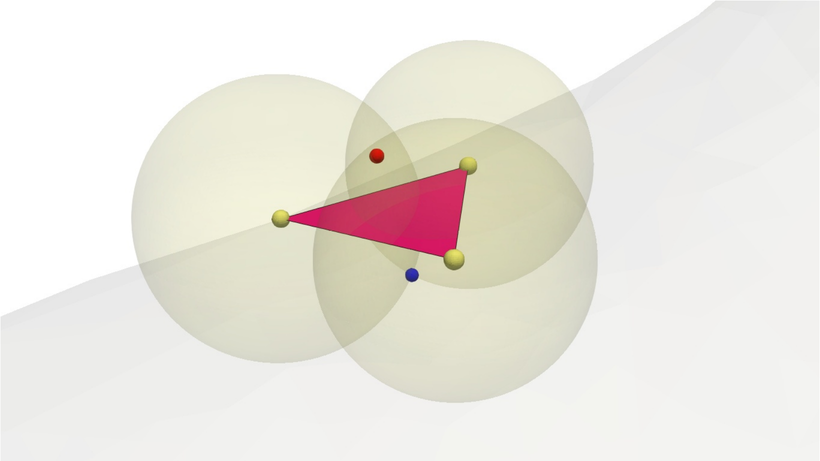

Case(1): singular facet of the α-shape

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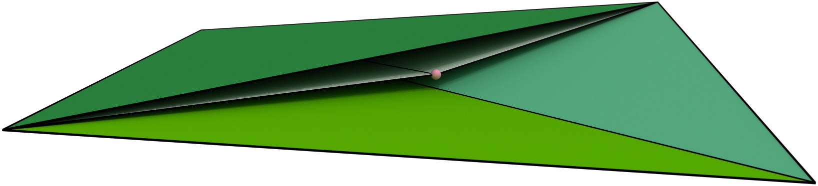


Case(2): regular component of the α-shape

Preliminaries - Isotopy

Isotopic surface reconstruction

Recover both surface topology and embedding

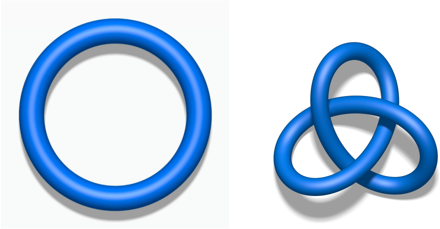
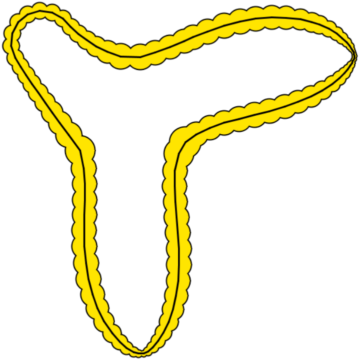


Figure from [Wikipedia]

Tools - Isotopic Surface Reconstruction

A purely topological condition [Chazal, Cohen-Steiner (2005)]

- $\mathcal{M}'$ is homeomorphic to $\mathcal{M}$,
- $\mathcal{M}'$ is included in a topological thickening $\mathbb{M}$ of $\mathcal{M}$,
- $\mathcal{M}'$ separates the sides of $\mathbb{M}$.



VoroCrust - The Abstract Algorithm

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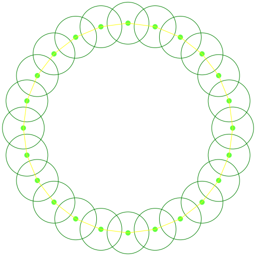

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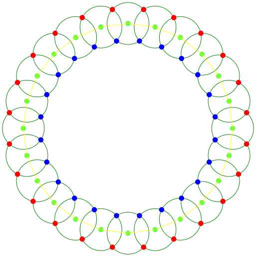

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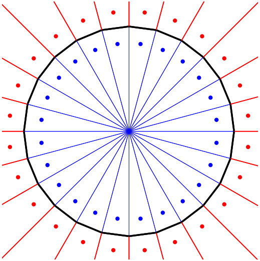

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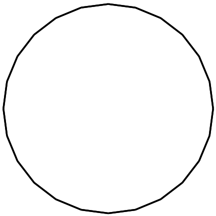
- Produce the surface approximation $\hat{\mathcal{M}}$ as the Vorono facets of separating $\mathcal{S}^{\uparrow}$ from $\mathcal{S}^{\downarrow}$.



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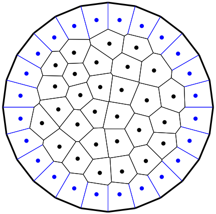

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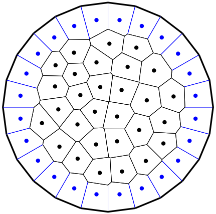
- Return the volume mesh $\hat{\mathcal{O}}$ as the Voronoi cells in $\mathit{Vor}(\mathcal{S}^{\updownarrow} \cup \mathcal{S}^{\downarrow\downarrow})$ with seeds in $\mathcal{S}^{\downarrow} \cup \mathcal{S}^{\downarrow\downarrow}$.



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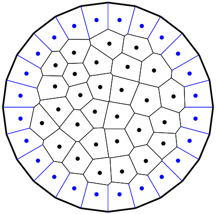
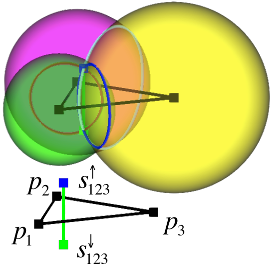
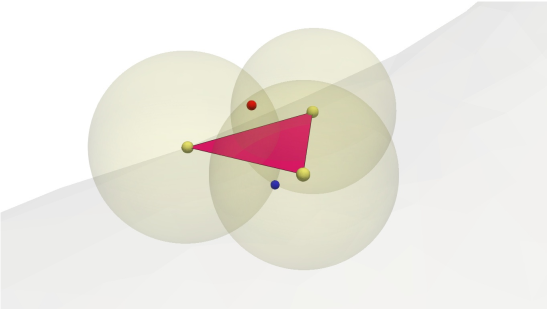


Theorem

$\hat{\mathcal{M}}$ is isotopic to $\mathcal{M}$. Hence, $\hat{\mathcal{O}}$ is isotopic to $\overline{\mathcal{O}}$.

VoroCrust - Ball Intersections

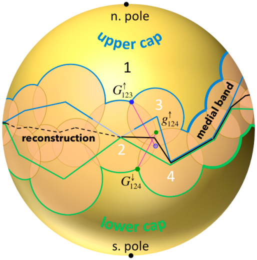




VoroCrust - Sampling Conditions

Requirement: disk caps

Each sample ball contributes exactly two caps, i.e., topological-disks, to the boundary of the union.



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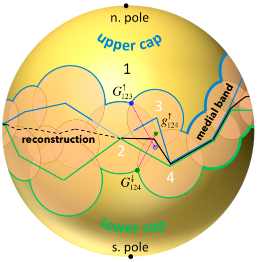
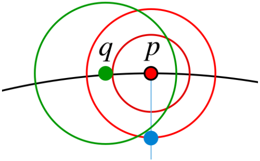


Sampling conditions

$$lfs(q) \geq lfs(p) \implies \|p - q\| \geq \sigma \cdot \epsilon lfs(p)$$



VoroCrust - Sampling Conditions

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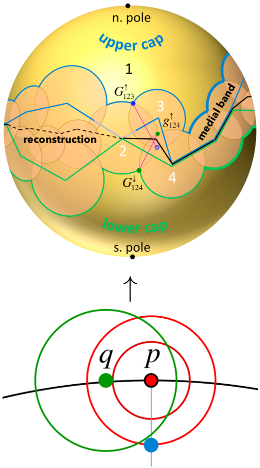
For sufficiently small ϵ, we may set $\sigma = 3/4$.



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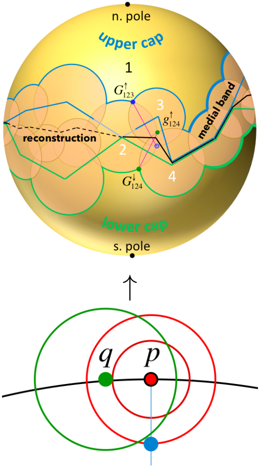
More geometric lemmata

- Samples appear as vertices in $\hat{\mathcal{M}}$.
- Bounds on angles and normal deviation of triangles in the weighted α-complex $\mathcal{K}$.



VoroCrust - Interior Sampling and Fatness

Lipschitz extension [Miller, Talmor, Teng (1999)]

For $x \in \mathcal{O}$ let $lfs(x) = \inf_{p \in \mathcal{M}} (lfs(p) + \|xp\|)$.

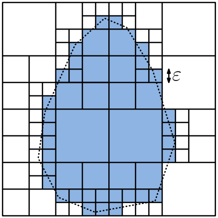


Figure from [David Mount]

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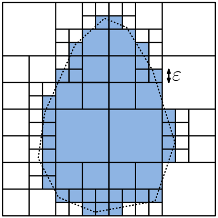
Sampling for fat cells

- Refine bounding box till *lfs* is satisfied
- Surface seeds are far from reconstruction $\hat{\mathcal{M}}$



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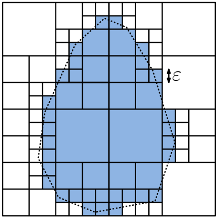
Quality for Voronoi cells: Fatness

Fix a Voronoi cell $v \in \hat{\mathcal{O}}$.

- R: radius of circumscribing sphere
- r: radius of inscribed sphere
- Fatness of $v = R/r$



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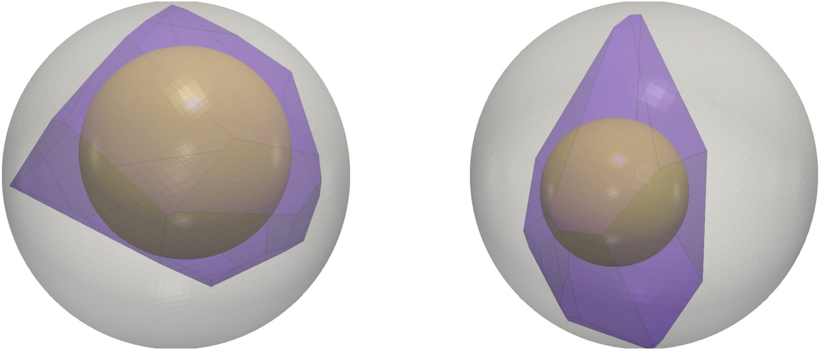

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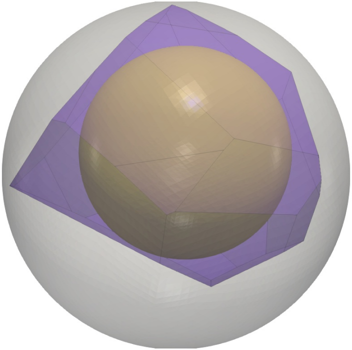
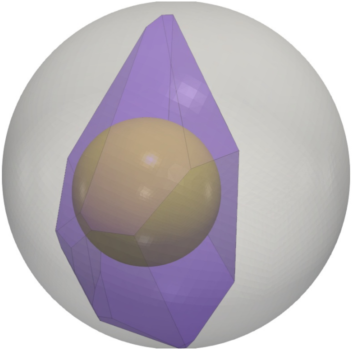
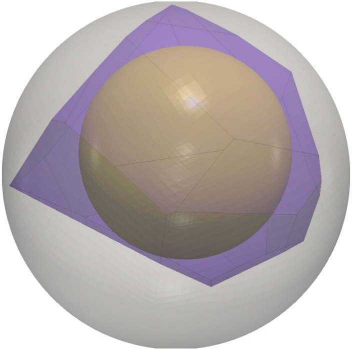


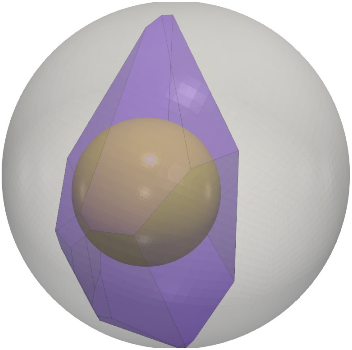


Lemma

- Out-radius to in-radius ratio is at most 15.

VoroCrust - Interior Sampling and Fatness



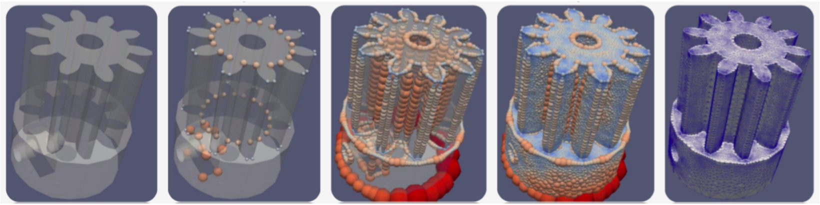


Lemma

- Out-radius to in-radius ratio is at most 15.
- Number of interior seeds $|\mathcal{S}^{\downarrow\downarrow}| = O(\epsilon^{-3} \cdot \int_{\text{vol}} \mathit{lfs}^{-3})$.

Summary

- A new Voronoi-based algorithm for isotopic surface reconstruction
- Conforming Voronoi meshing of volumes bounded by smooth surfaces
- Lower-bound on in-radius to out-radius ratio of all Voronoi cells
- For all matters related to the VoroCrust software
  - Please contact: Mohamed S. Ebeida (msebeid@sandia.gov)



Thanks for listening