Sampling Conditions for Conforming Voronoi Meshing by the VoroCrust Algorithm

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Motivation

Meshing

Partition into “simple” elements
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Partition into “simple” elements

Tetrahedral mesh
Motivation

Finite Element Method (FEM)

\[ \text{PDE} \xrightarrow{\Delta} \text{Mesh} \rightarrow \text{Algebra} \rightarrow \text{Discrete Approximation} \rightarrow \text{Interpolation} \]
Motivation

What type of element to use?
No silver bullet..

Tetrahedral mesh vs. Voronoi mesh
Motivation

Why polyhedral meshing?

- Less sensitive to stretching
  - Efficient meshing of complicated domains
- Higher node degree, even at boundaries
  - Better approximations of gradients

Why Voronoi meshing?

- Convex elements
- Positive Jacobians
- Orthogonal dual: a Delaunay mesh
Related Work

Voronoi meshing by “clipping”

Initial Voronoi mesh → Truncate cells by bounding surface → Defects
Related Work

Voronoi meshing by “mirroring”
Pair seeds naïvely across surface
Related Work

Voronoi meshing by “mirroring”
Pair seeds naively across surface → Bad surface normals
Voronoï meshing by “mirroring”

Pair seeds naîvely across surface → Bad surface normals
Related Work

Voronoi meshing

VoroCrust is a principled approach to mirroring
Preliminaries

Notation

- $\mathcal{O}$: bounded open set in $\mathbb{R}^3$; the volume to be meshed
- $\mathcal{M}$: boundary of $\mathcal{O}$; a smooth surface
- $\mathcal{P}$: input sample from $\mathcal{M}$
- $\hat{\mathcal{M}}, \hat{\mathcal{O}}$: surface and volume meshes

Fillette aux tourterelles [Luigi Pampaloni]
Preliminaries

Local features size (lfs)

At any $x \in \mathcal{M}$, $\text{lfs}(x)$ is the distance from $x$ to the medial axis of $\mathcal{M}$. 
\( \epsilon \)-sample

A set of points \( P \) on \( M \) such that \( \forall x \in M \ \exists p \in P \) s.t. \( \|px\| \leq \epsilon \cdot lfs(x) \).
VoroCrust Intuition - A 2D Example

(1) Weighted samples (balls)
VoroCrust Intuition - A 2D Example

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(2) Collect intersection points
VoroCrust Intuition - A 2D Example

1. Weighted samples (balls)
2. Collect intersection points
3. Compute Voronoi diagram
VoroCrust Intuition - A 2D Example

(1) Weighted samples (balls)

(2) Collect intersection points

(3) Compute Voronoi diagram

(4) Keep the separating facets
From balls to surfaces [Chazal, Lieutier (2006)]

Let $\mathcal{P}$ be an $\epsilon$-sample of $\mathcal{M}$ and define $b_p$ as the ball centered at $p \in \mathcal{P}$ with radius $\delta(\epsilon) \cdot lfs(p)$. Then, $\mathcal{M}$ is a deformation retract of $\bigcup b_p$. 
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VoroCrust step (1)
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**VoroCrust step (1)**

**Corollary**

For sufficiently small $\epsilon$, we may set $\delta(\epsilon) = c$; we take $c = 2$. 
For a ball $b \in B$ centered at $c$ with radius $r$, $\pi(b, x) = \|cx\|^2 - r^2$.

$V_b = \{x \in \mathbb{R}^d \mid \pi(b, x) \leq \pi(b', x) \ \forall b' \in B\}$.
Power distance and cell

For a ball $b \in B$ centered at $c$ with radius $r$, $\pi(b, x) = \|cx\|^2 - r^2$.

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Figures from [Edelsbrunner]
Weighted $\alpha$-complex and $\alpha$-shape [Edelsbrunner (1992-1995)]

Define $\mathcal{K} = \text{Nerve}(\{V_b \cap b \mid b \in \mathcal{B}\})$ and $S$ as the underlying space $|\mathcal{K}|$. 

Homotopy-equivalence

The nerve theorem implies $S = |\mathcal{K}|$ has the same homotopy-type as $\bigcup \mathcal{B}$. 

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Sampling Conditions for VoroCrust
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Medial Axis of a Union of Balls [Amenta, Kolluri (2001)]

Let $\mathcal{U}$ be a union of balls in $\mathbb{R}^d$, let $V$ be the vertices of $\partial \mathcal{U}$ and let $S$ be the $\alpha$-shape of $\mathcal{U}$. The medial axis of $\mathcal{U}$ consists of:

1. the singular faces of $S$ and
2. the subset of $\text{Vor}(V)$ which intersects the regular components of $S$. 
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VoroCurst steps (2-4)
Case(1): singular facet of the $\alpha$-shape
Case(2): regular component of the $\alpha$-shape
Isotopic surface reconstruction
Recover both surface topology and embedding

Figure from [Wikipedia]
A purely topological condition [Chazal, Cohen-Steiner (2005)]

- $\mathcal{M}'$ is homeomorphic to $\mathcal{M}$,
- $\mathcal{M}'$ is included in a topological thickening $\mathbb{M}$ of $\mathcal{M}$,
- $\mathcal{M}'$ separates the sides of $\mathbb{M}$. 
Start with an \( \epsilon \)-sample \( \mathcal{P} \subset \mathcal{M} \) with weights 
\[ r_i = 2 \cdot lfs(p_i) \]
defining the associated balls \( B \).
VoroCrust - The Abstract Algorithm

- Start with an $\epsilon$-sample $\mathcal{P} \subset \mathcal{M}$ with weights $r_i = 2 \cdot lfs(p_i)$ defining the associated balls $\mathcal{B}$.
- Collect the corners of $\bigcup \mathcal{B}$, $S^\downarrow = S^\uparrow \cup S^\downarrow$.
VoroCrust - The Abstract Algorithm

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- Collect the corners of \( \bigcup B \), \( S^\uparrow = S^\uparrow \cup S^\downarrow \).
- Compute the Voronoi diagram of \( S^\uparrow \), \( Vor(S^\uparrow) \).
VoroCrust - The Abstract Algorithm

- Start with an $\epsilon$-sample $P \subset M$ with weights $r_i = 2 \cdot lfs(p_i)$ defining the associated balls $B$.
- Collect the corners of $\bigcup B$, $S^\downarrow = S^\uparrow \cup S^\downarrow$.
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- Produce the surface approximation $\hat{M}$ as the Vorono facets of separating $S^\uparrow$ from $S^\downarrow$. 

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Collect the corners of $\cup \mathcal{B}$, $S^\downarrow = S^\uparrow \cup S^\downarrow$.

Compute the Voronoi diagram of $S^\downarrow$, $Vor(S^\downarrow)$.

Produce the surface approximation $\hat{\mathcal{M}}$ as the Vorono facets of separating $S^\uparrow$ from $S^\downarrow$.

Generate additional seeds $S^\downarrow\downarrow$ from $\mathcal{O}$.
Start with an $\epsilon$-sample $P \subset M$ with weights $r_i = 2 \cdot lfs(p_i)$ defining the associated balls $B$.

Collect the corners of $\bigcup B$, $S^\downarrow = S^\uparrow \cup S^\downarrow$.

Compute the Voronoi diagram of $S^\downarrow$, $\text{Vor}(S^\downarrow)$.

Produce the surface approximation $\hat{M}$ as the Vorono facets of separating $S^\uparrow$ from $S^\downarrow$.

Generate additional seeds $S^\downarrow\downarrow$ from $O$.

Return the volume mesh $\hat{O}$ as the Voronoi cells in $\text{Vor}(S^\uparrow \cup S^\downarrow\downarrow)$ with seeds in $S^\downarrow \cup S^\downarrow\downarrow$. 

Theorem $\hat{M}$ is isotopic to $M$. Hence, $\hat{O}$ is isotopic to $O$. 

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Start with an $\epsilon$-sample $\mathcal{P} \subset \mathcal{M}$ with weights $r_i = 2 \cdot lfs(p_i)$ defining the associated balls $\mathcal{B}$.

Collect the corners of $\bigcup \mathcal{B}$, $S^{\uparrow \downarrow} = S^{\uparrow} \cup S^{\downarrow}$.

Compute the Voronoi diagram of $S^{\uparrow \downarrow}$, $\text{Vor}(S^{\uparrow \downarrow})$.

Produce the surface approximation $\hat{M}$ as the Vorono facets of separating $S^{\uparrow}$ from $S^{\downarrow}$.

Generate additional seeds $S^{\downarrow \downarrow}$ from $\mathcal{O}$.

Return the volume mesh $\hat{O}$ as the Voronoi cells in $\text{Vor}(S^{\uparrow \downarrow} \cup S^{\downarrow \downarrow})$ with seeds in $S^{\downarrow} \cup S^{\downarrow \downarrow}$.

**Theorem**

$\hat{M}$ is isotopic to $M$. Hence, $\hat{O}$ is isotopic to $\overline{O}$.
VoroCrust - Ball Intersections

\[ p_1 p_2 p_3 s_{123} \]

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Requirement: disk caps

Each sample ball contributes exactly two caps, i.e., topological-disks, to the boundary of the union.
VoroCrust - Sampling Conditions

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**Sampling conditions**

\[ \text{lfs}(q) \geq \text{lfs}(p) \implies \| p - q \| \geq \sigma \cdot \epsilon \text{lfs}(p) \]
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Lemma
For sufficiently small \( \epsilon \), we may set \( \sigma = 3/4 \).
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More geometric lemmata
- Samples appear as vertices in \( \hat{\mathcal{M}} \).
- Bounds on angles and normal deviation of triangles in the weighted \( \alpha \)-complex \( \mathcal{K} \).
Lipschitz extension [Miller, Talmor, Teng (1999)]

For \( x \in \mathcal{O} \) let \( lfs(x) = \inf_{p \in \mathcal{M}} (lfs(p) + \|xp\|) \).

Figure from [David Mount]
Lipschitz extension [Miller, Talmor, Teng (1999)]

For $x \in \mathcal{O}$ let $lfs(x) = \inf_{p \in \mathcal{M}} (lfs(p) + \|xp\|)$.

Sampling for fat cells

- Refine bounding box till $lfs$ is satisfied
- Surface seeds are far from reconstruction $\hat{\mathcal{M}}$

Figure from [David Mount]
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**Sampling for fat cells**

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- Surface seeds are far from reconstruction \( \hat{\mathcal{M}} \)

**Quality for Voronoi cells: Fatness**

Fix a Voronoi cell \( v \in \hat{\mathcal{O}} \).
- \( R \): radius of circumscribing sphere
- \( r \): radius of inscribed sphere
- Fatness of \( v = R/r \)

Figure from [David Mount]
Lemma
Out-radius to in-radius ratio is at most 15.
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- Number of interior seeds $|S_{\downarrow\downarrow}| = O(\epsilon^{-3} \cdot \int_{\text{vol}} lfs^{-3})$. 
Summary

- A new Voronoi-based algorithm for isotopic surface reconstruction
- Conforming Voronoi meshing of volumes bounded by smooth surfaces
- Lower-bound on in-radius to out-radius ratio of all Voronoi cells
- For all matters related to the VoroCrust software
  - Please contact: Mohamed S. Ebeida (msebeid@sandia.gov)

Thanks for listening