VoroCrust Geometry:
3D polyhedral meshing
with true Voronoi cells
conforming to prescribed surface points.

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(speaker instead of Scott)
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Related Prior USNCCM Talk

• Mohamed’s talk covered algorithm, reconstruction properties, implementation, and example meshes
• This talk covers geometric primal-dual math, algorithmic correctness

VOROCRUST
Volumetric Voronoi Meshing that Conforms to Surface Samples

• Mohamed Ebeida
• Optimization and Uncertainty Quantification Department
• Sandia National Laboratories
VoroCrust

ALGORITHM OVERVIEW
‘Classic’ VoroCrust Primal-Dual-Primal Dance

**Input:** primal mesh of the input surface, points and edges in 2d and triangles in 3d
**Primal-Dual-Primal Dance**

Create weighted Voronoi balls around primal points

**Properties:**
- Balls overlap if points share an edge
- Balls protect points, as no seeds will be inside balls
Primal-Dual-Primal Dance

Create dual unweighted seed pairs at weighted Voronoi sphere intersections

red = exterior
blue = interior

2D: 2 circles intersect at two points
3D: 3 spheres intersect at two points

Require: both intersection points lie outside all other balls

Want seeds sufficient to reconstruct surface
surface mesh nodes are Voronoi vertices
surface mesh edges are Voronoi edges
surface mesh triangles are Voronoi facets
Primal-Dual-Primal Dance

Property: Unweighted Voronoi diagram reconstructs the surface mesh

Want seeds sufficient to reconstruct surface
- surface mesh nodes are Voronoi vertices
- surface mesh edges are Voronoi edges
- surface mesh triangles are Voronoi facets
Enhancement

**Freedom:** Additional seeds outside balls, added for mesh quality.
E.g. could use a (graded) disk packing or lattice points for a hex-dominant mesh.
Enhancement

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Challenge

• Problem:
  – Sometimes one intersection point is covered, and the other in the pair is uncovered
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• Solution:
  – “Basic” algorithm adds additional seeds
    • reconstructs vertices, edes, facets one by one
    • reconstructs a refinement of the input mesh
VoroCrust

BASIC ALGORITHM
Basic VoroCrust Algorithm

• Step (0): input mesh
Basic VoroCrust Algorithm

• Step (1): Spheres around each mesh node
Basic VoroCrust Algorithm

- Step (2): Circles around each mesh edge, on spheres
Basic VoroCrust Algorithm

• Step (3): Four seeds on each circle
VoroCrust

ANALYSIS OF BASIC ALGORITHM
Basic VoroCrust Correctness

• Reconstruction of all input Vertices
  – By construction
    • ≥ 3 non-collinear samples in 2D
    • ≥ 4 non-cocircular samples in 3D
Basic VoroCrust Correctness

• Reconstruction of all input **Edges**
  – Requirement: empty diametric spheres
  • Also implies non-obtuse triangulation

\[ \forall p \in \overline{ab}, \forall v \notin \{a, b\}, \min\{d(p, a), d(p, b)\} < d(p, v) \]
Basic VoroCrust Correctness

• Reconstruction all input **Faces**
  – Requirement: empty diametric and circum- spheres
  • Also implies non-obtuse Delaunay triangulation

\[ \forall p \in \triangle abc, \forall v \notin \{a, b, c\}, \min\{d(p, a), d(p, b), d(p, c)\} < d(p, v) \]
Basic VoroCrust Analysis

- Edge midpoints
- Circumcenters
- Steiner points
- Angle bisectors
Basic VoroCrust Analysis
Basic VoroCrust Analysis

Non-Zero Radii

Zero Radii
VoroCrust

OPTIMIZATIONS OF BASIC ALGORITHM
Basic VoroCrust Optimization-1

Edge circles do not intersect

Circles intersect cyclically
Basic VoroCrust Optimization-1
Basic VoroCrust Optimization-1

[Diagram of Basic VoroCrust and Optimization-1 (all-quad)]
Basic VoroCrust Optimization-1

Optimization-1
Non-zero Radii

Zero Radii
Basic VoroCrust Optimization-2
Basic VoroCrust Optimization-3
Conclusion

- Work in progress
- Status
- Open

Thank you!

Questions?
Thoughts?
Comments?