Introduction to Computational Topology

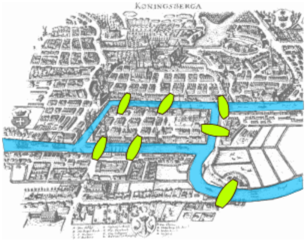
Ahmed Abdelkader

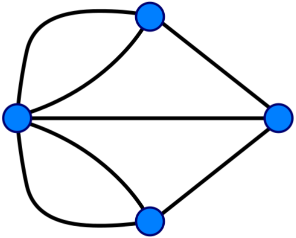
Guest Lecture

CMSC 754 – Spring 2020

May 7th, 2020

Early Topological Insights





Seven Bridges of Königsberg: find a path that crosses each bridge exactly once

The Origins of Graph Theory

- Euler observed that the subpaths within each land mass are irrelevant.
- Use an abstract model of land masses and their connectivity – a graph!
- A path enters a node through an edge, and exits through another edge.
- The solution exists if there are exactly 0 or 2 nodes of odd degree.

Figures from Wikipedia [1, 2]

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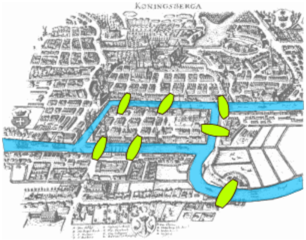

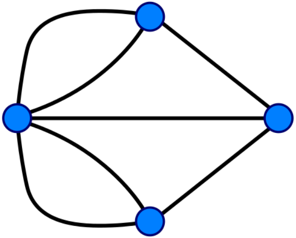

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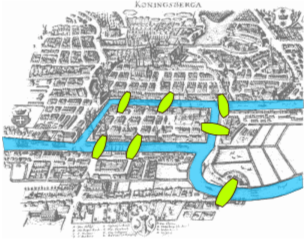
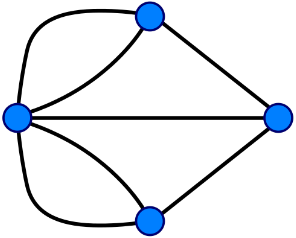


$\Rightarrow$



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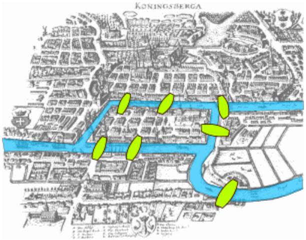

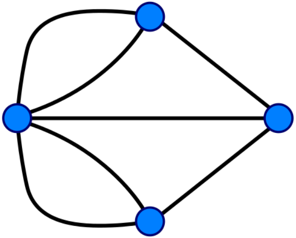

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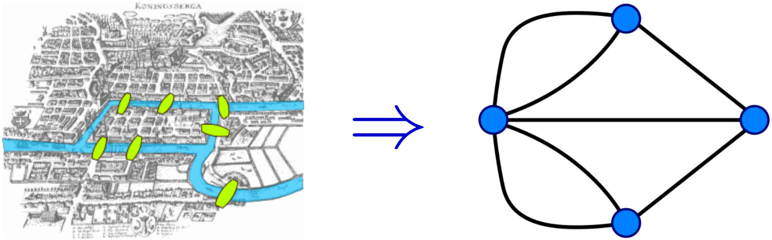

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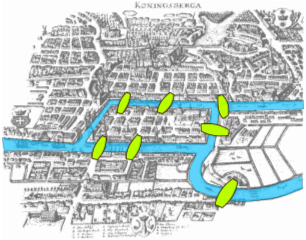

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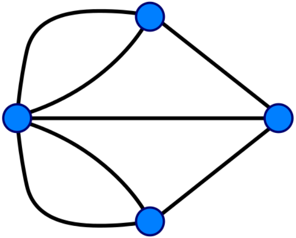

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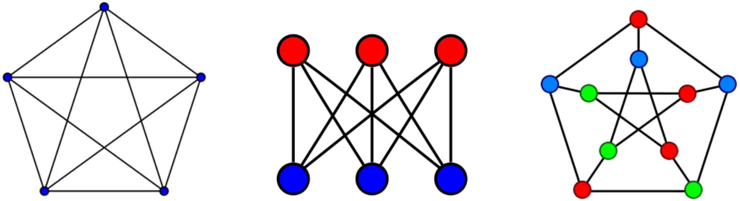
More Graph Theory



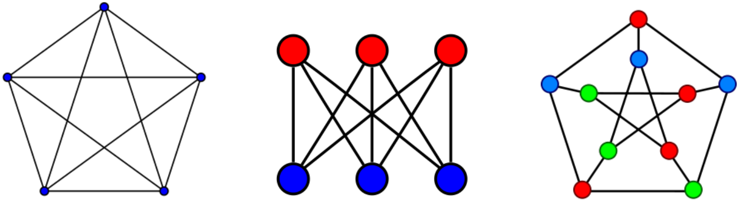
Complete graph K_5, complete bipartite graph $K_{3,3}$, and the Petersen graph

Forbidden Graph Characterizations

- A minor H of a graph G is the result of a sequence of operations:
  - Contraction (merge two adjacent vertices), edge and vertex deletion.
- A graph if planar iff it does not have any K_5 or $K_{3,3}$ minors.
- Hadwiger conjecture: a graph is t-colorable iff it does not have any K_t minors.

Figures from Wikipedia [3, 4, 5]

More Graph Theory



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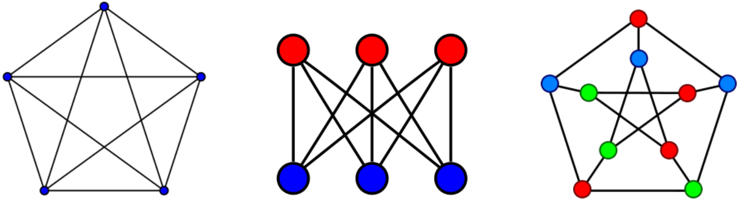

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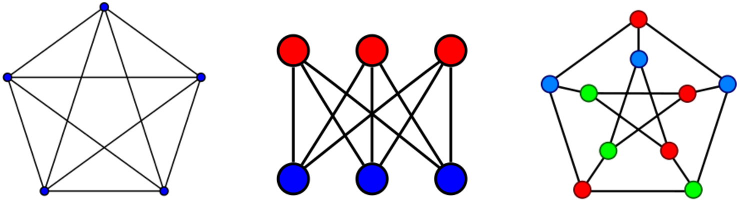

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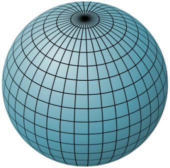
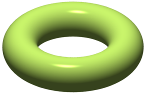
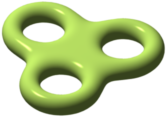


0 holes



1 hole

2 handles



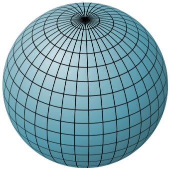
3 handles

Topological Invariants

- Instead of edge deletion and contraction for graphs, we study surfaces under continuous deformations that do not *tear* or *pinch* the surface.
- The genus corresponds to the number of *holes* or *handles*.
- Joke: a *topologist* cannot distinguish his coffee mug from his doughnut!
  - Topology as rubber-sheet geometry

Figures from Wikipedia [6, 7, 8, 9]

Surfaces



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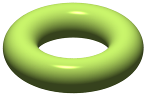

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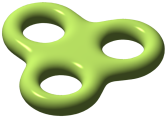

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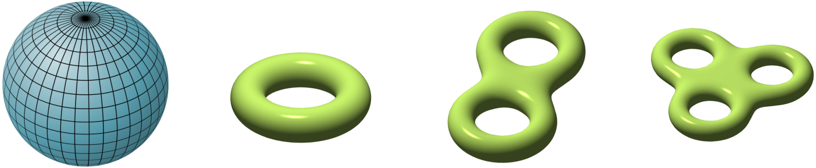

0 holes — 1 hole — 2 handles — 3 handles

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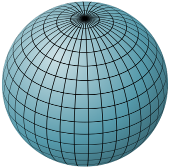

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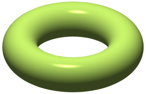
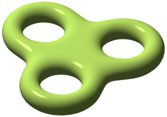


1 hole

2 handles



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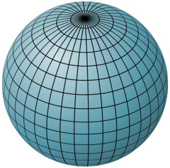

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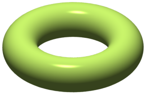

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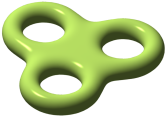

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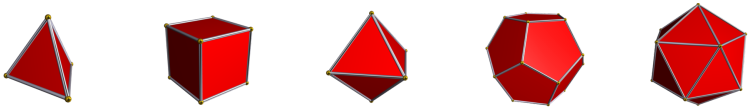
How do you compute the genus *without looking*?

Convex Polytopes



$4 - 6 + 4$ $8 - 12 + 6$ $6 - 12 + 8$ $20 - 30 + 12$ $12 - 30 + 20$

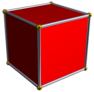
Euler's Polyhedron Formula

- Alternating sum of the number of vertices (V), edges (E), and facets (F)
$$\chi = V - E + F$$
- As spheres can be continuously deformed into convex polytopes, they also have an Euler characteristic of 2.
- Unlike the genus, this is easily computed by simple counting or algebra.

Figures from Wikipedia [10, 11, 12, 13, 14]

Convex Polytopes


$4 - 6 + 4$


$8 - 12 + 6$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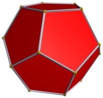
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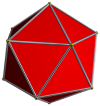
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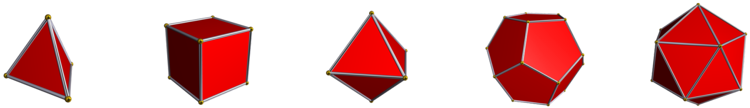

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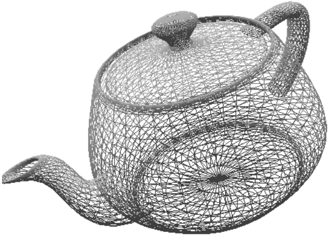
What about non-convex surfaces?

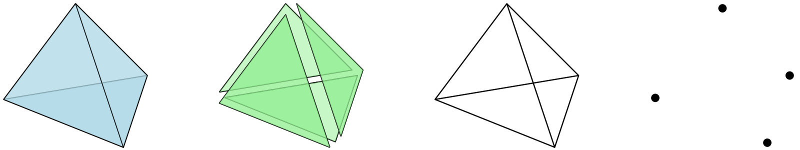
Wireframes

Rendering all triangles



Wireframe, edges only

Simplicial Complexes

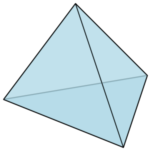


A 3-simplex — Four 2-simplices — Six 1-simplices — Four 0-simplices

Definitions

- A *p-simplex* is the convex hull of $(p+1)$ affinely-independent points.
- We write this as $\sigma = [v_0, \ldots, v_p] = \mathrm{conv}\{v_0, \ldots, v_p\}$ and say $\dim \sigma = p$.
- A *simplicial complex* K is a set of simplices closed under intersection, and its dimension $\dim K$ is the maximum dimension of its simplices.
- If $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2 \in K$. The (-1)-simplex $\emptyset$ is always in K.

Simplicial Complexes



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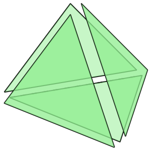
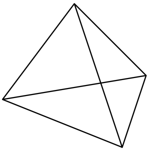


Four 2-simplices



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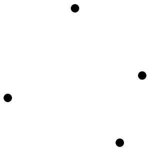

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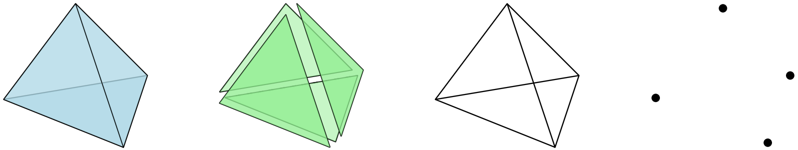

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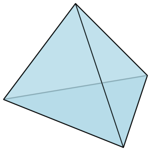

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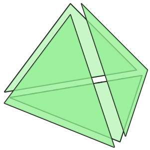

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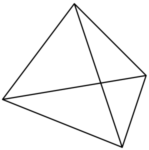

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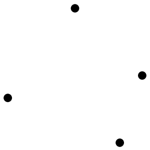

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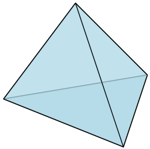

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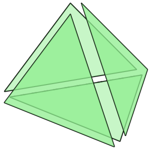

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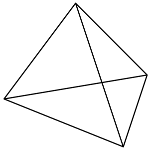

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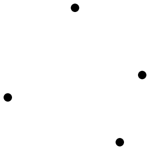

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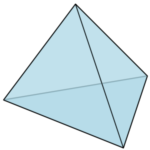
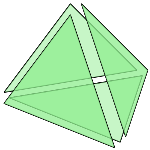
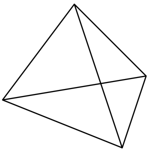
Simplicial Complexes



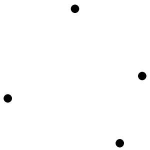
A 3-simplex



Four 2-simplices



Six 1-simplices



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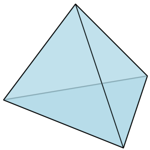

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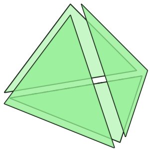

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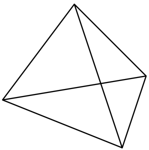

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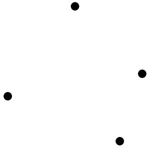

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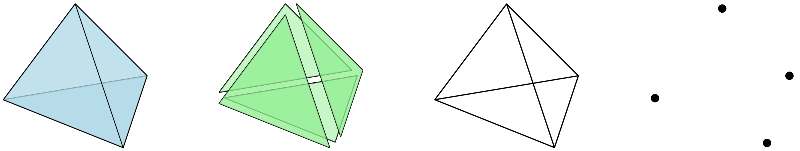

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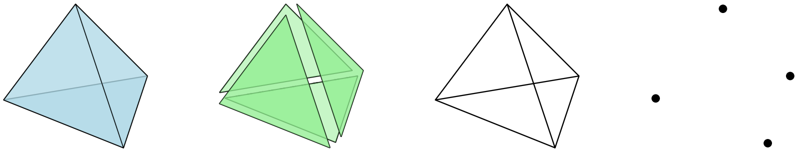

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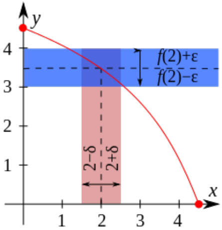
How to represent a *mapping* between two surfaces?

Continuous Deformations

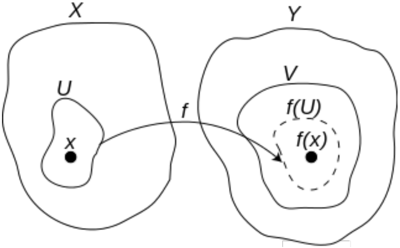
A continuous deformation of a cow model into a ball

Figure from Wikipedia [15]

Continuous Maps



Continuity at $x = 2$ by (ε, δ)



Continuity at $x \in X$ using neighborhoods

Definition of Continuity

- Small changes in the input yield small changes in the output.
- Calculus formalizes this notion using the (ε, δ)-definition of the limit.
- For general topologies, we use neighborhoods instead of (ε, δ) intervals.

Figures from Wikipedia [16, 17]

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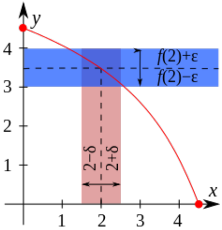

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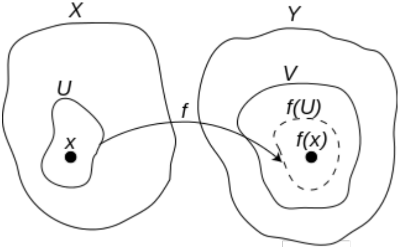

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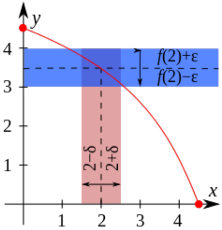

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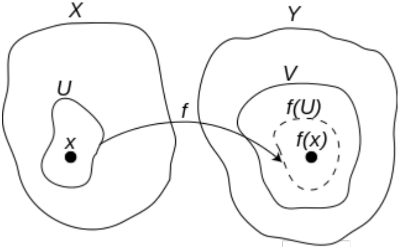

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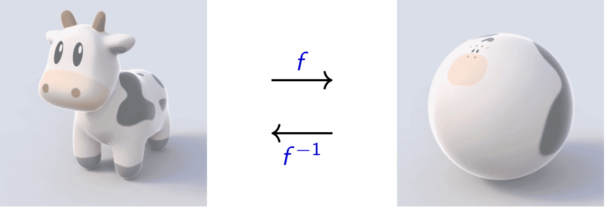
Homeomorphisms



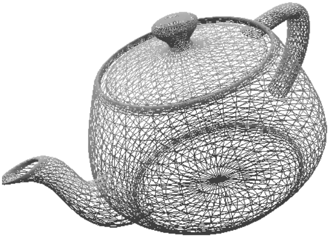
Definition

Two topological spaces X and Y are said to be *homeomorphic* whenever there exists a continuous map $f : X \to Y$ with a continuous inverse $f^{-1} : Y \to X$. Such a function f is called a *homeomorphism*.

Figure from Wikipedia [15]

But, we will be using the triangulations rather than the surfaces ...

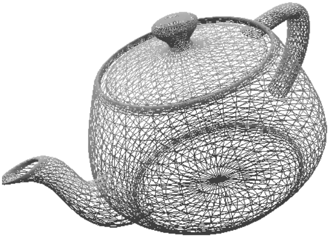
Triangulations



Definition

- A *triangulation* of a topological space X is a simplicial complex $\hat{X}$ such that X and $|\hat{X}|$ are homeomorphic.
- A topological space is *triangulable* if it admits a triangulation.

Triangulations



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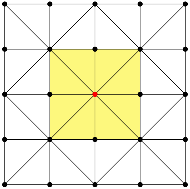
Continuous Maps between Simplicial Complexes



Simplicial Neighborhoods

- Fix a simplicial complex K.
- The *star* of σ is the collection its cofaces:

$$\mathrm{St}_K(\sigma) = \{\tau \in K \mid \sigma \preceq \tau\}.$$

- The *star neighborhood* of σ is the union of the interior of its cofaces:

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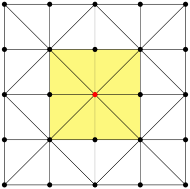

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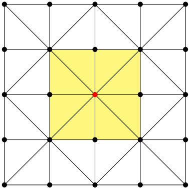

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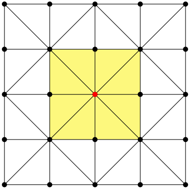
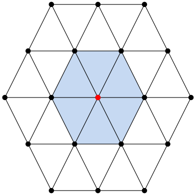


$$\hat{X} \mapsto \hat{Y}$$



The Star Condition

- Fix two simplicial complexes $\hat{X}$ and $\hat{Y}$ and a map $\hat{f} : |\hat{X}| \to |\hat{Y}|$.
- We say that $\hat{f}$ satisfies the star condition if for all vertices $v \in \hat{X}$

$$\hat{f}\left(N_{\hat{X}}(v)\right) \subseteq N_{\hat{Y}}(u) \quad \text{for some vertex } u = \phi(v) \in \hat{Y}.$$

- The map $\phi : \text{Vert } \hat{X} \to \text{Vert } \hat{Y}$ extends to a *simplicial map* that maps every simplex $\sigma \in \hat{X}$ to some simplex $\tau \in \hat{Y}$.
- The simplicial map induces *a simplicial approximation*: a piecewise-linear map $\hat{f}_\Delta : \hat{X} \to \hat{Y}$ that approximates the original function f.

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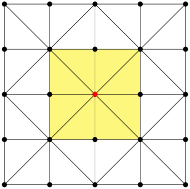

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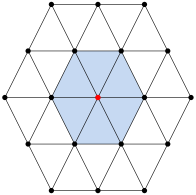

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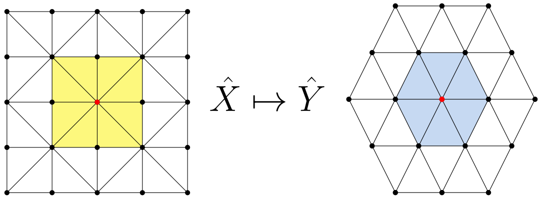

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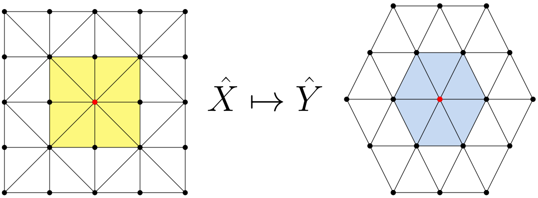

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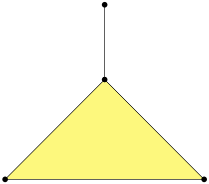
What if $\hat{f} : |\hat{X}| \to |\hat{Y}|$ fails the star condition?

Simplicial Approximation Theorem



Barycentric Subdivisions

- If there exists a vertex $v \in \hat{X}$ such that $\hat{f}\left(N_{\hat{X}}(v)\right)$ is not contained in $N_{\hat{Y}}(u)$ for any vertex $u \in \hat{Y}$, then $N_{\hat{X}}(v)$ is too large!
- Solution: refine $\hat{X}$ without changing $\hat{f} : \hat{X} \to \hat{Y}$.
- The *barycenter* of $\sigma = [v_0, \ldots, v_p]$ is defined as $\frac{1}{p+1} \sum_{i=0}^{p} v_i$.
- Repeated subdivisions eventually achieve the star condition.

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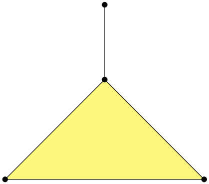

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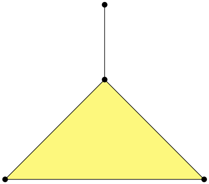

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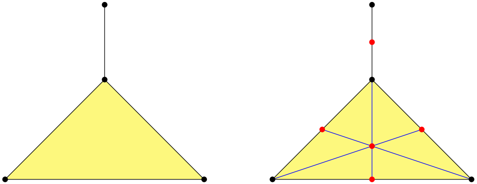

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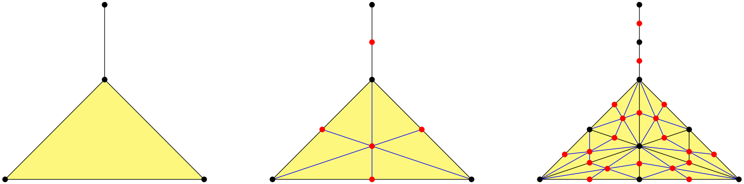

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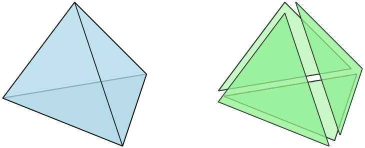
From Convex Polyhedra to Simplicial Complexes



Simplicial Counting

- Recall the alternating sum used to compute the Euler characteristic χ.
- We would like to derive a similar computation on a simplicial complex K.
- But, a single simplex can be shared among multiple cofaces.
- How do we keep track of the correct count?

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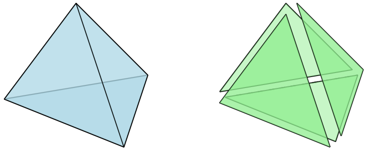

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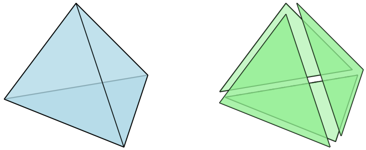

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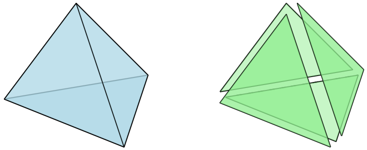

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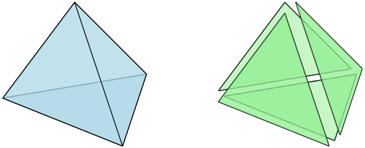

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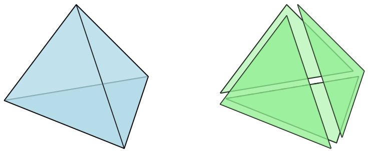
Chains



Counting Modulo 2

- Define a p-chain as a subset of the p-simplices in the complex K.
- We write a p-chain as a formal sum $c = \sum_i a_i \sigma_i$, where σ_i ranges over the p-simplices and a_i is a coefficient.
- We will work with coefficients in $\mathbb{F}_2 = \{0, 1\}$ with *addition modulo 2*.

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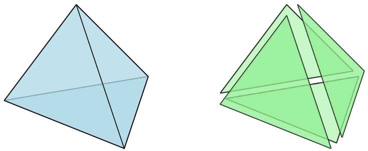

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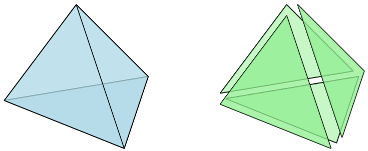

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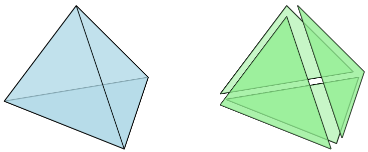

Counting Modulo 2

- Two p-chains can be added to obtain a new p-chain.
- Letting $c_1 = \sum_i a_i\sigma_i$ and $c_2 = \sum_i b_i\sigma_i$. Then, $c_1 + c_2 = \sum_i (a_i + b_i)\sigma_i$.
- As $a_i + b_i \in \mathbb{F}_2$ for all i, we get that $c_1 + c_2$ is a chain.
- Regarding p-chains as sets, we can interpret that $c_1 + c_2$ with modulo 2 coefficients is the *symmetric difference* between the two sets.

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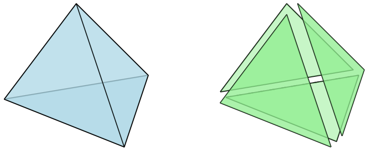

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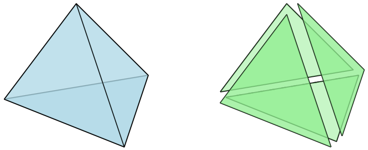

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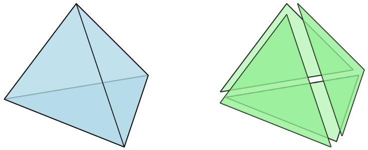

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Chain Groups

Algebra I

A *group* $(A, \bullet)$ is a set A together with a *binary operation* satisfying:

- Closure: for all $\alpha, \beta \in A$, we have that $\alpha \bullet \beta \in A$.
- Associativity: so that for all $\alpha, \beta, \gamma \in A$ we have $\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma$.
- A has an *identity element* ω such that $\alpha + \omega = \alpha$ for all $\alpha \in A$.

If, in addition, $\bullet$ is *commutative*, we have that $\alpha \bullet \beta = \beta \bullet \alpha$ for all $\alpha, \beta \in A$, and we say the group $(A, \bullet)$ is *abelian*.

Chains as Groups

We can now recognize p-chains $(C_p, +)$ as abelian groups.

Chains as Vector Spaces

If the complex K has n_p p-simplices, then C_p is (isomorphic to) the set of binary vectors of length n_p, i.e., $\{0, 1\}^{n_p}$, with the exclusive-or operation $\oplus$.

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Boundary of a Chain

Linear Extensions

- Fix a p-simplex $\sigma = [v_0, \ldots, v_p]$ in the complex K.
- Recall that the boundary of σ is the collection of its proper faces, which we denoted by $\partial\sigma$.
- We can now express the boundary elements as a single $(p-1)$-chain

$$\partial_p \sigma = \sum_{i=0}^{p} [v_0, \ldots, \hat{v}_i, \ldots, v_p],$$

  where $\hat{v}_i$ indicates that v_i is excluded in the corresponding face.
- Notice that we used the subscript to qualify the *boundary operator* as the one acting on the p-th chain group.
- For any p-chain $c = \sum_i a_i \sigma_i$, its boundary is the $(p-1)$-chain

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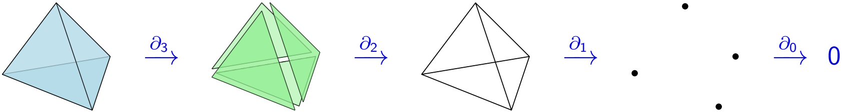
The Chain Complex



Boundary Homomorphisms

- The boundary operator ∂_p commutes with the group operations.
- If c_1 and c_2 are p-chains, then: $\partial_p(c_1 +_{(p)} c_2) = \partial_p c_1 +_{(p-1)} \partial_p c_2$, where we qualify the addition operators on each side of the equation.
- This means that ∂_p induces a *group homomorphism* or a mapping between groups that preserves the group structures: $\partial_p : C_p \to C_{p-1}$.
- We can arrange the chain groups into a chain complex, effectively replacing the geometric complex K with a series of algebraic modules.

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots$$

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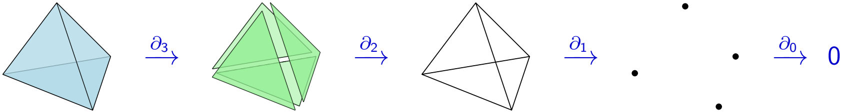

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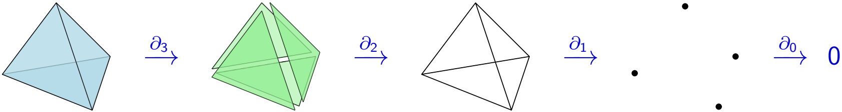

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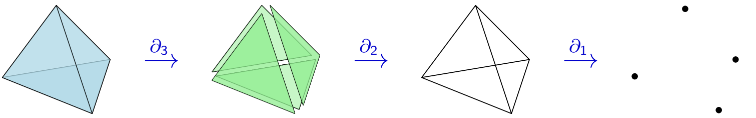

$$\xrightarrow{\partial_0} 0$$

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But like .. what's the point?

Boundary Matrices

Chains Groups as Vector Spaces

- Let $\{\sigma_i\}_i$ and $\{\tau_j\}_j$ denote the p-simplices and $(p-1)$-simplices of K.
- The boundary of a p-chain $c = \sum_i a_i \sigma_i$ is the $(p-1)$-chain

$$\partial_p c = \partial_p \left(\sum_i a_i \sigma_i \right) = \sum_i a_i \partial_p \sigma_i = \sum_i a_i \sum_j \partial_p^{j,i} \tau_j = \sum_j b_j \tau_j,$$

where $b_i = \sum_i \left(a_i \partial_p^{j,i} \right)$, and $\partial_p^{j,i}$ is 1 if $\tau_j \in \partial_p \sigma_i$ and 0 otherwise.

- With that, we can express the boundary operator ∂_p in matrix form.

$$\partial_p c = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_{p-1}} \end{bmatrix}, \quad \partial_p = \begin{bmatrix} \partial_p^{1,1} & \partial_p^{1,2} & \cdots & \partial_p^{1,n_p} \\ \partial_p^{2,1} & \partial_p^{2,2} & \cdots & \partial_p^{2,n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_p^{n_{p-1},0} & \partial_p^{n_{p-1},2} & \cdots & \partial_p^{n_{p-1},n_p} \end{bmatrix}, \quad c = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_p} \end{bmatrix}$$

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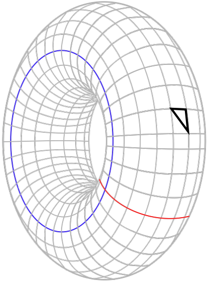
Boundaries and Cycles

Which Boundaries are Useful?

Consider the 1-chains on the torus to the right.

- We have a blue and a red loop.
- Also the boundary of the black triangle.
- Which of those help distinguish the torus from a sphere?



Chains with No Boundary

- Any such chain is called a p-cycle.
- A p-cycle that arises as the boundary of a $(p+1)$-chain is a p-boundary.
- We need a way to count distinct p-cycles while ignoring all p-boundaries.
- Observe that $\partial_p \circ \partial_{p+1} = 0$.

Figure from Wikipedia [18]

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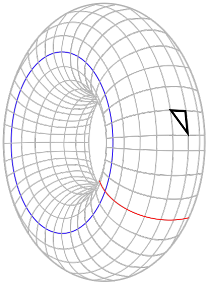

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- We are particularly interested in p-chains c satisfying $\partial_p c = \emptyset$.
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Boundaries and Cycles

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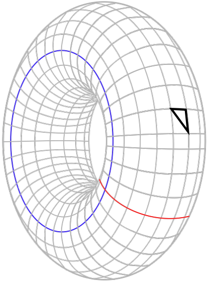

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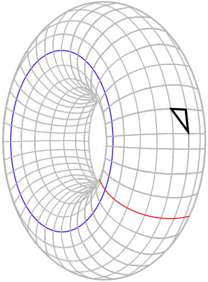

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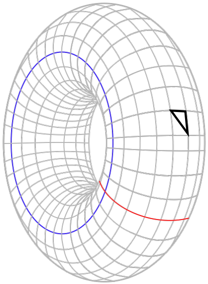

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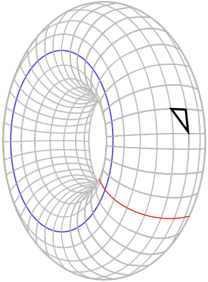

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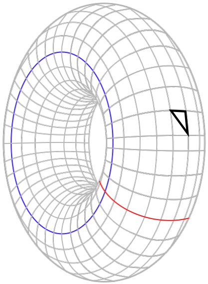

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Equivalence and Quotients

Boundaries and Cycles as Subgroups

- Denote all p-cycles by Z_p and all p-boundaries by B_p.
- As the boundary map commutes with addition, Z_p is a subgroup of C_p.
- Likewise, B_p is a subgroup of Z_p.
- For any p-cycle $\alpha \in Z_p$ and a p-boundary β, we get that $\alpha + \beta \in Z_p$.

Algebra II

- We define an *equivalence relation* that identifies a pair of elements $\alpha, \alpha' \in Z_p$ whenever $\alpha' = \alpha + \beta$ for some $\beta \in B_p$.
- The equivalence relation partitions Z_p into *equivalence classes* or *cosets*; the coset $[\alpha]$ consists of all the elements identified with α.
- Then, the collection of cosets together with the addition operator give rise to the *quotient group* Z_p/B_p of the elements in Z_p *modulo* the elements in B_p.

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Homology

Algebra III

- Take a group $(A, \bullet)$.
  - The order of the group is the cardinality of A.
  - The rank of the group is the cardinality of a minimal *generator*.
- For a set of binary vectors, such as C_p or Z_p
  - The order is the number of distinct binary vectors.
  - The rank is the number basis vectors that span the entire set.

Homology Groups and Betti Numbers

- We can now defined the p-th homology group as $H_p = Z_p / B_p$.
- The rank of H_p is known as the *p-th Betti number* β_p

$$\beta_p = \text{rank } H_p = \text{rank } Z_p - \text{rank } B_p.$$

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Rank-Nullity

Algebra IV

- Let V and W be vector spaces and $T : V \to W$ a *linear transformation*.
- We define the *kernel* of T as the subspace of V, denoted $\mathrm{Ker}(T)$ of all vectors v such that $T(v) = 0$.
- The remaining elements $v \in V$ for which $T(v) \neq 0$ are mapped to a subspace of W, i.e., the *image* of T.
- The *rank-nullity theorem* states that

$$\dim V = \dim \mathrm{Image}(T) + \dim \mathrm{Ker}(T).$$

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- Z_p is the kernel of ∂_p, while B_{p-1} is its image.
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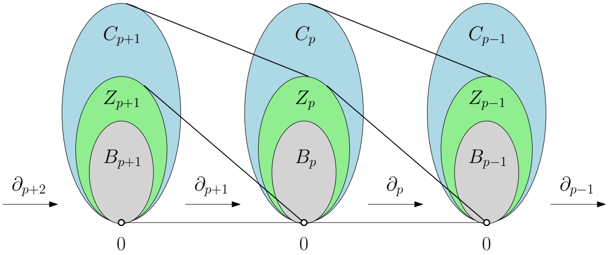

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The Euler Characteristic Revisited

A Generalized Formula

Recalling the alternating sum in Euler's polyhedron formula, we may write

$$\begin{aligned}
\chi = \sum_{p\geq 0}(-1)^p \text{ rank } C_p &= \sum_{p\geq 0}(-1)^p(\text{rank } Z_p + \text{rank } B_{p-1}) \\
&= (\text{rank } Z_0 + \cancel{\text{rank } B_{-1}}) - (\text{rank } Z_1 + \text{rank } B_0) + (\text{rank } Z_2 + \text{rank } B_1) - \dots \\
&= (\text{rank } Z_0 - \text{rank } B_0) - (\text{rank } Z_1 - \text{rank } B_1) + (\text{rank } Z_2 - \text{rank } B_2) - \dots \\
&= \sum_{p\geq 0}(-1)^p(\text{rank } Z_p - \text{rank } B_p) \\
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Note that the homology groups and the Betti numbers do not depend on the specific triangulation of the underlying space, i.e., they are indeed topological invariants.

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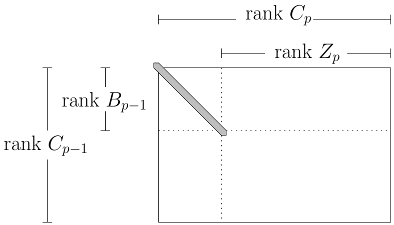
Rank Computations

- To compute β_p as the difference between $\text{rank } Z_p$ and $\text{rank } B_p$ we work with the matrix representation of the boundary map ∂_p.
- Using a sequence of row/column operations, the matrix is reduced without changing its rank into a simple form easily providing the ranks.
- A variant of *Gaussian elimination* is used to get the Smith normal form.



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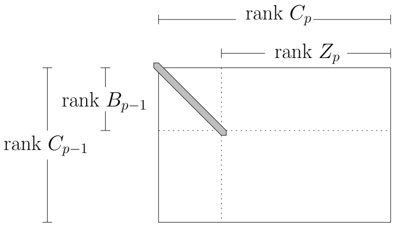
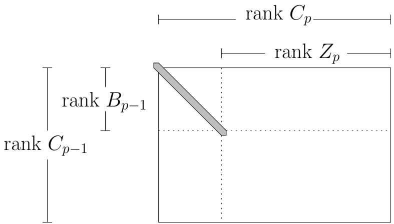


Matrix Reduction

Rank Computations

- To compute β_p as the difference between $\mathrm{rank}\ Z_p$ and $\mathrm{rank}\ B_p$ we work with the matrix representation of the boundary map ∂_p.
- Using a sequence of row/column operations, the matrix is reduced without changing its rank into a simple form easily providing the ranks.
- A variant of *Gaussian elimination* is used to get the Smith normal form.



What about the maps between spaces?

Induced Maps on Homology

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{\hat{X}}} & C_{p+1}(\hat{X}) & \xrightarrow{\partial_{\hat{X}}} & C_p(\hat{X}) & \xrightarrow{\partial_{\hat{X}}} & C_{p-1}(\hat{X}) & \xrightarrow{\partial_{\hat{X}}} & \cdots \\
 & & \downarrow{\scriptstyle \hat{f}_\#} & & \downarrow{\scriptstyle \hat{f}_\#} & & \downarrow{\scriptstyle \hat{f}_\#} & & \\
\cdots & \xrightarrow{\partial_{\hat{Y}}} & C_{p+1}(\hat{Y}) & \xrightarrow{\partial_{\hat{Y}}} & C_p(\hat{Y}) & \xrightarrow{\partial_{\hat{Y}}} & C_{p-1}(\hat{Y}) & \xrightarrow{\partial_{\hat{Y}}} & \cdots
\end{array}$$

Functoriality

- A simplicial map $\hat{f}_\Delta : \hat{X} \to \hat{Y}$ maps simplices in $\hat{X}$ to simplices in $\hat{Y}$.
- A simplicial map extends to a map from the chains of $\hat{X}$ to the chains of $\hat{Y}$, which we denote by $\hat{f}_\# : C_p(\hat{X}) \to C_p(\hat{Y})$, as shown in the diagram.
- As $\hat{f}_\#$ commutes with boundary maps, it also maps the cycles and boundaries of $\hat{X}$ to the cycles and boundaries of $\hat{Y}$, respectively.
- Hence, $\hat{f}_\#$ maps the homology groups of $\hat{X}$ to the homology groups of $\hat{Y}$, i.e., it induces a map on homology denoted by $H(\hat{f}) : H_p(\hat{X}) \to H_p(\hat{Y})$.

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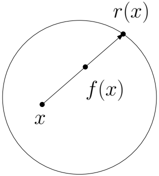
Applications of $H(\hat{f}) : H_p(\hat{X}) \to H_p(\hat{Y})$

$$\begin{tikzcd}
Y_1 \arrow[rr, "f"] \arrow[dr, dashed, "f_1"'] & & Y_2 \\
& X \arrow[ur, dashed, "f_2"'] &
\end{tikzcd}$$

Indirect Inference

If a map $f : Y_1 \to Y_2$ *factors through* $f_1 : Y_1 \to X$ and $f_2 : X \to Y_2$ such that $f = f_2 \circ f_1$, then we can infer the homology groups of X using knowledge of the homology groups of Y_1 and Y_2.

Brouwer's Fixed Point Theorem



$$\begin{tikzcd} \partial\mathbb{D} \arrow[r, "\iota"'] \arrow[rr, bend left, "\mathrm{Id}"] & \mathbb{D} \arrow[r, "r"'] & \partial\mathbb{D} \end{tikzcd}$$

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Every continuous map from the disc to itself has a fixed point

- Assume that $f : \mathbb{D} \to \mathbb{D}$ is continuous and has no fixed point.
- Define $r : \mathbb{D} \to \partial\mathbb{D}$ as the intersection of the ray form x to $f(x)$ with $\partial\mathbb{D}$.
- As f is continuous, so is r. Hence, the diagram in the middle *commutes*.
- Passing through homology, as shown to the right, we get that
  - $H_1(\partial\mathbb{D}) \cong \mathbb{F}_2$ while $H_1(\mathbb{D}) = 0$.
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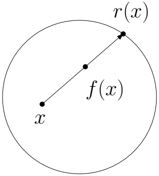

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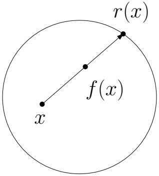

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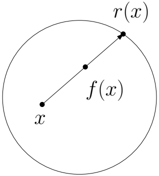

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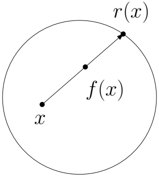

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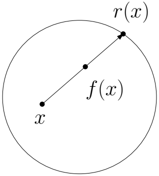

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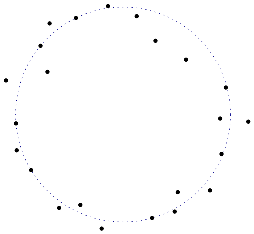
But, how do we get triangulations in the first place?

Sampled Data and Noise



The Cêch Complex

- We are given a collection of sample points from an unknown underlying manifold or surface in $\mathbb{R}^d$.
- We would like to infer some of the properties of the manifold.
- To do so, we grow a ball at each sample and take the union.
- Then, we derive an abstract simplicial complex from the union of balls.

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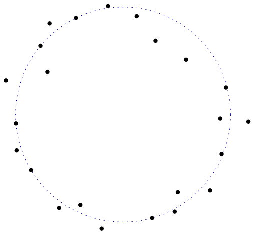

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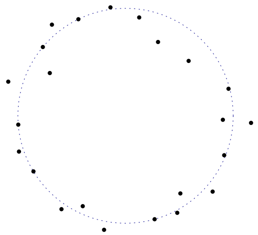

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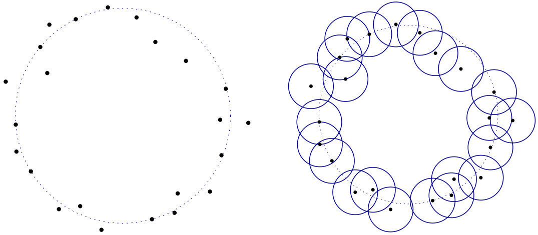

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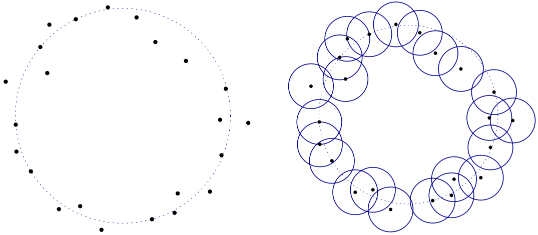

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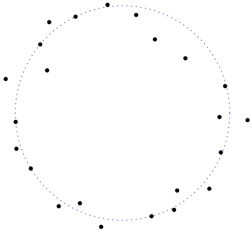

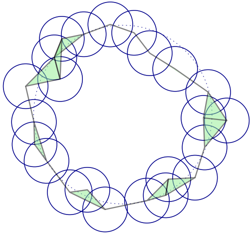

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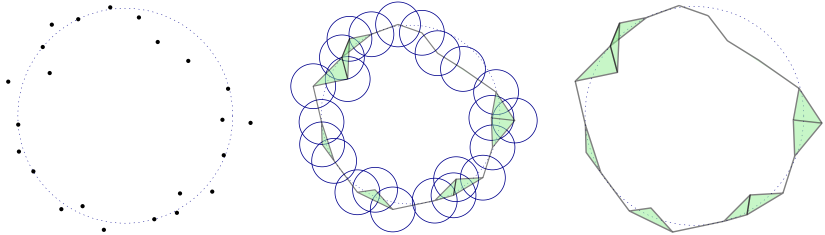

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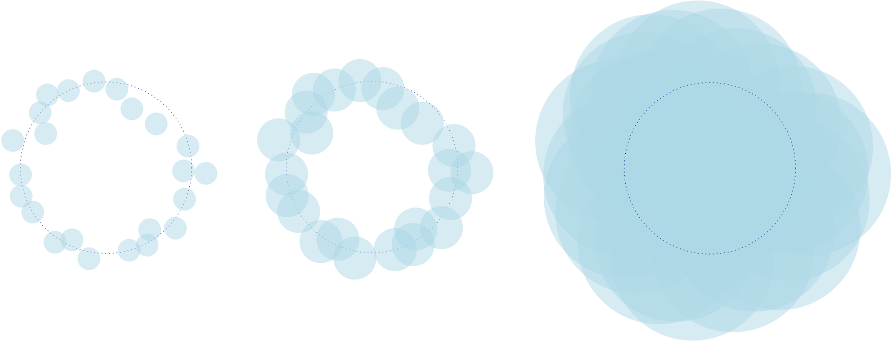
But, how do we choose the radii of the balls?

Scale and Persistence



Examining All Scales at Once

- As shown above, different radii may result in very different results.
- Imagine a continuous process growing the radii from $r = 0$ to $r = \infty$.
- Each topological feature will be present over an interval $[a, b)$.

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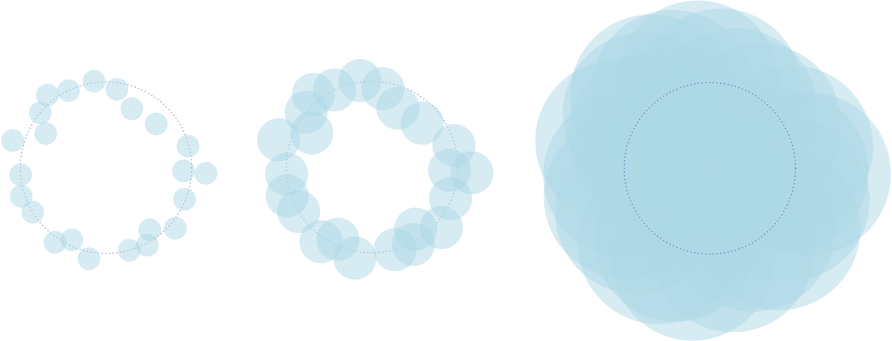

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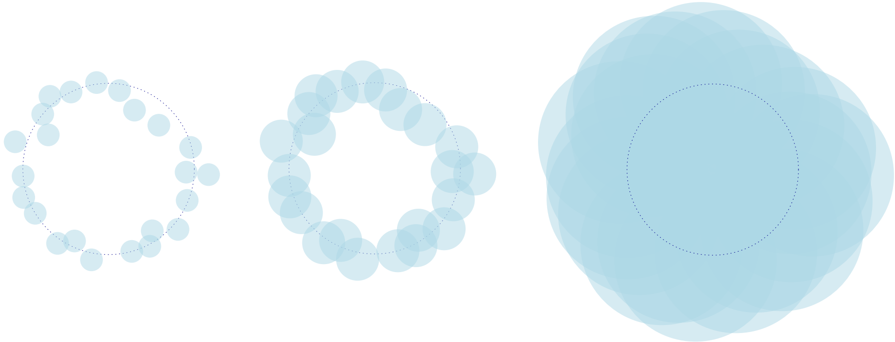

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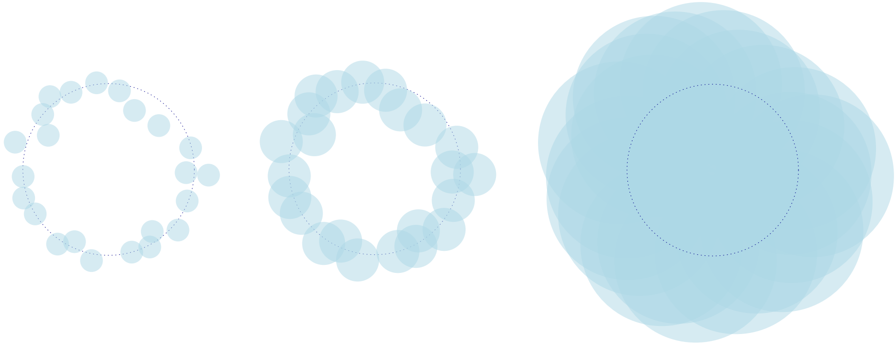

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- Features of high persistence are salient, while noise has low persistence.

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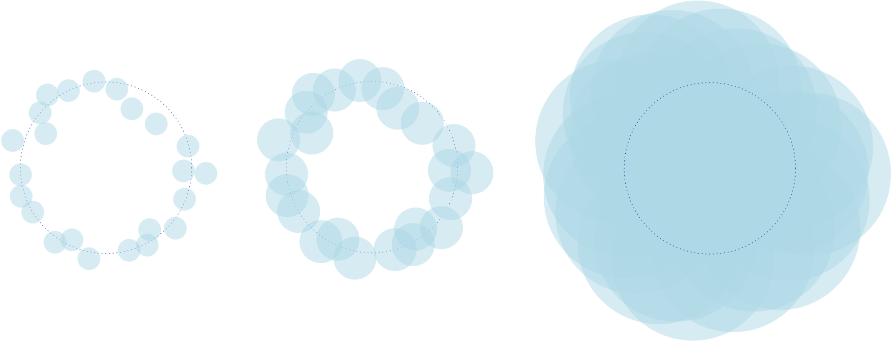

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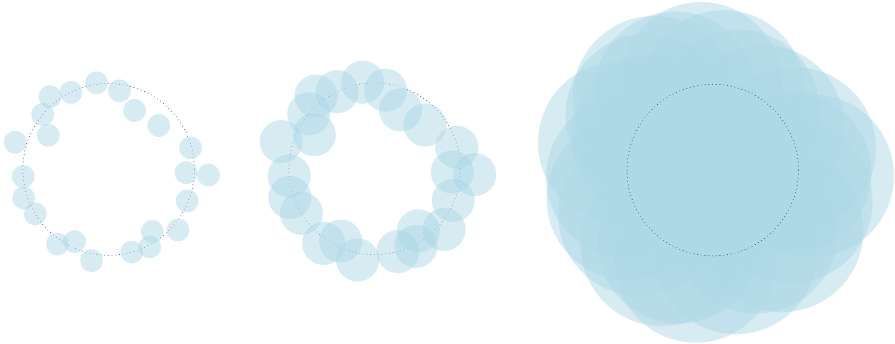

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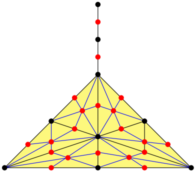


$$\partial_p = \begin{bmatrix} \partial_p^{1,1} & \partial_p^{1,2} & \cdots & \partial_p^{1,n_p} \\ \partial_p^{2,1} & \partial_p^{2,2} & \cdots & \partial_p^{2,n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_p^{n_{p-1},0} & \partial_p^{n_{p-1},2} & \cdots & \partial_p^{n_{p-1},n_p} \end{bmatrix}$$

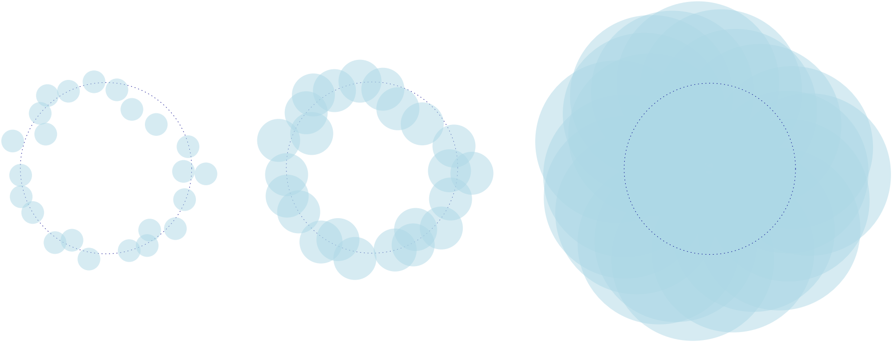
$$H_p = Z_p / B_p$$

Main Concepts Introduced

- Continuous deformations
- Simplicial approximations
- Chain algebra and homology
- Functoriality

Figure from Wikipedia [15]

Summary



Key Concepts Missing

- Persistent homology
- Persistence diagrams and barcodes
- Simplicial collapses
- Sparse filtrations and interleaving