Spatial search by quantum walk

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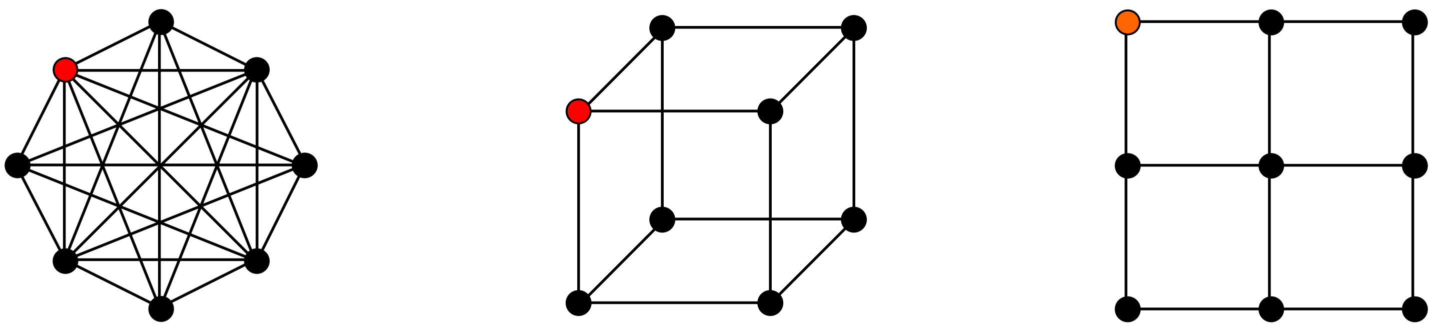


quant-ph/0306054

Unstructured search

- N items $\{1,2,\ldots,N\}$
- One "marked item" w
- Query: "is $w=x$?"
  I.e., black box function $f(x) = \begin{cases} 0 & x \neq w \\ 1 & x = w \end{cases}$

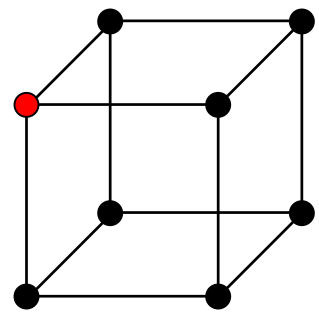
- Classical: $\Theta(N)$
- Grover 1996: $O(N^{1/2})$ quantum algorithm
- BBBV 1996: This is optimal

Combinatorial search vs. spatial search

- Combinatorial search: $f(x)$ is an efficiently computable function

- Spatial search: N items distributed in space (e.g., a physical database)

Model: N-vertex graph G



Algorithm must be *local* with respect to this graph.

Grover's algorithm in *d* dimensions

- One dimension: no speedup; $\Theta(N)$

- Benioff 00: searching a *d*-dimensional grid with a "quantum robot"
  - Each iteration takes $O(N^{1/d})$ steps to traverse the grid
  - $N^{1/2}$ Grover iterations $\Rightarrow$ $O(N^{1/2+1/d})$ algorithm

- Can we do better?

Aaronson-Ambainis algorithm

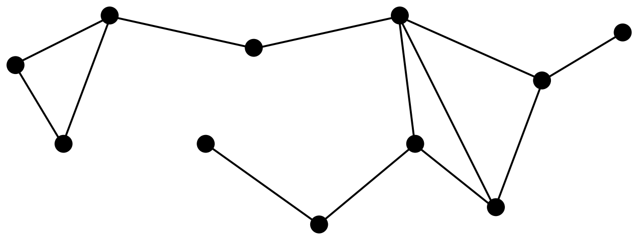
- Recursive search of subcubes with amplitude amplification
- Results
  - $d>2$: $O(N^{1/2})$ algorithm
  - $d=2$: $O(N^{1/2} \log^2 N)$ algorithm

Quantum walk search algorithm

- Simple Hamiltonian dynamics
- Applicable to any graph G
- Results
  - Complete graph="analog analogue" [FG96]; run time $O(N^{1/2})$
  - Hypercube: $O(N^{1/2})$ by previous results
  - d-dimensional lattice
    - $d>4$: $O(N^{1/2})$
    - $d=4$: $O(N^{1/2} \log^{3/2} N)$
    - $d<4$: no speedup

Graphs and matrices

Undirected graph G with no self loops



- Adjacency matrix: $A_{jk} = \begin{cases} 1 & (j,k) \in G \\ 0 & \text{otherwise} \end{cases}$
- Laplacian: $L = A - D$
  D diagonal, $D_{jj} = \deg(j)$

  Ex: For a line in one dimension, $L|j\rangle = |j+1\rangle + |j-1\rangle - 2|j\rangle$
  (Discrete approximation to ∇^2)

Random walk	Quantum walk
State space	
N vertices $j=1,\ldots,N$	N basis states $\lvert 1\rangle,\ldots,\lvert N\rangle$
$p_j(t)$ = probability of being at vertex j at time t	$q_j(t) = \langle j \vert \psi(t)\rangle$ = amplitude to be at vertex j at time t
Differential equation	
$\dfrac{\mathrm{d}p_j}{\mathrm{d}t} = \gamma \sum_k L_{jk}\, p_k$	$i\dfrac{\mathrm{d}q_j}{\mathrm{d}t} = \sum_k H_{jk}\, q_k$
Generator	
L = Laplacian of G	Can choose $H = -\gamma L$
Probability conservation	
$\sum_j L_{jk} = 0 \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}t}\sum_j p_j = 0$	$H = H^\dagger \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}t}\sum_j \lvert q_j\rvert^2 = 0$

Quantum walk search algorithm

- Marked state identified by "oracle Hamiltonian" $H_w = -|w\rangle\langle w|$

Algorithm

- Start in state $|s\rangle = \frac{1}{\sqrt{N}} \sum_j |j\rangle$
- Schrödinger evolve for time T using Hamiltonian $H = -\gamma L + H_w$
- Measure position

- Goal: Choose γ, T so that $|\langle w|e^{-i\,H\,T}|s\rangle|^2$ is as close to 1 as possible (for T not too big)

Why might this work?

$$H = -\gamma L - |w\rangle\langle w|$$

critical γ

ground state $\sim |s\rangle + |w\rangle$

first excited state $\sim |s\rangle - |w\rangle$

time $\sim 1/(E_1 - E_0)$

$\gamma \to 0$

$H \sim -|w\rangle\langle w|$

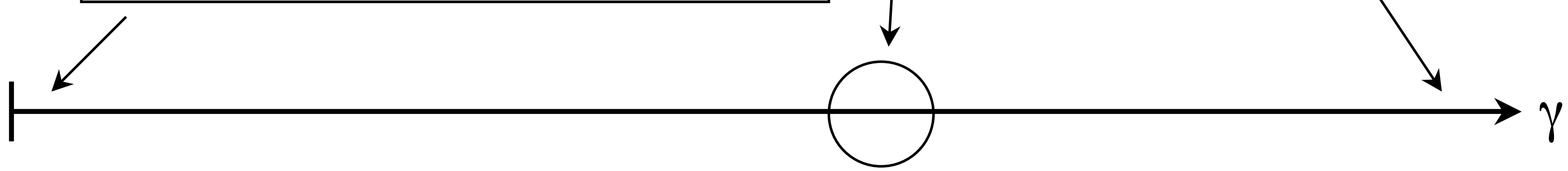
ground state $\sim |w\rangle$

first excited state $\sim |s\rangle$

$\gamma \to \infty$

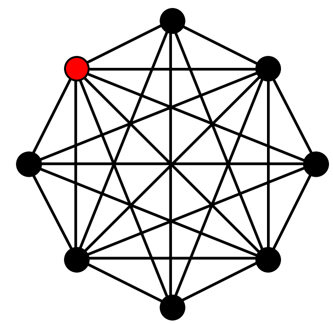
$H \sim -\gamma L$

ground state $\sim |s\rangle$



Complete graph

$$L + NI = N|s\rangle\langle s| = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$



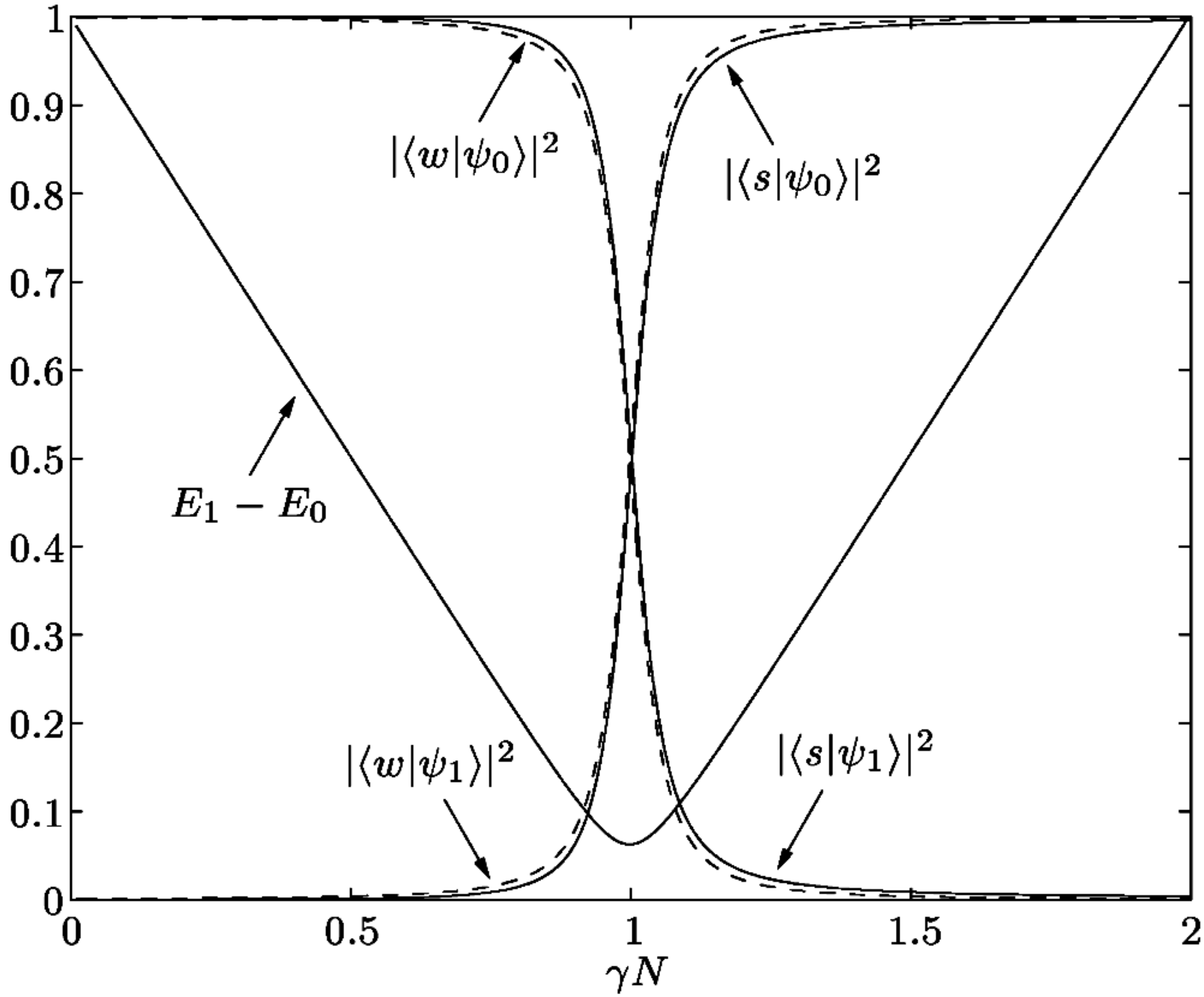
$$H = -\gamma N|s\rangle\langle s| - |w\rangle\langle w|$$

γN = 1 is the "analog analogue" of Grover's algorithm

Eigenstates ~ $|s\rangle \pm |w\rangle$

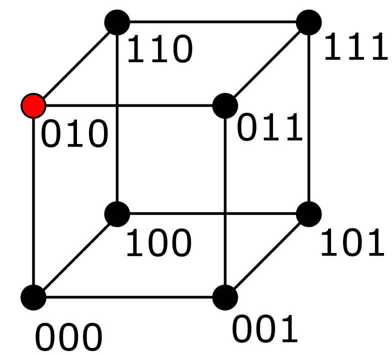
Gap $2N^{-1/2}$

Complete graph



N=1024

Hypercube



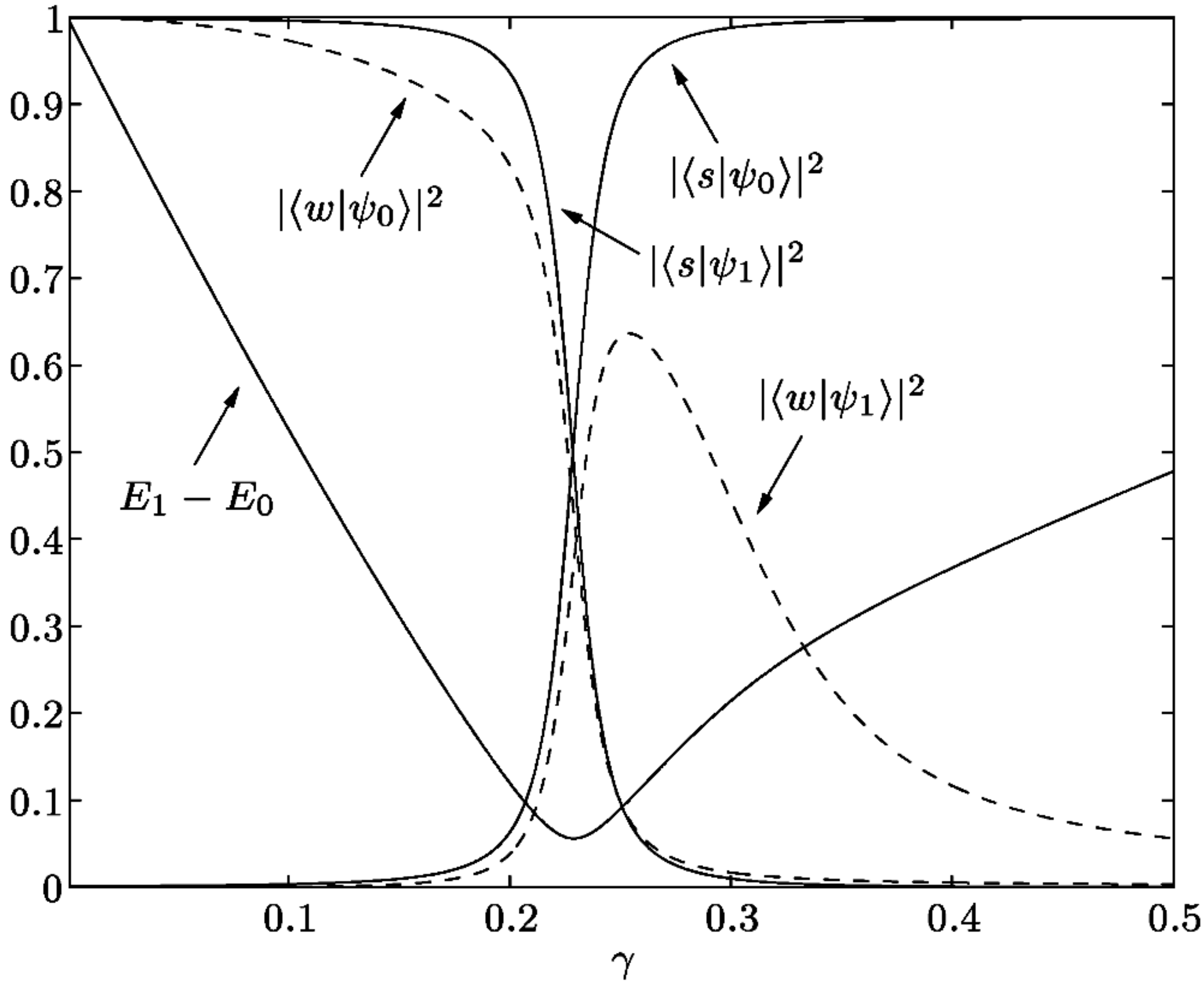
Vertices labelled by n-bit strings
$N=2^n$

Adjacency matrix: $A = \sum_{j=1}^{n} \sigma_x^{(j)}$

Hamiltonian: $H = -\gamma A - |w\rangle\langle w|$

Analyze using total spin operators [FGGS00]

Hypercube



$N=2^{10}=1024$

d-dimensional lattice

- Periodic cubic lattice with N sites, size $N^{1/d}$ in each dimension (d independent of N)

- Algorithm works if d is large enough ($d>4$)

Analysis

Success amplitude: $\langle w|e^{-iHt}|s\rangle = \sum_a \langle w|\psi_a\rangle\langle\psi_a|s\rangle e^{-iE_at}$

Eigenvectors: $H|\psi_a\rangle = (-\gamma L - |w\rangle\langle w|)|\psi_a\rangle = E_a|\psi_a\rangle$

so $(-\gamma L - E_a)|\psi_a\rangle = |w\rangle\langle w|\psi_a\rangle$

$$|\psi_a\rangle = \frac{\langle w|\psi_a\rangle}{-\gamma L - E_a}|w\rangle$$

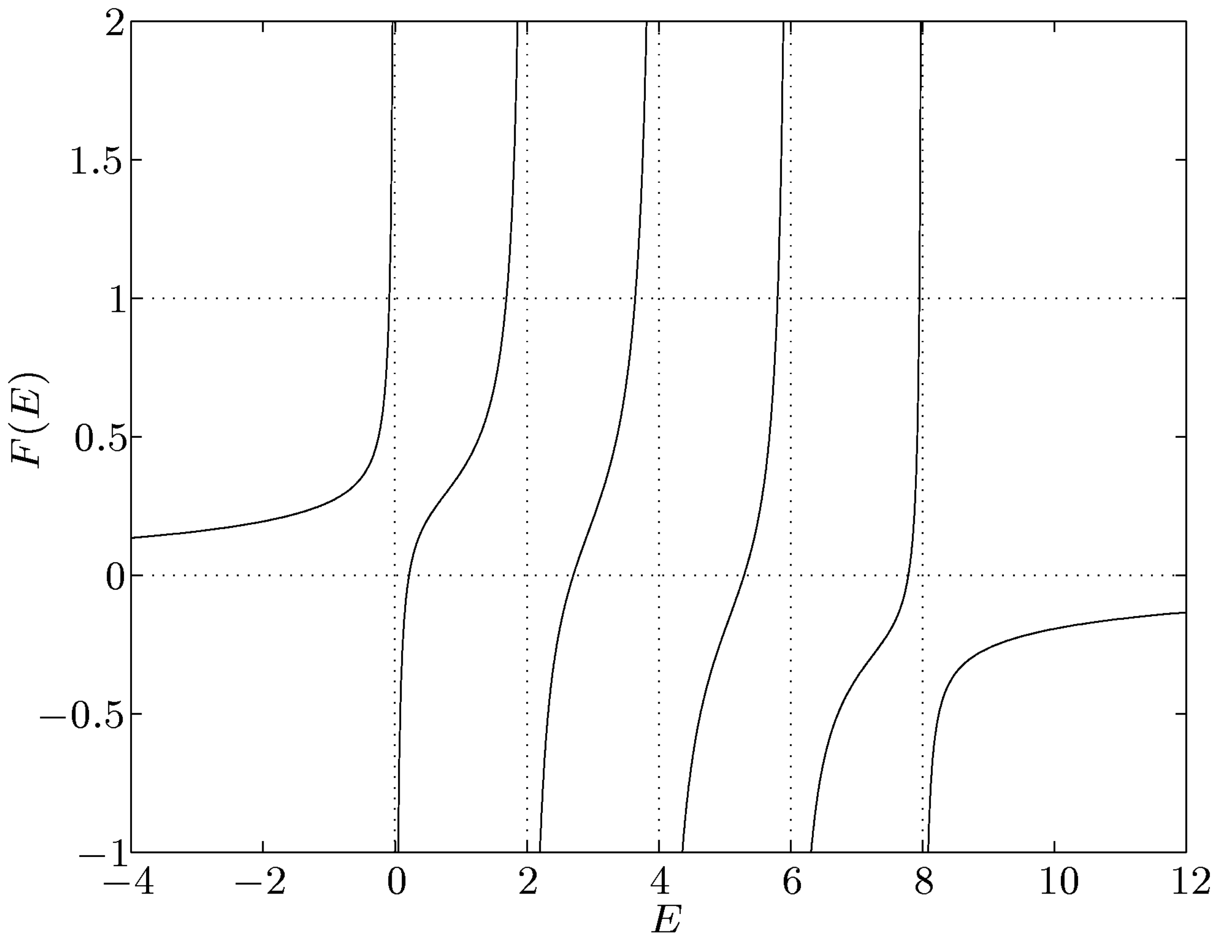
$$1 = \langle w|\frac{1}{-\gamma L - E_a}|w\rangle$$

Eigenvectors of –*L*: $\langle x|\phi(\vec{k})\rangle = \frac{1}{\sqrt{N}}e^{i\vec{k}\cdot\vec{x}} \qquad k_j = \frac{2\pi m_j}{N^{1/d}}$

$$\mathcal{E}(\vec{k}) = 2\left(d - \sum_{j=1}^{d}\cos\left(k_j\right)\right)$$

Eigenvalue condition

$$F(E_a) = 1 \quad \text{where} \quad F(E) = \frac{1}{N} \sum_{k} \frac{1}{\gamma \mathcal{E}(k) - E}$$



Critical point in *d*>4

$$F(E) = -\frac{1}{NE} + \frac{1}{N}\sum_{k\neq 0}\frac{1}{\gamma\mathcal{E}(k) - E}$$

$$= -\frac{1}{NE} + \frac{1}{N}\sum_{k\neq 0}\frac{1}{\gamma\mathcal{E}(k)} + \frac{E}{N}\sum_{k\neq 0}\frac{1}{[\gamma\mathcal{E}(k)]^2} + \cdots$$

$$\Big[|E| \ll \gamma\mathcal{E}(k)\Big]$$

$$\approx -\frac{1}{NE} + \frac{1}{\gamma}I_{1,d} + \frac{E}{\gamma^2}I_{2,d}$$

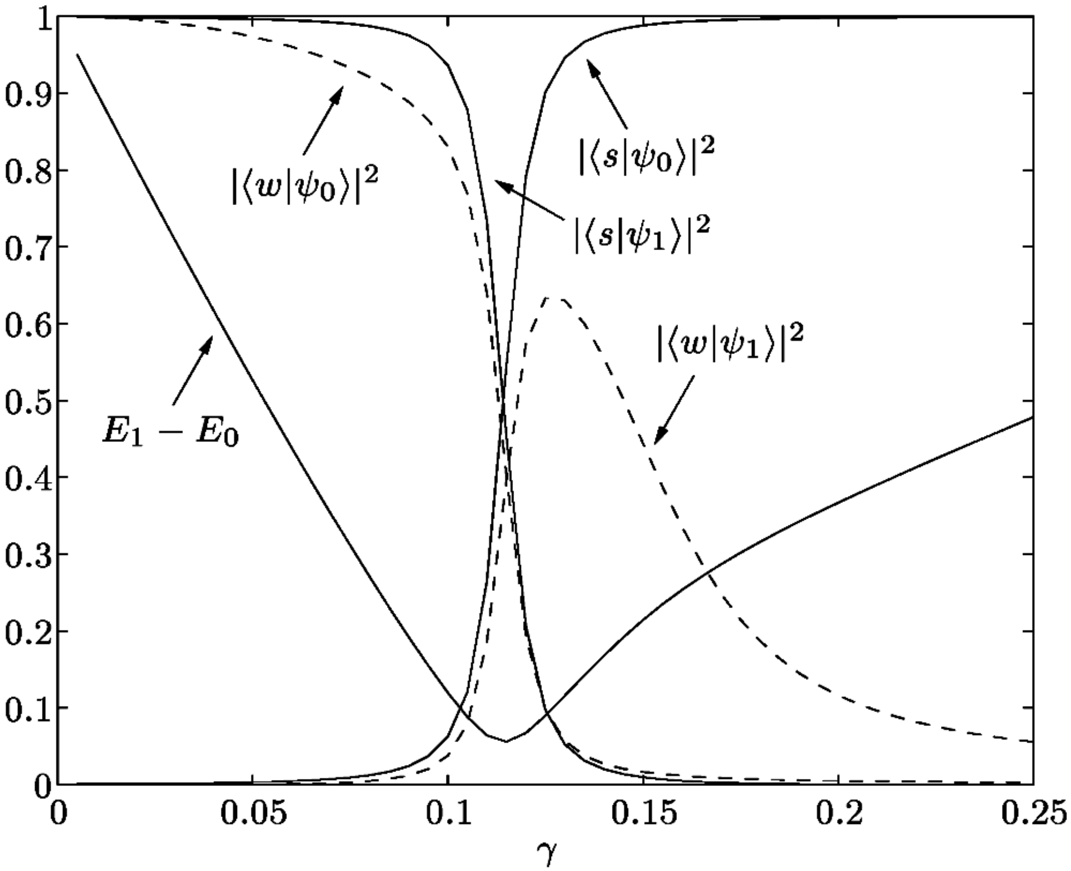
converge for d>4

where $I_{j,d} = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\frac{\mathrm{d}^d k}{[\mathcal{E}(k)]^j}$

Set $\gamma \approx I_{1,d}$; then $F(E)=1$ for $E_{0,1} \approx \pm\frac{I_{1,d}}{\sqrt{I_{2,d}N}}$

By similar analysis, $\langle s|\psi_{0,1}\rangle$ and $\langle w|\psi_{0,1}\rangle$ are $O(1)$ here.

(d>4)-dimensional lattice



d=5
N=4^5=1024

Critical region

$\gamma = \gamma^* \pm O(N^{-1/2})$

$\gamma < \gamma^*$

$|s\rangle \sim |\psi_1\rangle$

$\gamma > \gamma^*$

$|s\rangle \sim |\psi_0\rangle$

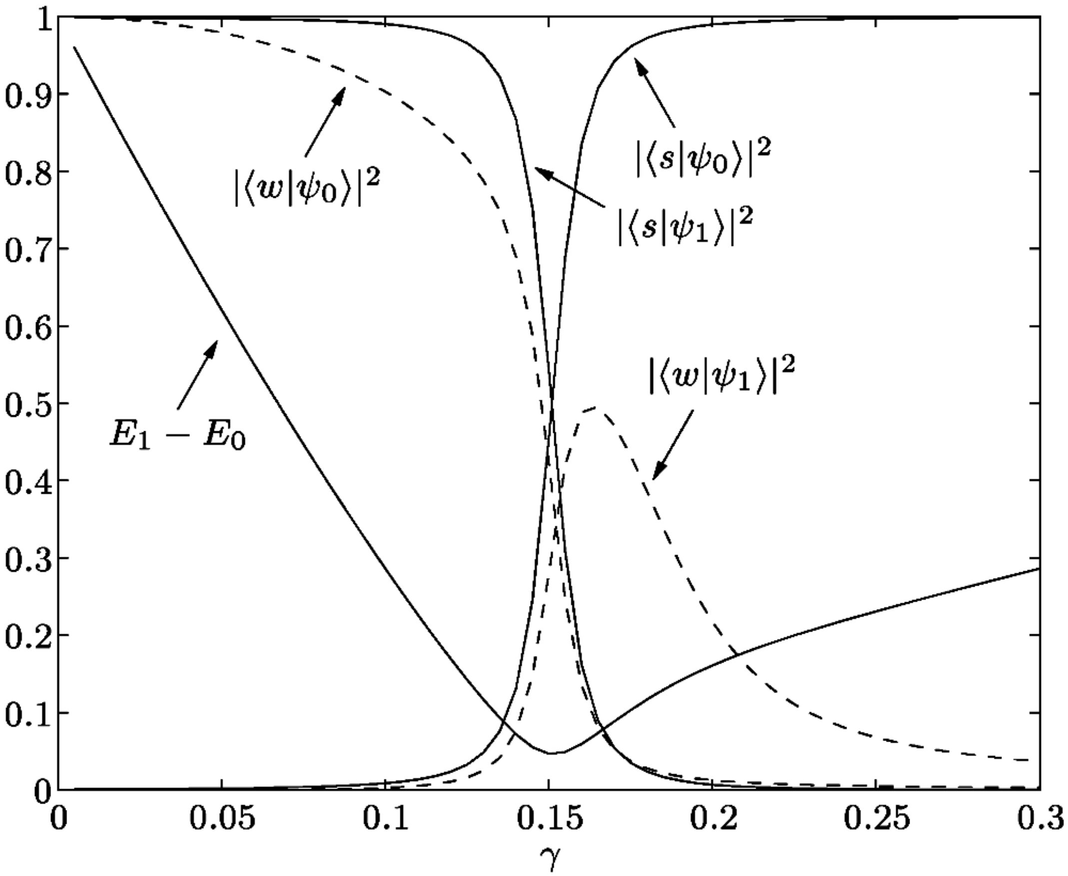
$\gamma \sim \gamma^*$

$E_1 - E_0 = O(N^{-1/2})$

$|\psi_{0,1}\rangle \sim |s\rangle \pm O(1)\,|w\rangle$

Run time $O(N^{1/2})$

4-dimensional lattice



d=4

N=6^4=1296

Critical region

$$\gamma = \gamma^* \pm O\left(\sqrt{\frac{\log N}{N}}\right)$$

$\gamma < \gamma^*$

$|s\rangle \sim |\psi_1\rangle$

$\gamma > \gamma^*$

$|s\rangle \sim |\psi_0\rangle$

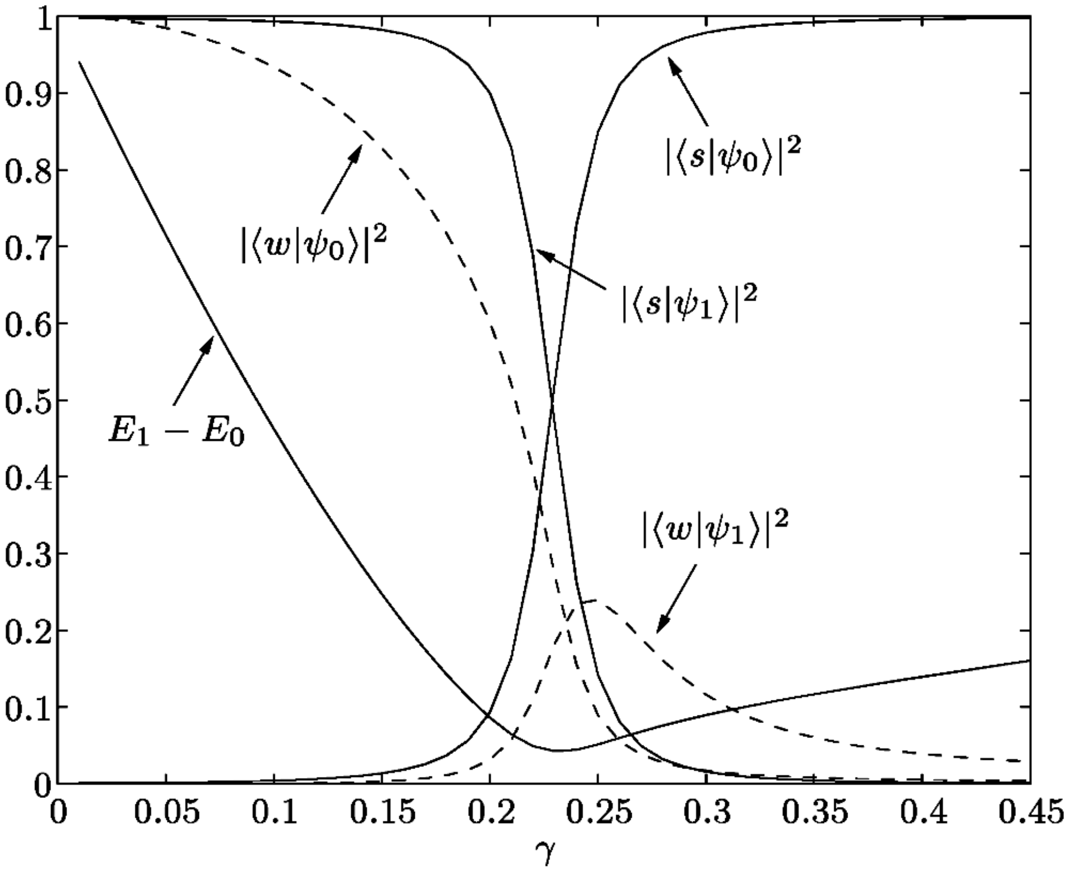
$\gamma \sim \gamma^*$

$$E_1 - E_0 = O\left(\frac{1}{\sqrt{N \log N}}\right)$$

$$|\psi_{0,1}\rangle \sim |s\rangle \pm O\left(\frac{1}{\sqrt{\log N}}\right)|w\rangle$$

Run time $O(\sqrt{N}\ \log^{3/2} N)$

3-dimensional lattice



d=3
$N=10^3=1000$

Critical region

$\gamma = \gamma^* \pm O\left(\frac{1}{N^{1/3}}\right)$

$\gamma < \gamma^*$

$|s\rangle \sim |\psi_1\rangle$

$\gamma > \gamma^*$

$|s\rangle \sim |\psi_0\rangle$

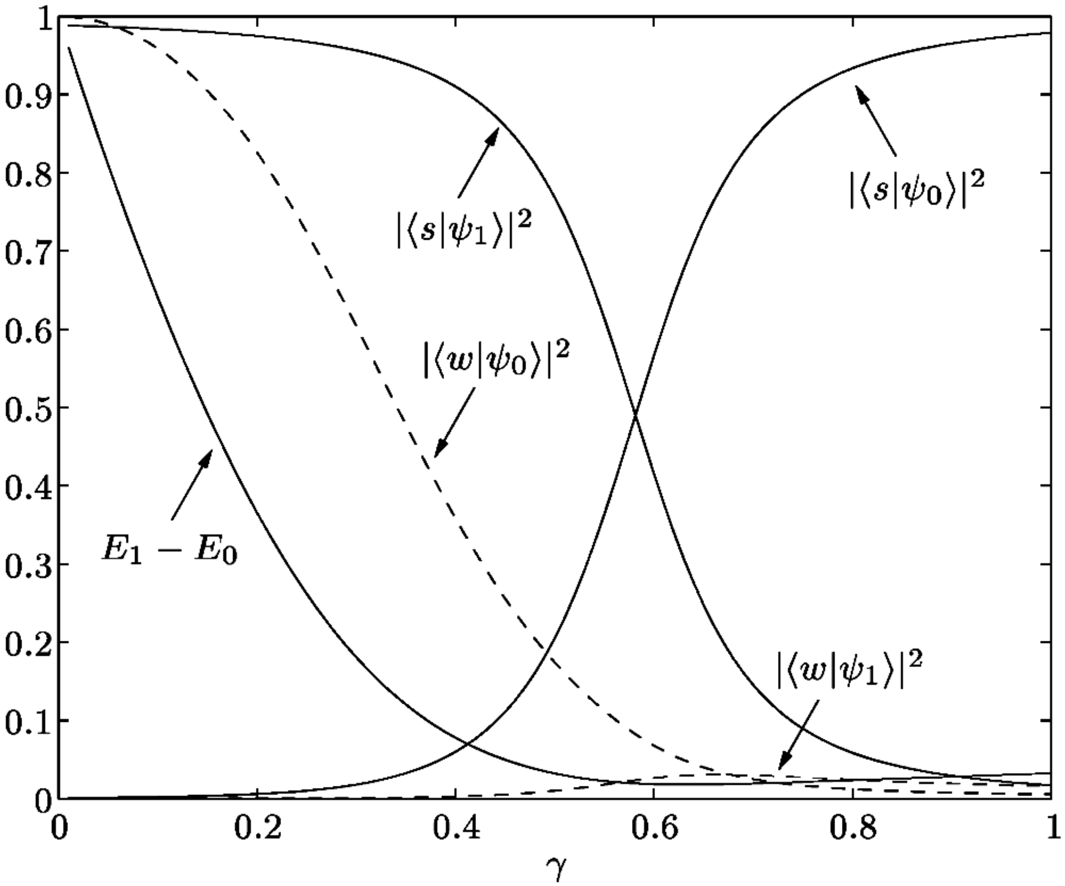
$\gamma \sim \gamma^*$

$E_1 - E_0 = O\left(\frac{1}{N^{2/3}}\right)$

$|\psi_{0,1}\rangle \sim |s\rangle \pm O\left(\frac{1}{N^{1/6}}\right)|w\rangle$

Run time $O(N)$

2-dimensional lattice



d=2
N=32^2=1024

Critical region

$\gamma = \gamma^* \log N \pm O(1)$

$\gamma < \gamma^* \log N$

$|s\rangle \sim |\psi_1\rangle$

$\gamma > \gamma^* \log N$

$|s\rangle \sim |\psi_0\rangle$

$\gamma \sim \gamma^* \log N$

$E_1 - E_0 = O\left(\frac{\log N}{N}\right)$

$|\psi_{0,1}\rangle \sim |s\rangle \pm O\left(\sqrt{\frac{\log N}{N}}\right) |w\rangle$

Run time $O(N^2/\log^2 N)$

Initial state

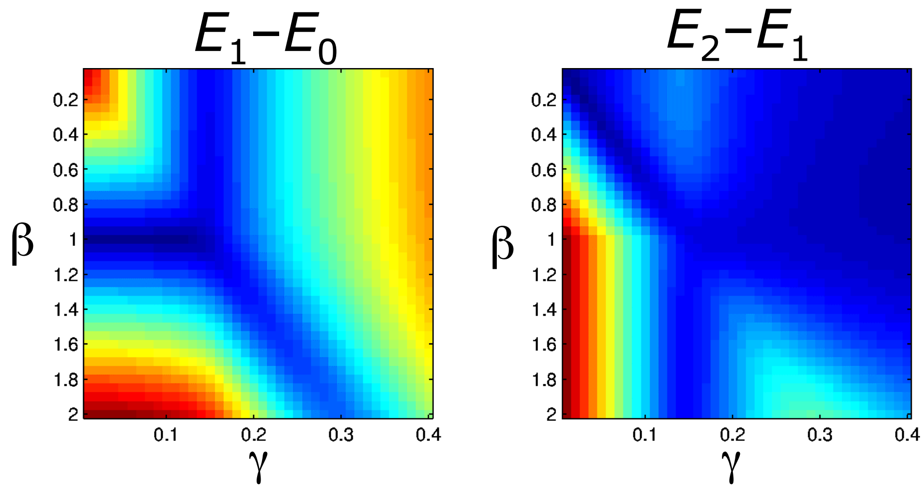
What if we cannot start in the state $|s\rangle$?

- The state $|s\rangle$ can be locally prepared from a known state $|x\rangle$ in time $O(N^{1/d})$
- Alternatively, run the algorithm backwards starting from $|x\rangle$, assuming $w=x$ (double the run time)
- Or, start from a random $|x\rangle$ with a modified Hamiltonian…

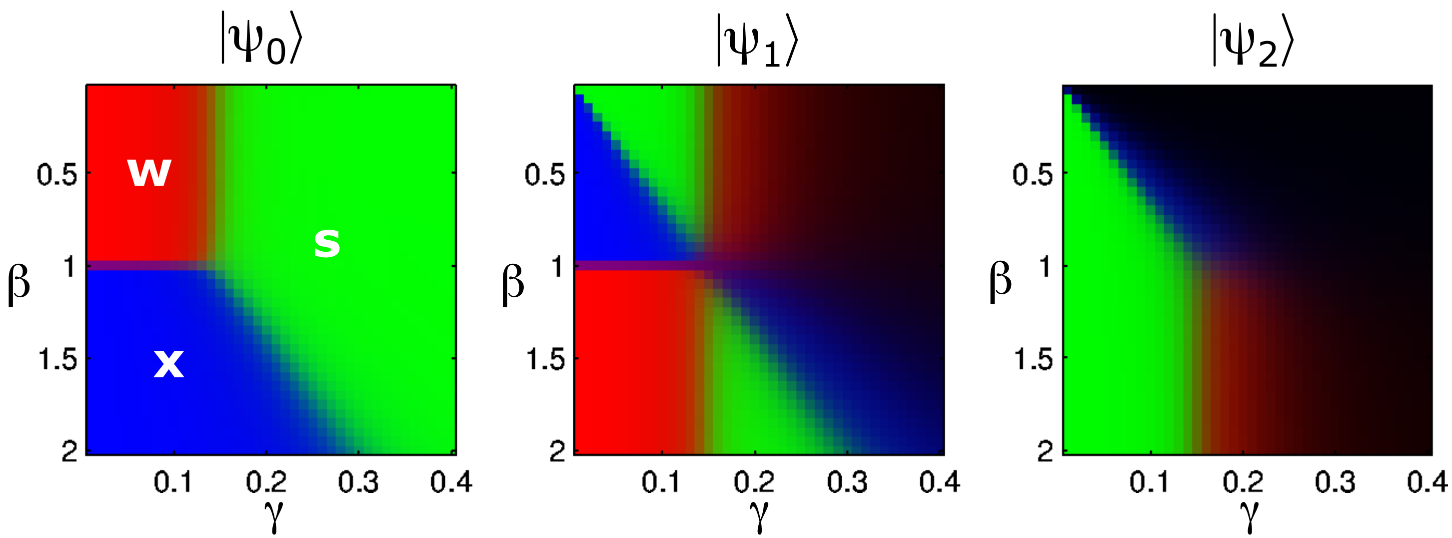
Starting from a random vertex

Start in state $|x\rangle$

$$H = -\gamma L - |w\rangle\langle w| - \beta|x\rangle\langle x|$$



d=4
N=4^4=256



Related algorithms

- Shenvi, Kempe, Whaley 02: discrete time quantum walk search algorithm on hypercube, O($N^{1/2}$)

  Behavior in constant dimensions?

- Adiabatic evolution [RC01, vDMV01]

- Measurement [CDFGGS02]

Open questions

- Find more applications of quantum walks
- Is there a time independent Hamiltonian algorithm that works in d=2,3?
- What is the actual complexity of the search problem in d=2?