Evaluating formulas on a quantum computer

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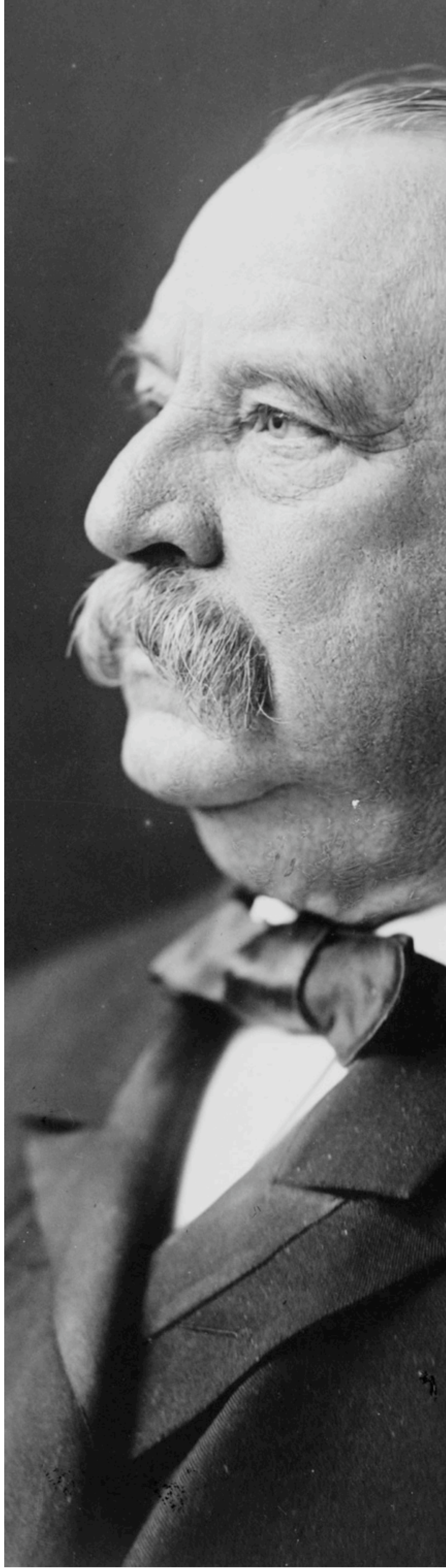
quant-ph/0703015

FOCS 2007

OR

How fast can we compute the OR of n bits?

Evaluate formula: x_1 OR x_2 OR $\ldots$ OR x_n



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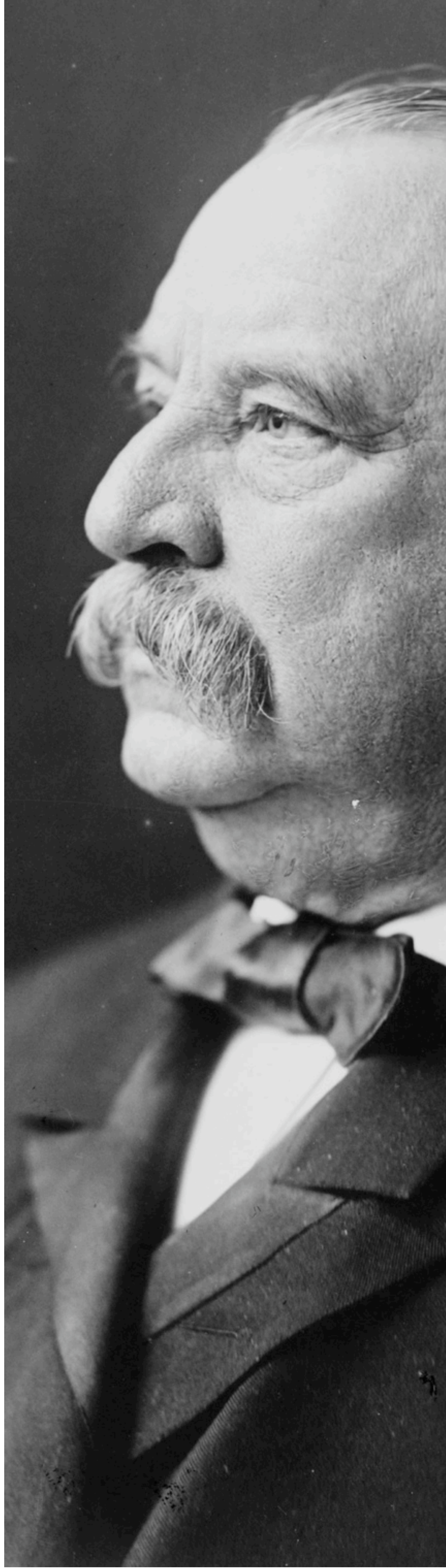

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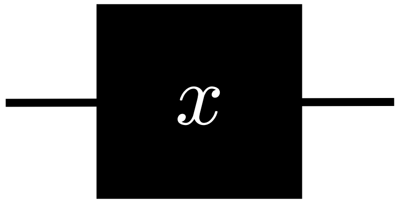

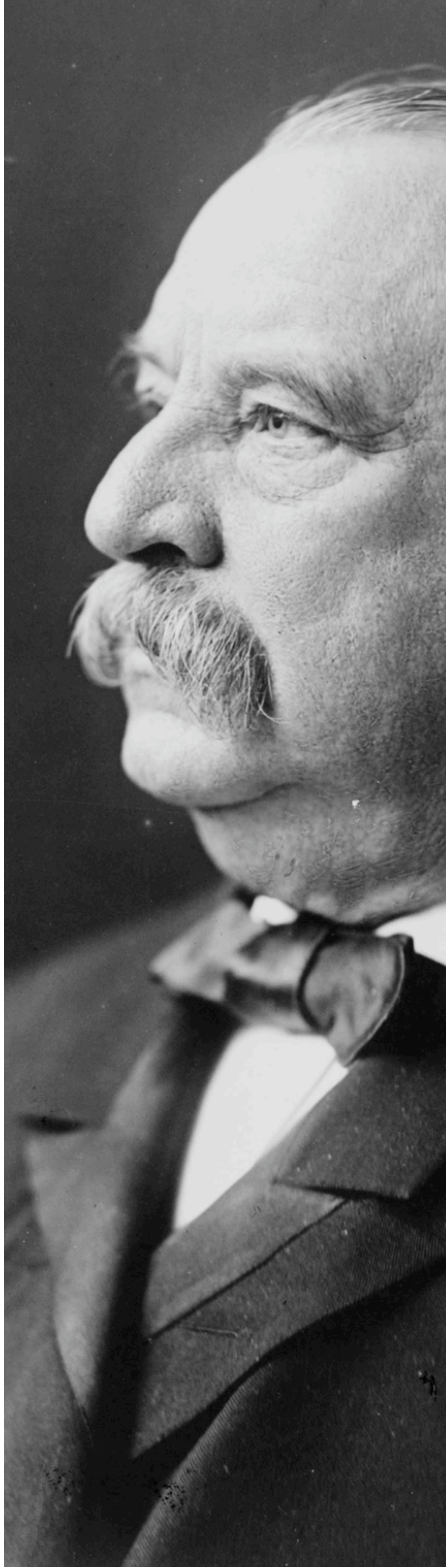

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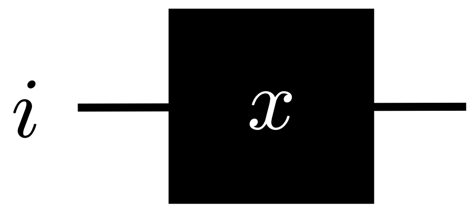

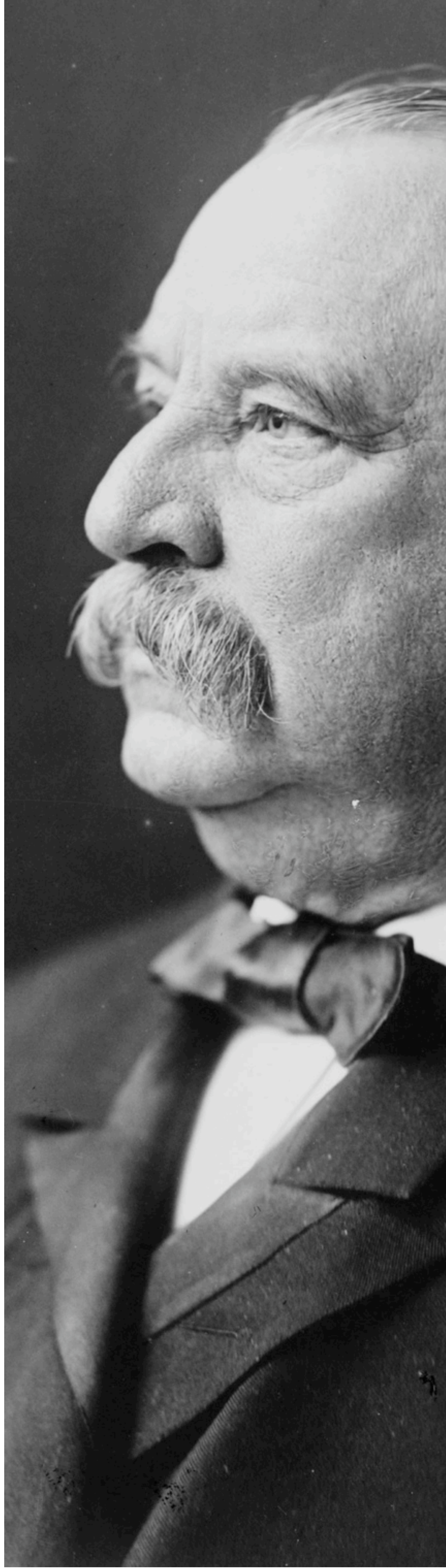

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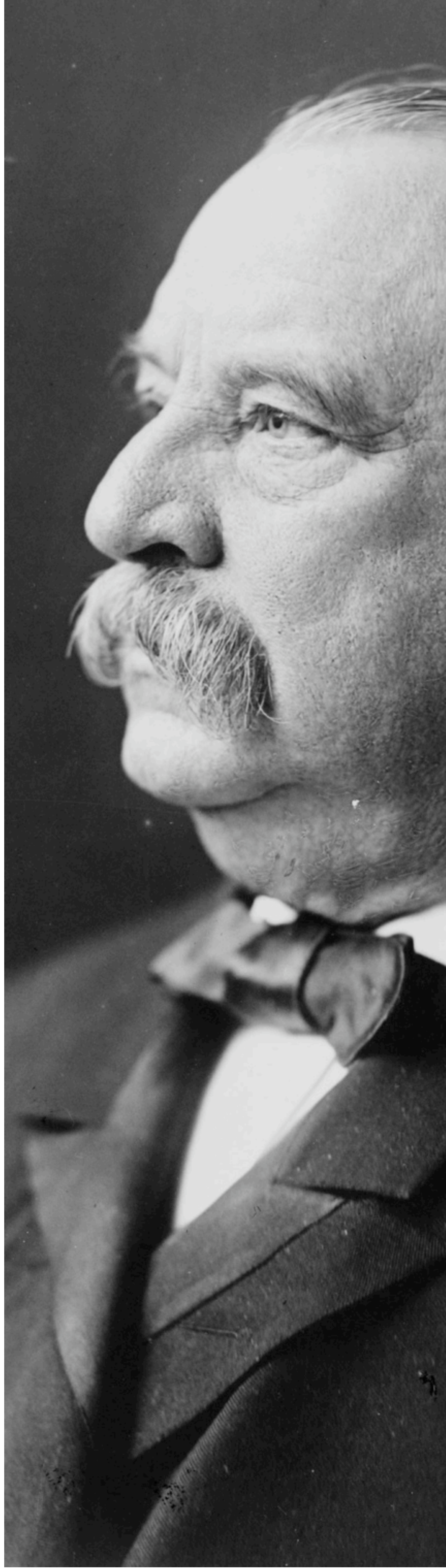

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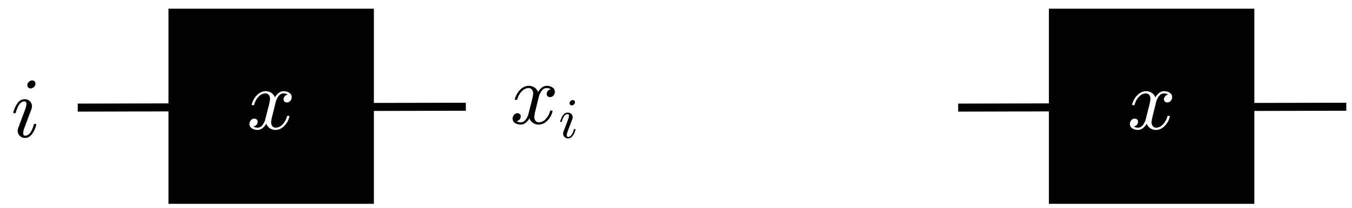

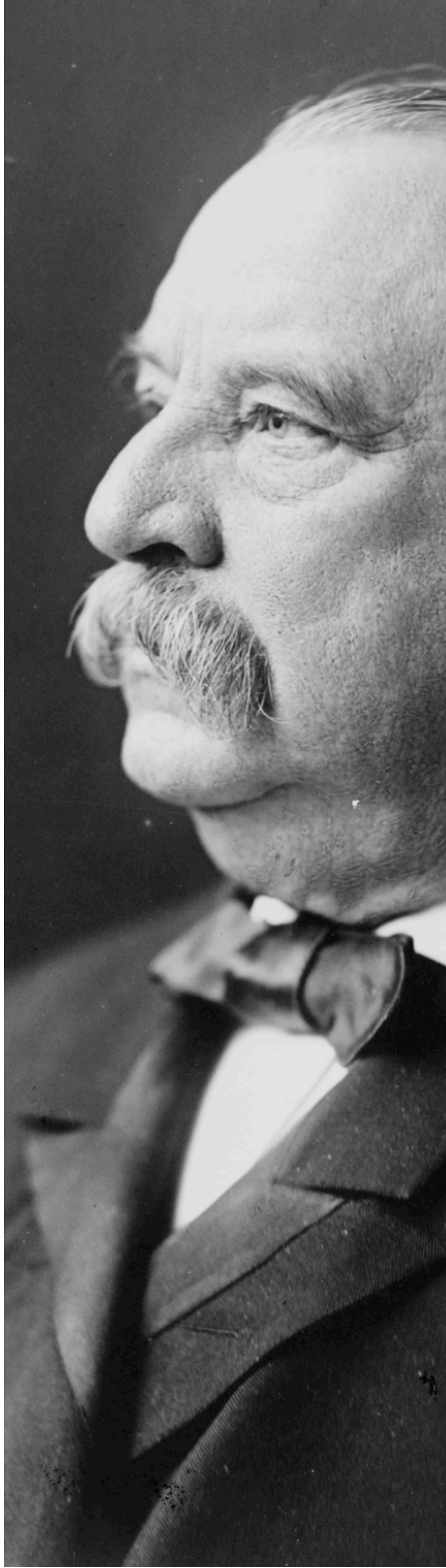

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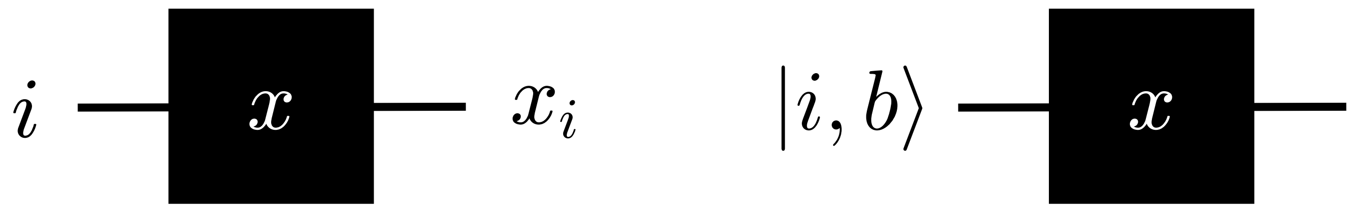

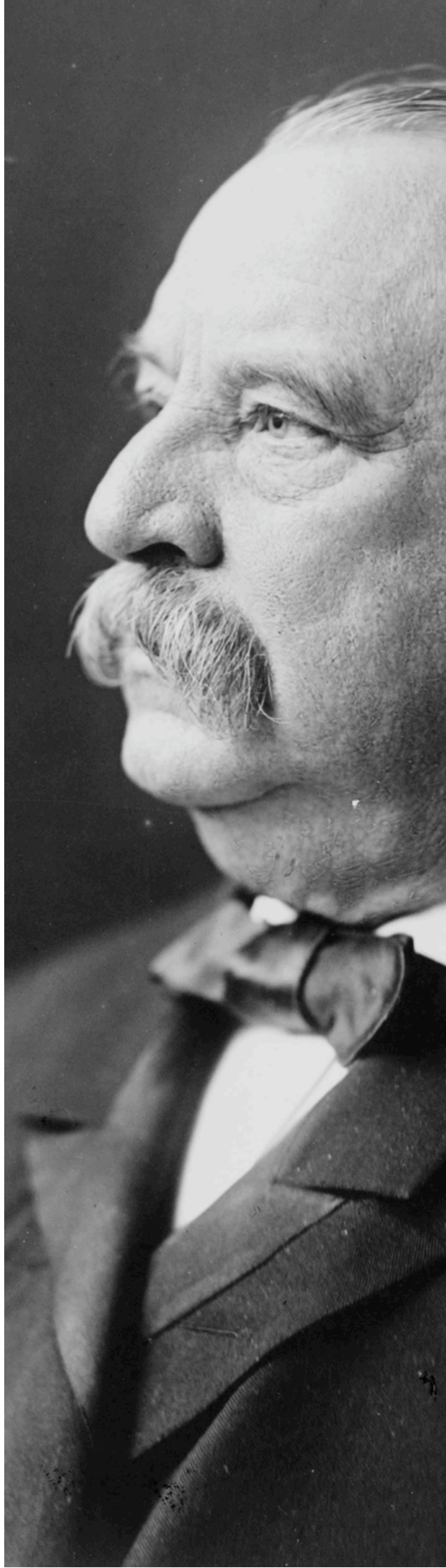

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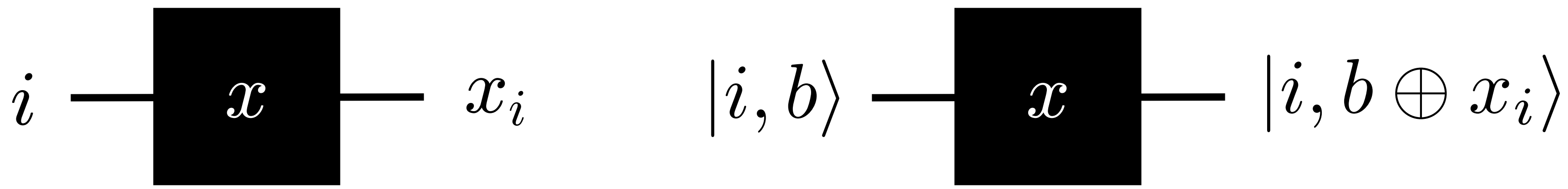

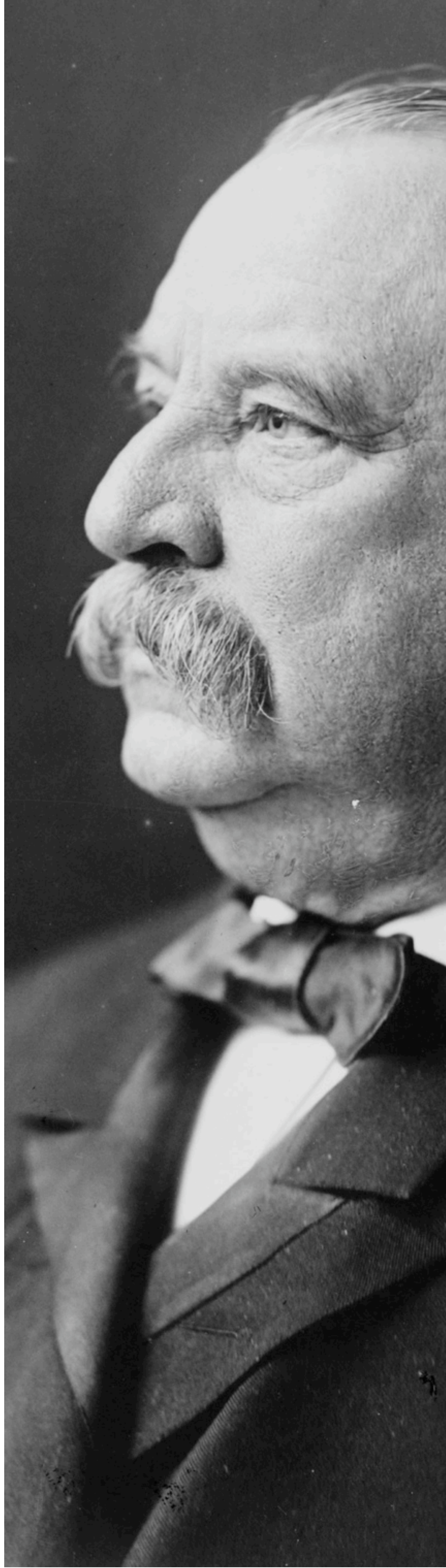

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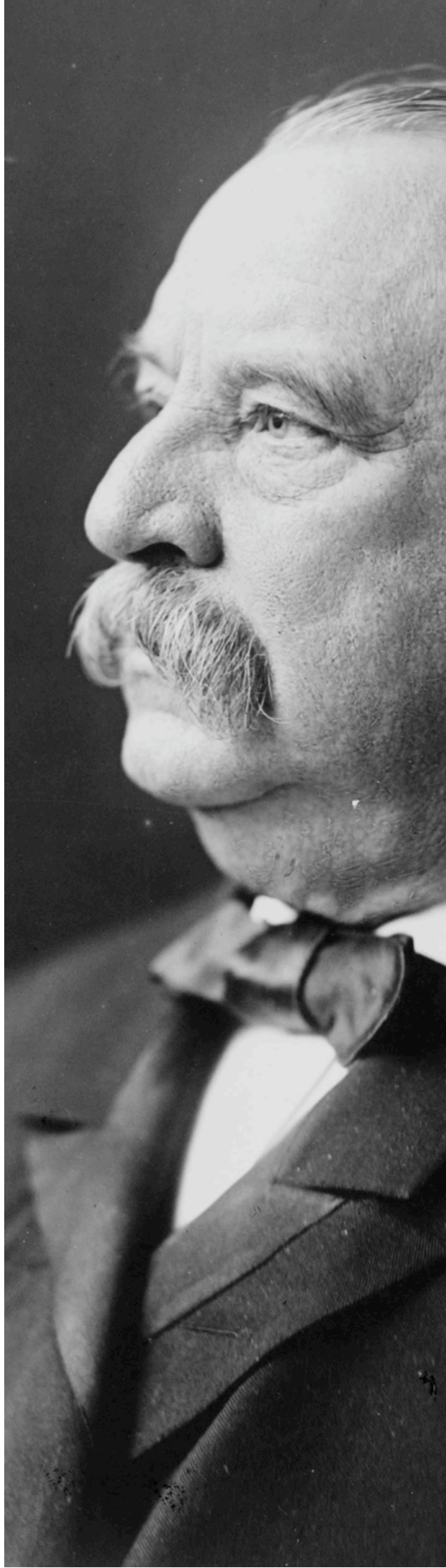
How many queries are required to evaluate OR?

Classical complexity: $\Theta(N)$

Quantum algorithm [Grover 1996]: $O(\sqrt{N})$

Quantum lower bound [BBBV 1996]: $\Omega(\sqrt{N})$



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We must evaluate a formula involving AND and OR gates:

0-player wins if she can make any move that gives 0
i.e., she only loses if all of her moves give 1 (AND)

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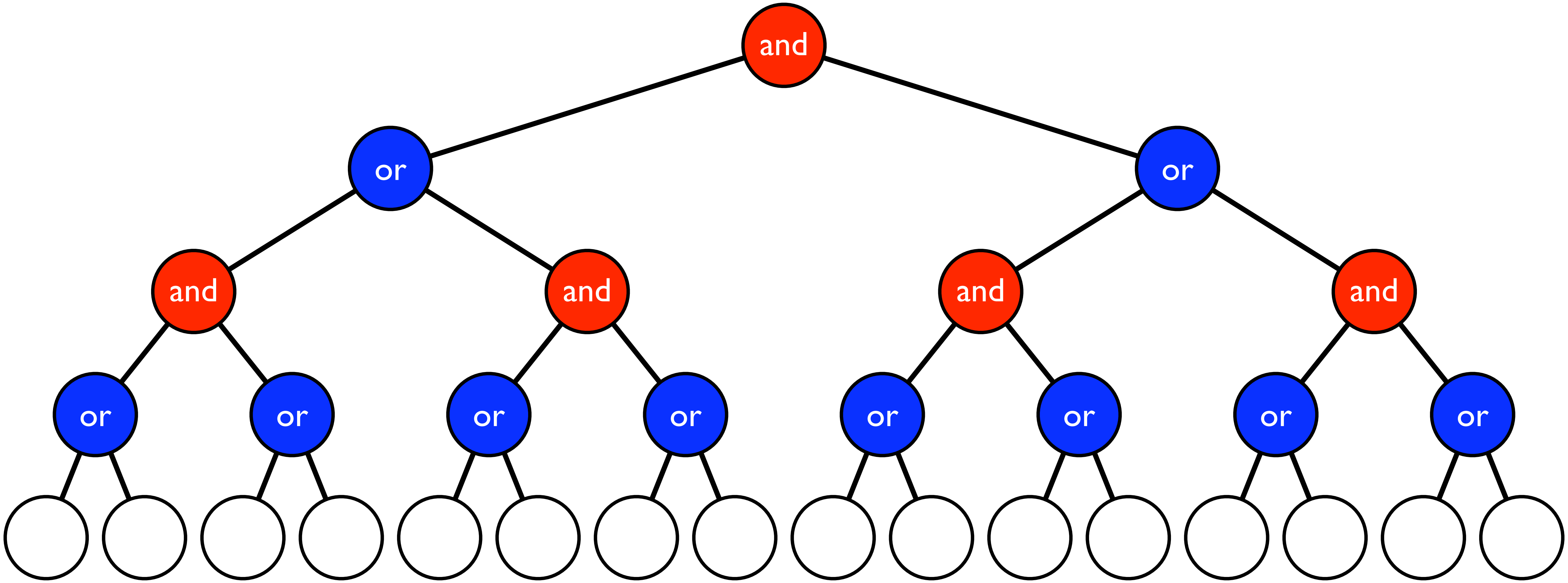
Game trees

Example: $d = 2, k = 4$



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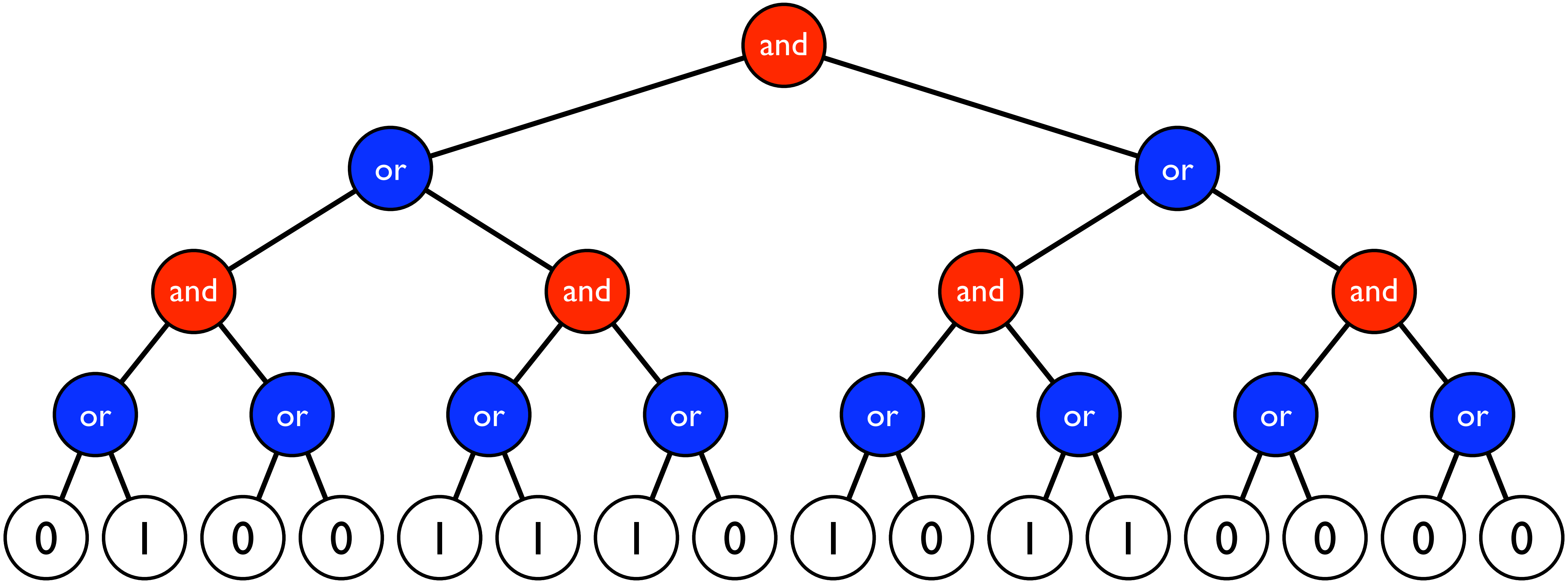

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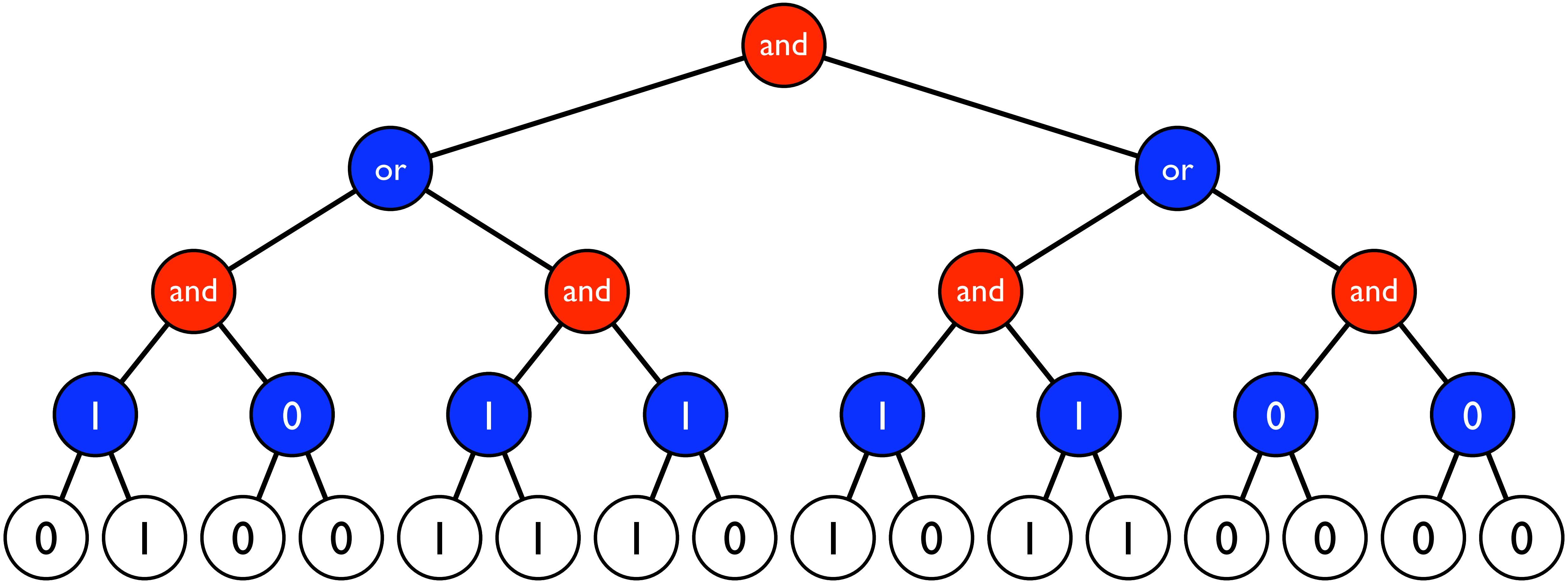

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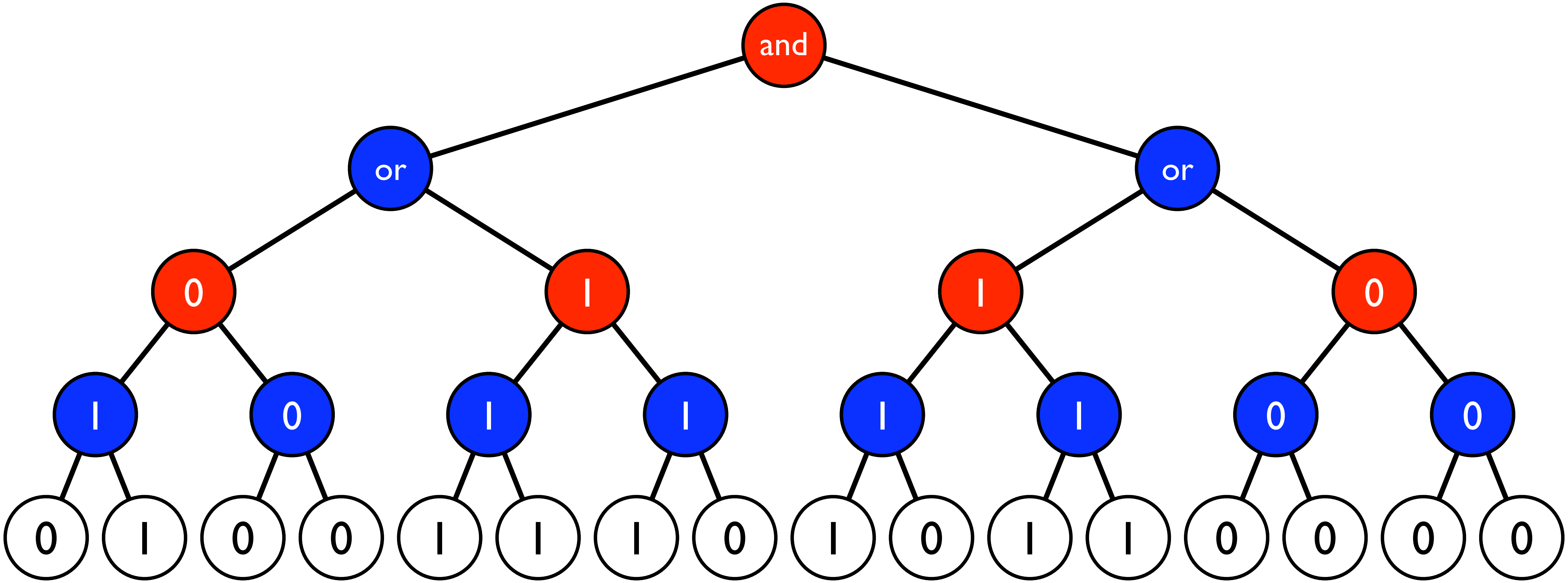

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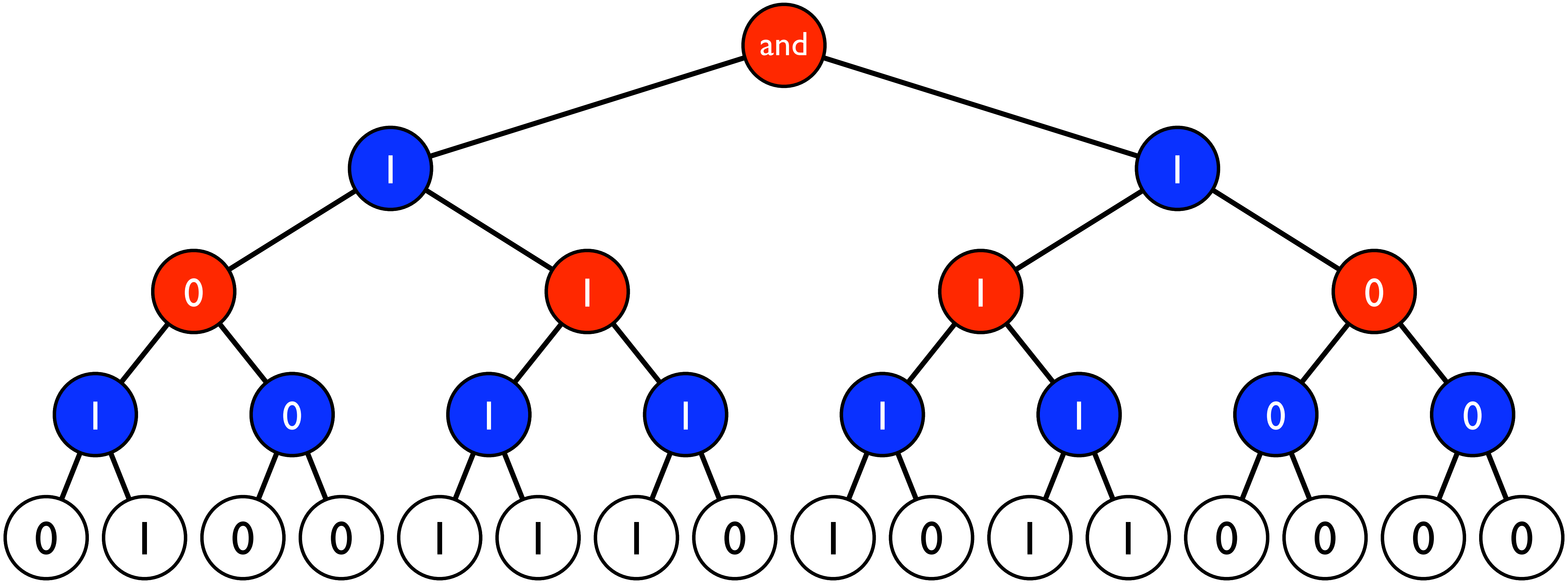

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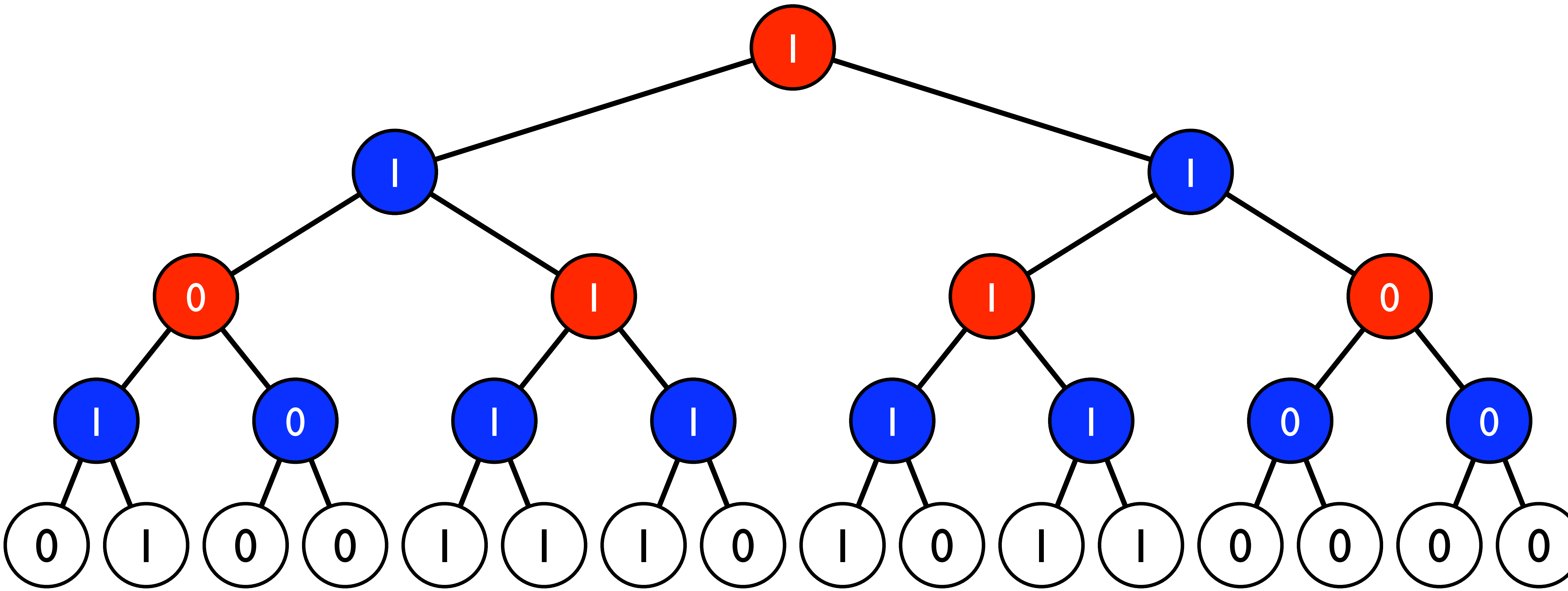

Evaluating game trees

Classical complexity: $\Theta\left(n^{\log_d \frac{d-1+\sqrt{d^2+14d+1}}{4}}\right)$ $\quad$ **(**$d = 2$**:** $\Theta(n^{0.753})$**)**

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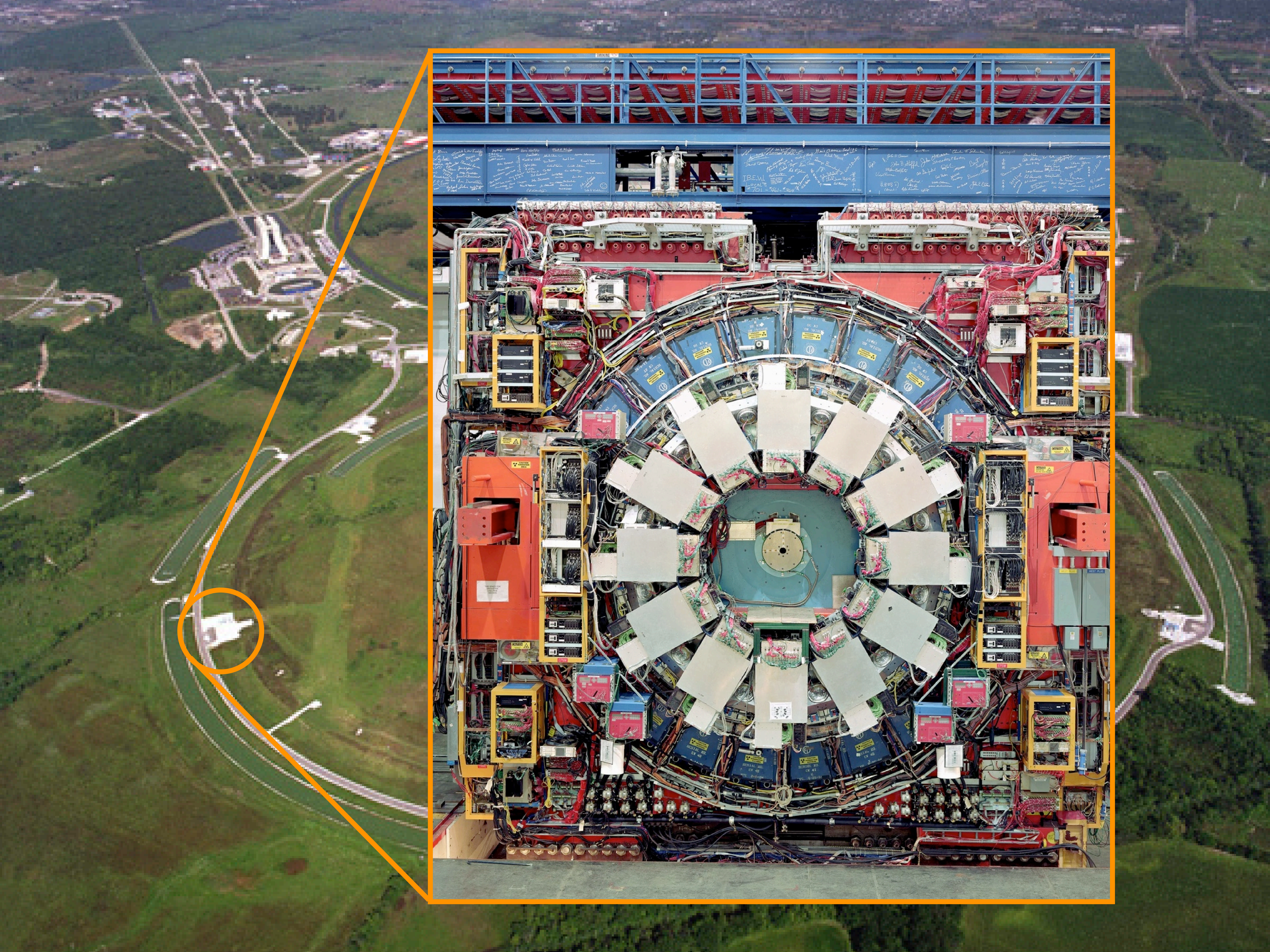
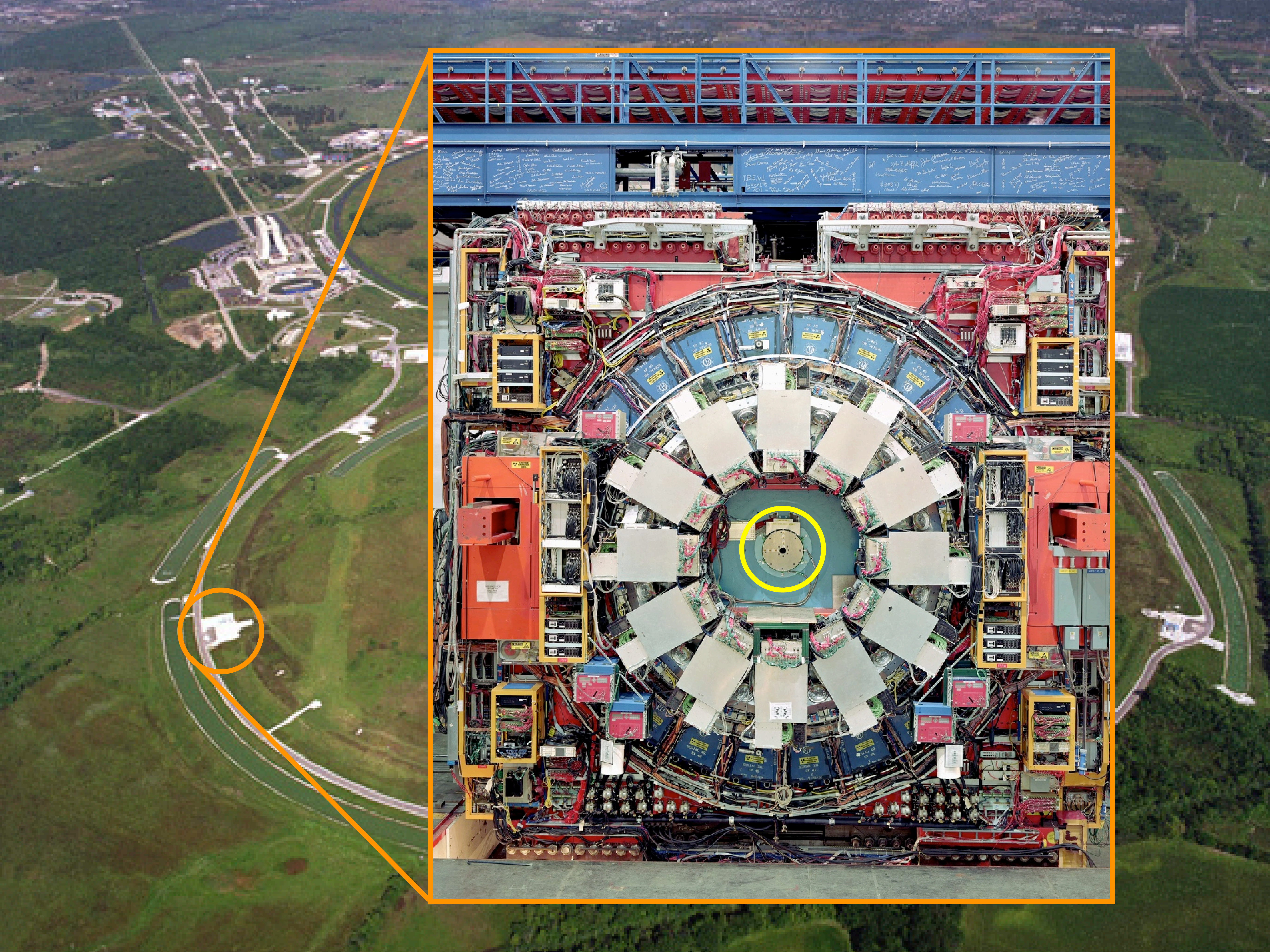
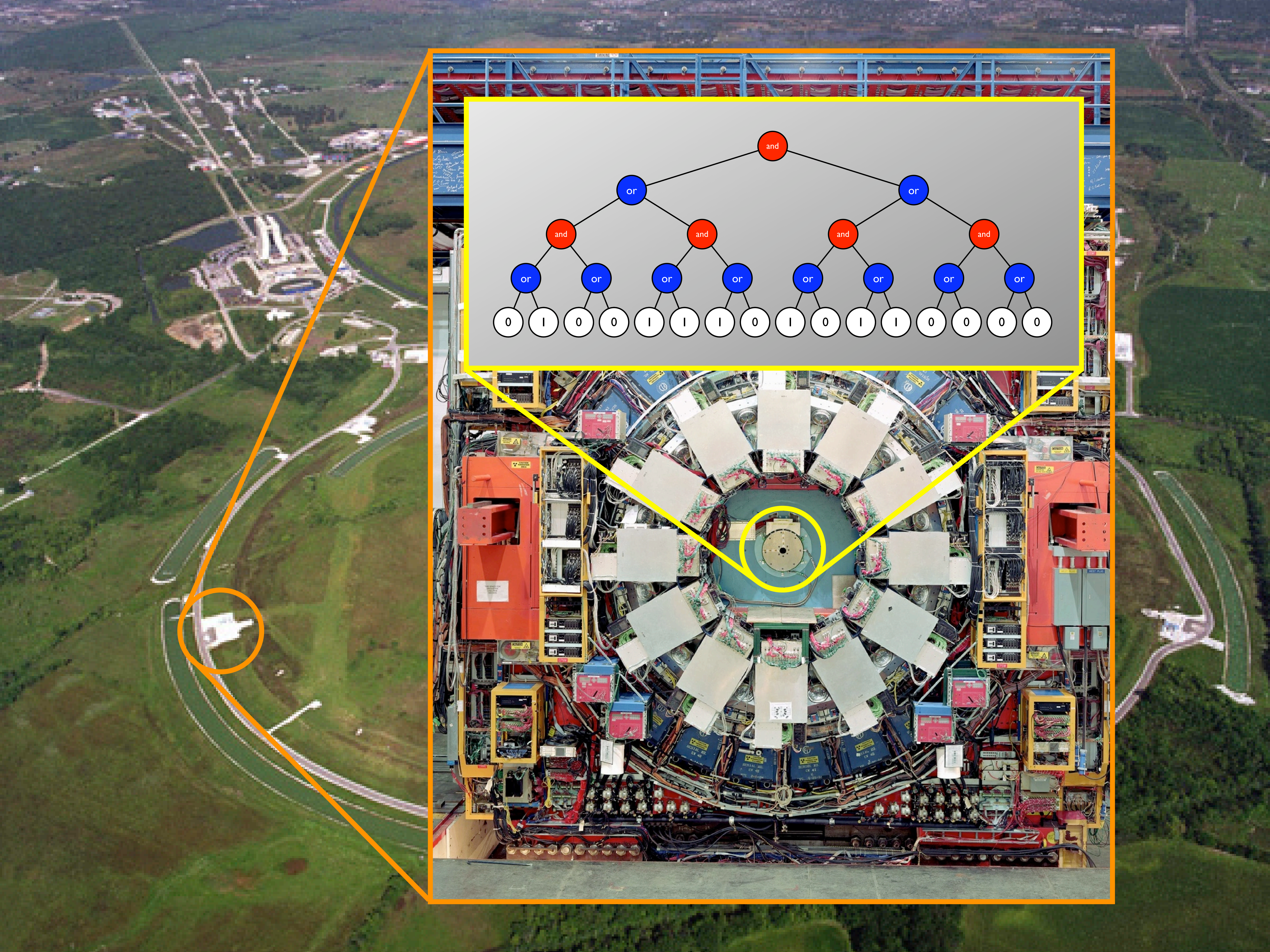
But these algorithms are only close to tight for k constant.

And for low degree (e.g., $d=2$), nothing better than classical was known until very recently!






and
or or
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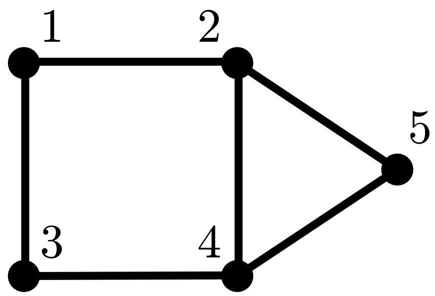
0 1 0 0 1 1 1 0 1 0 1 1 0 0 0 0

Continuous-time quantum walk

Graph G:



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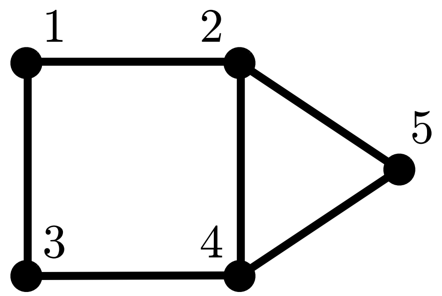

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

adjacency matrix

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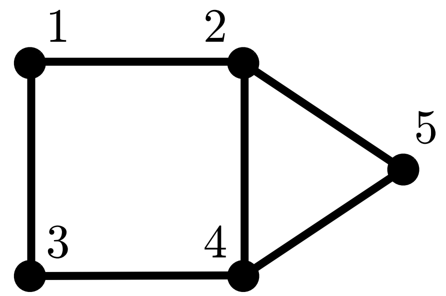

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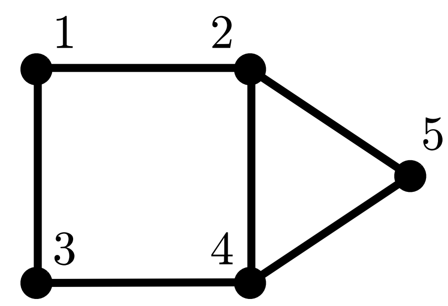
Graph G: 

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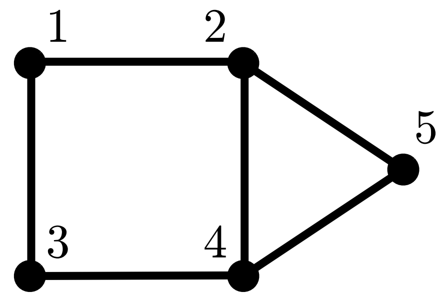
Graph G: 

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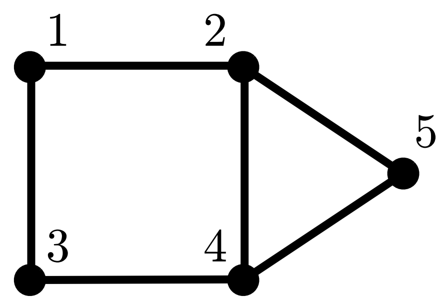
Graph G: 

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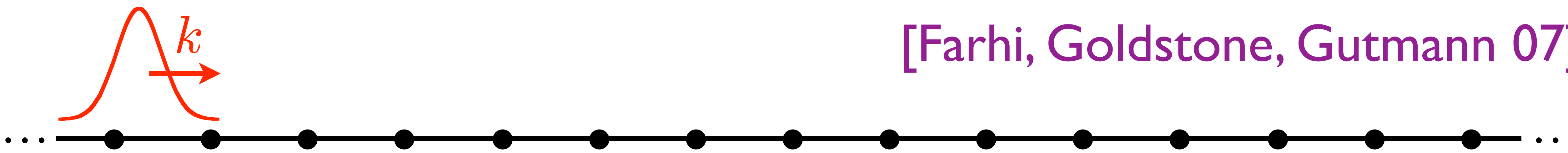
Evaluating AND-OR trees by scattering

[Farhi, Goldstone, Gutmann 07]



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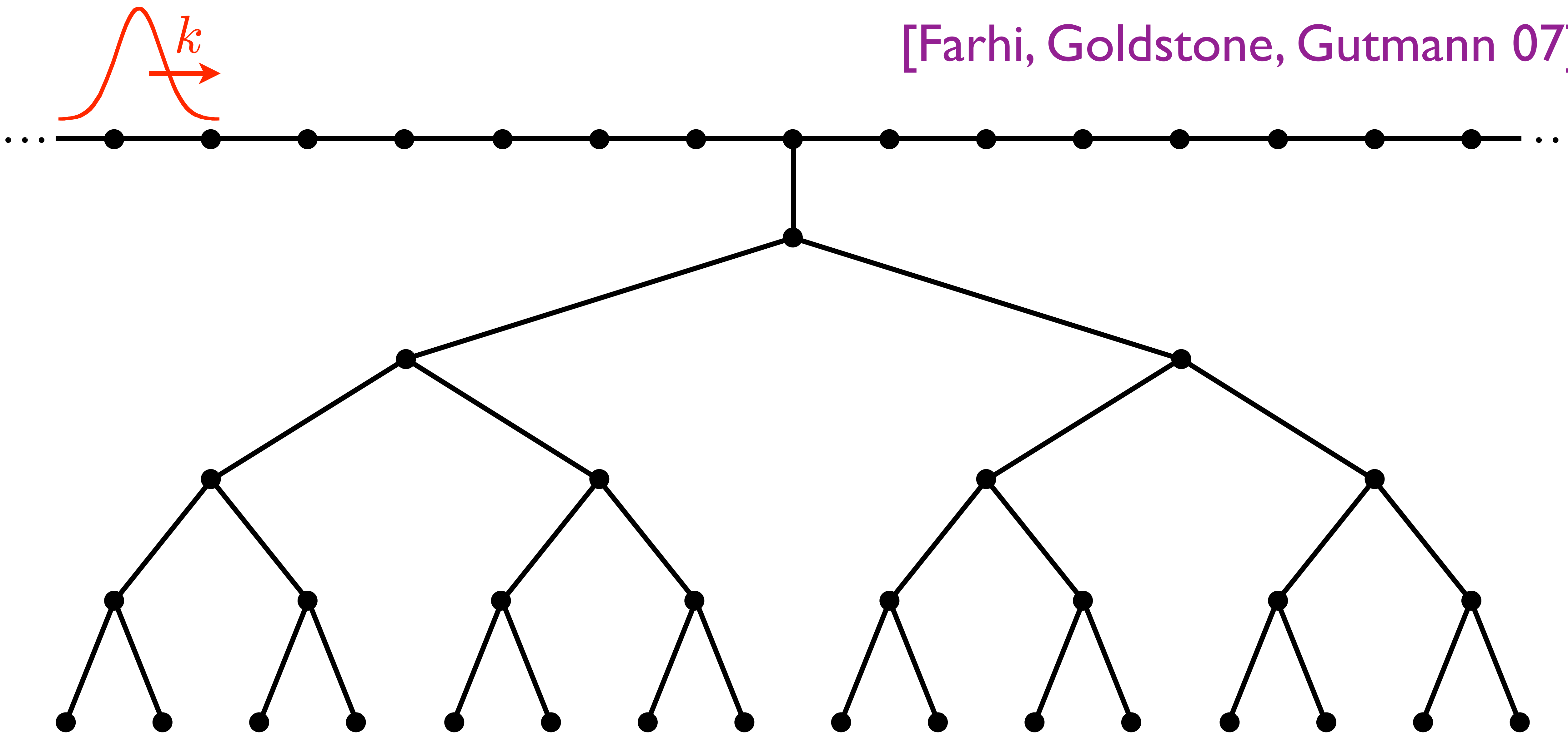

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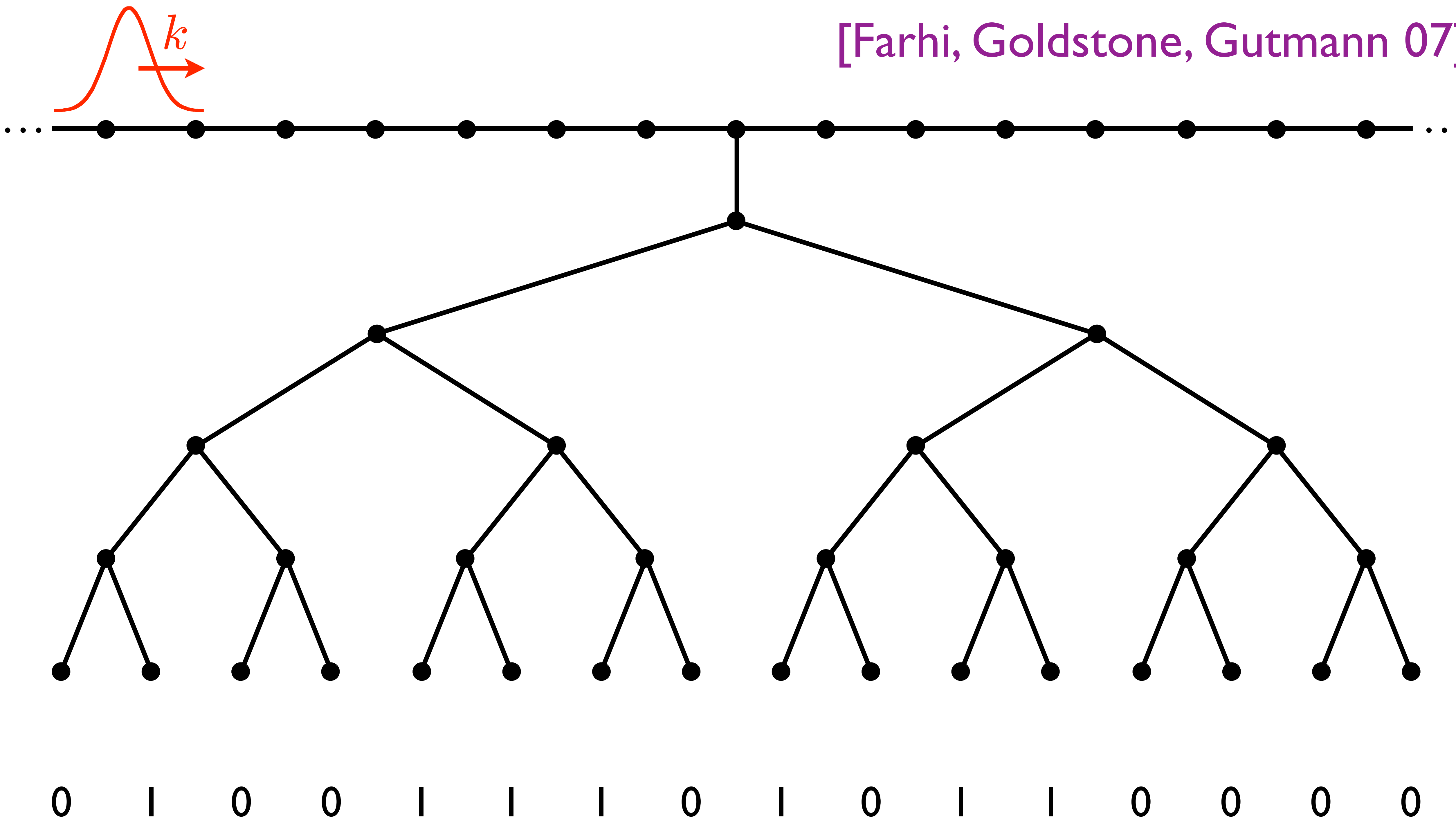

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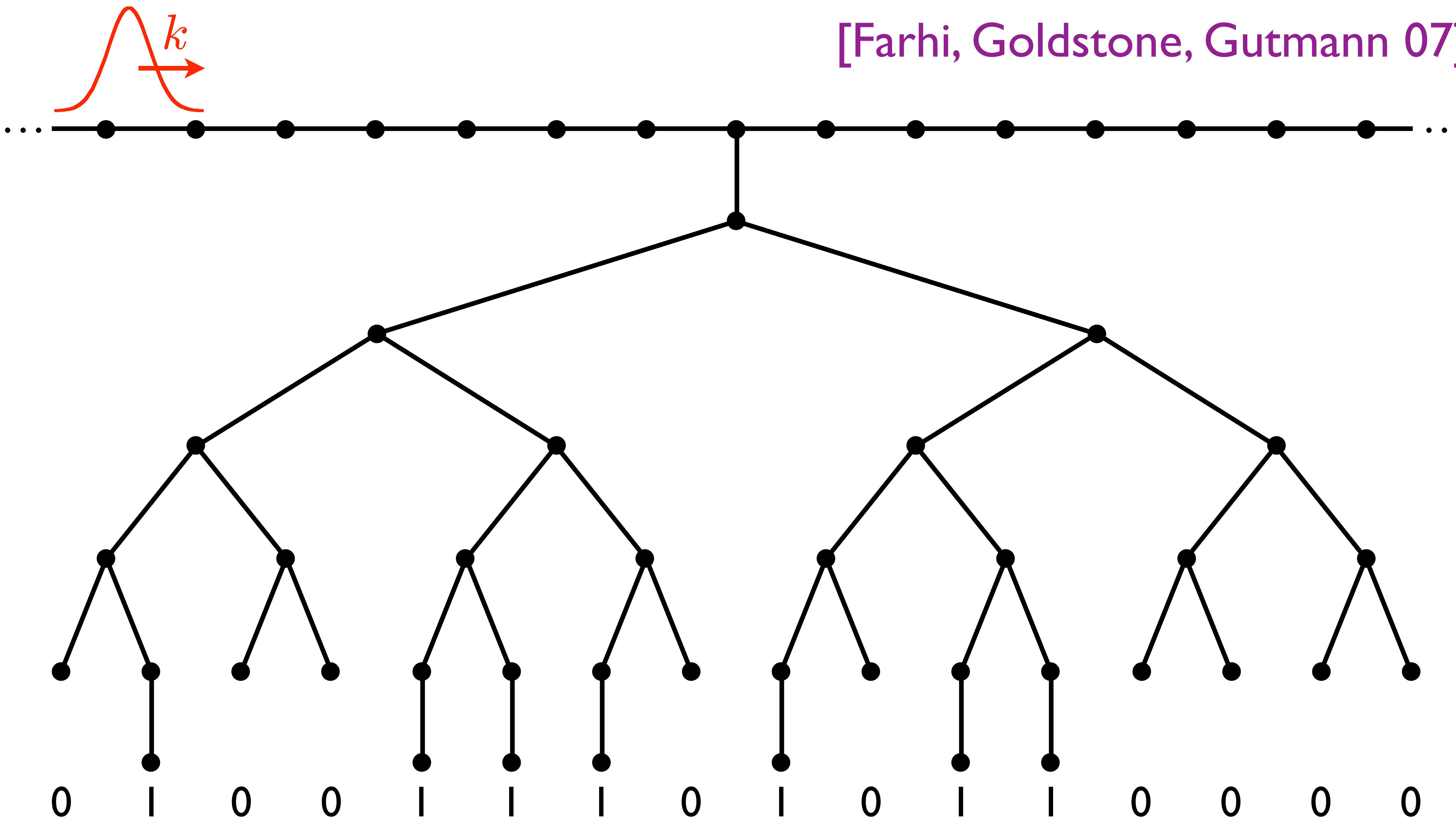

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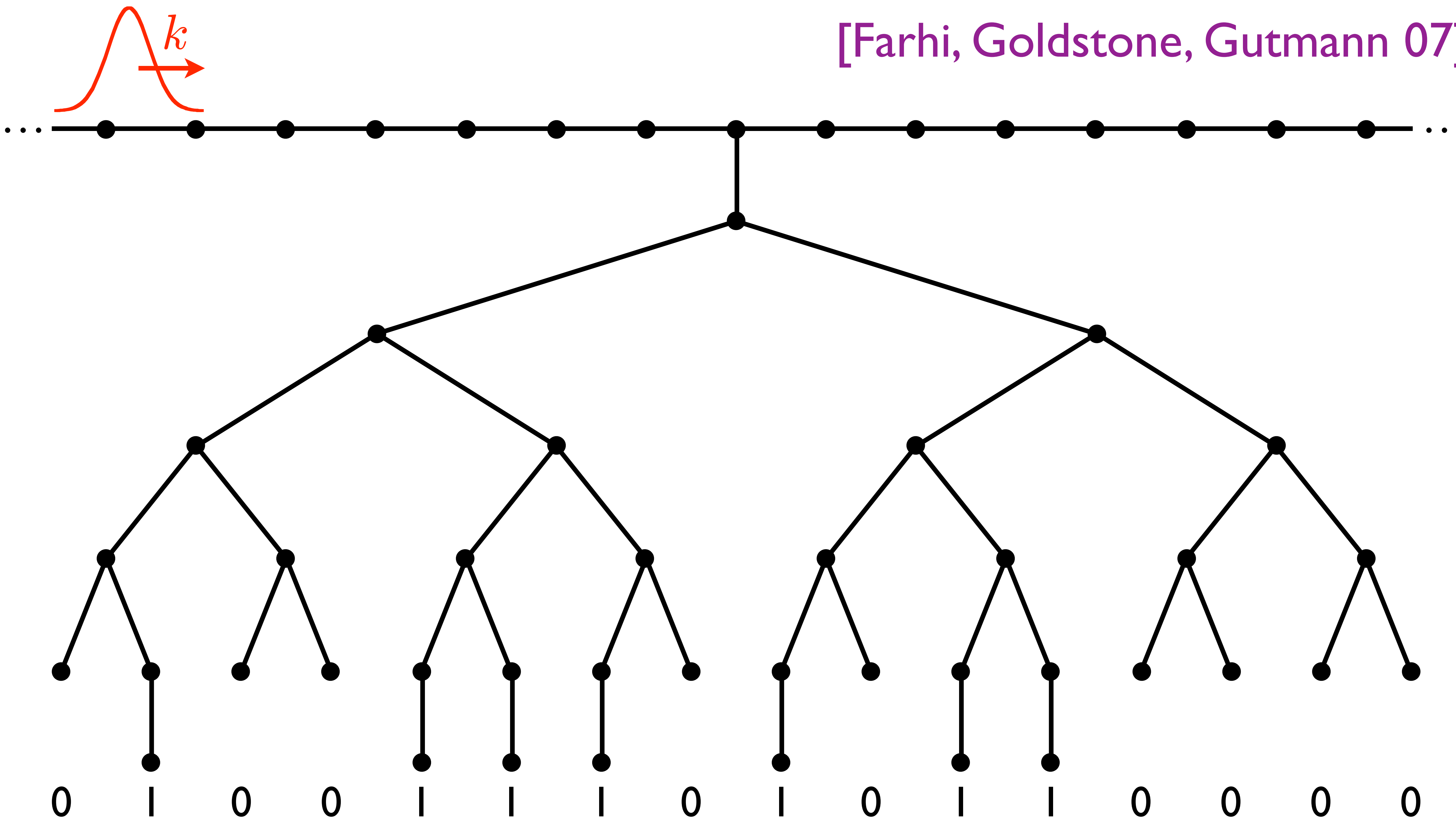

Claim: For small k, the wave is transmitted if the formula (translated into NAND gates) evaluates to 0, and reflected if it evaluates to 1.

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{AND, OR, NOT} equivalent to {NAND}

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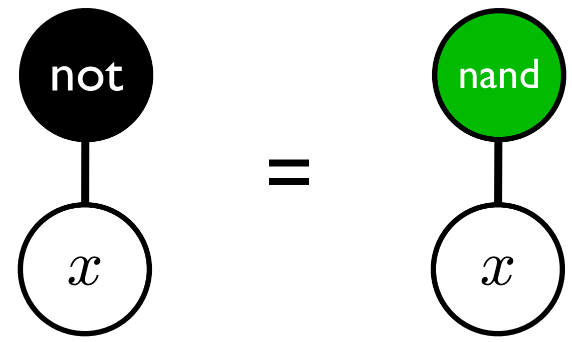

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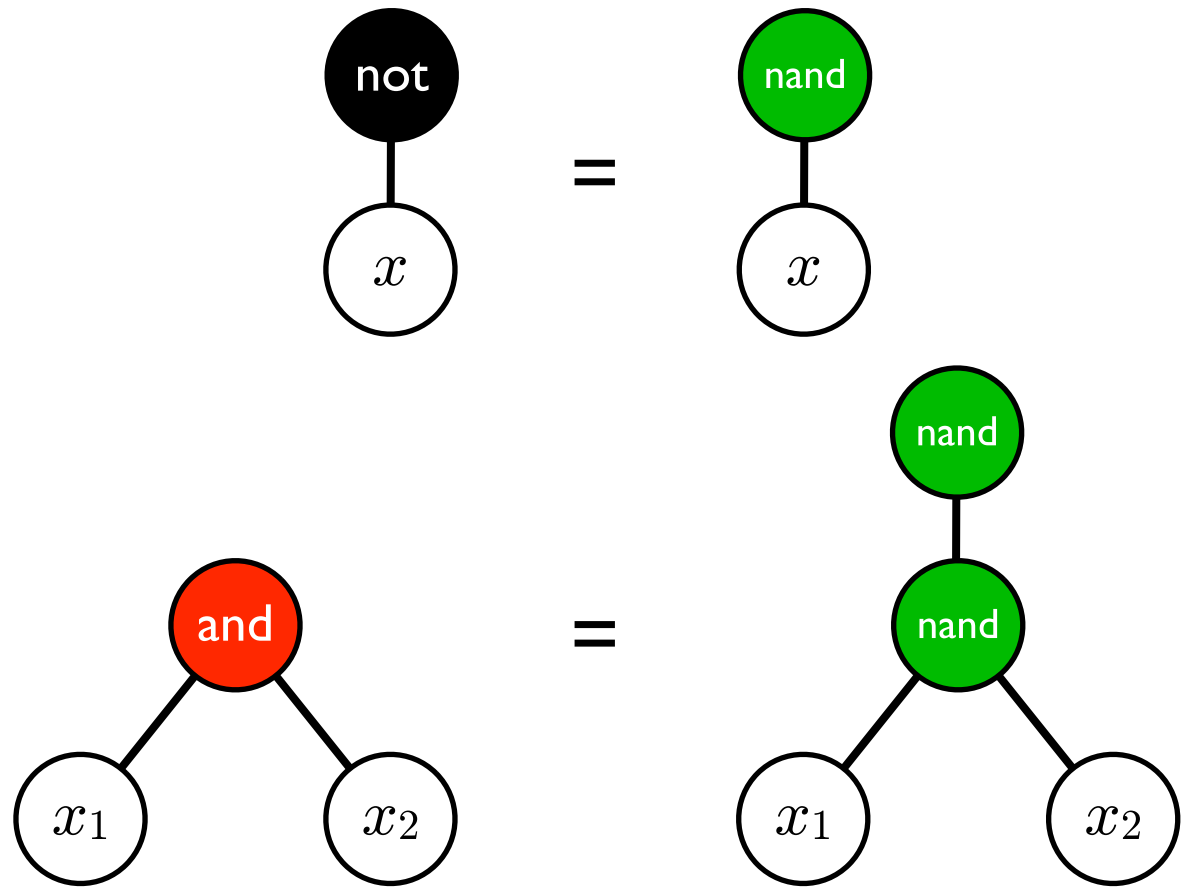

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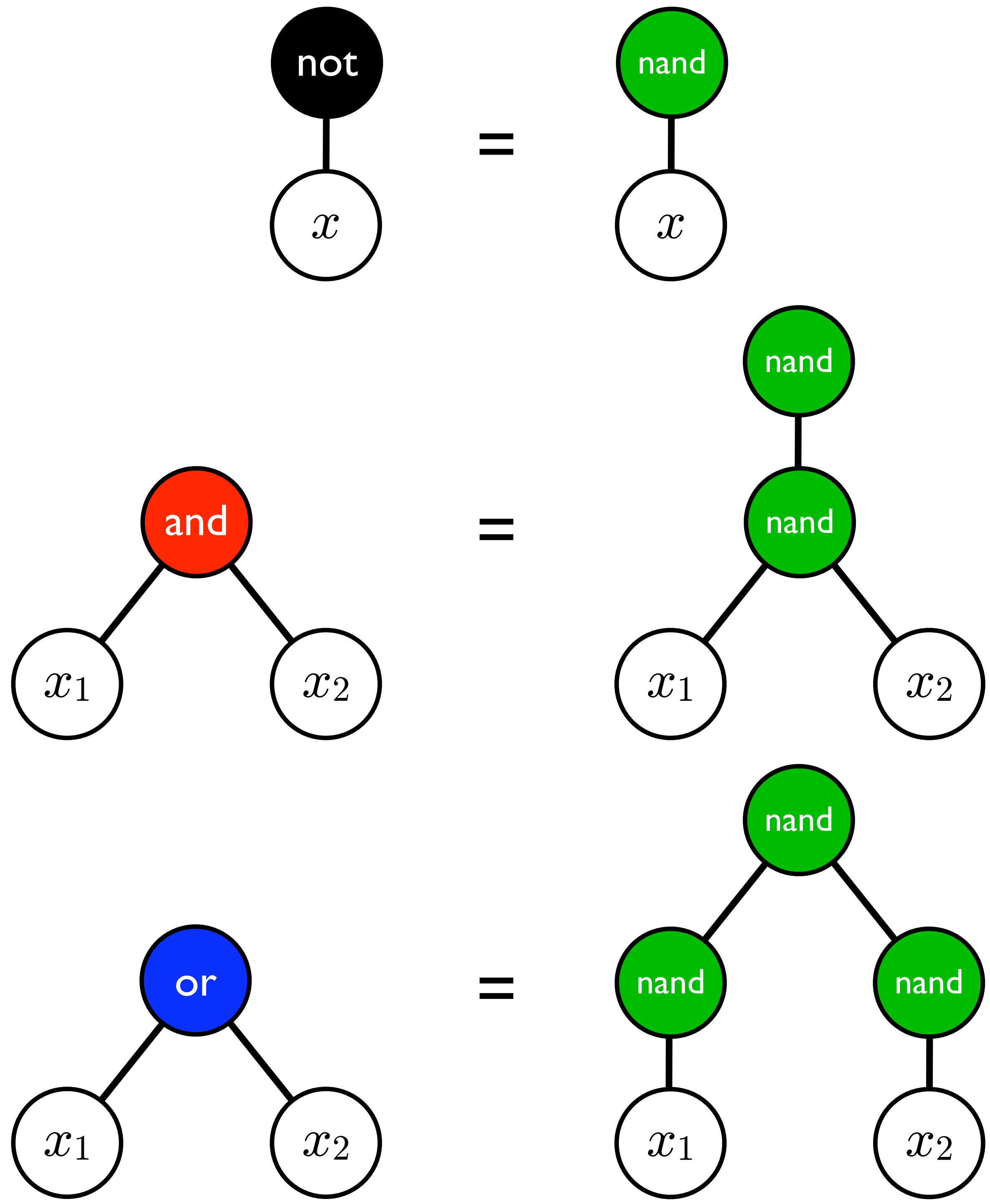

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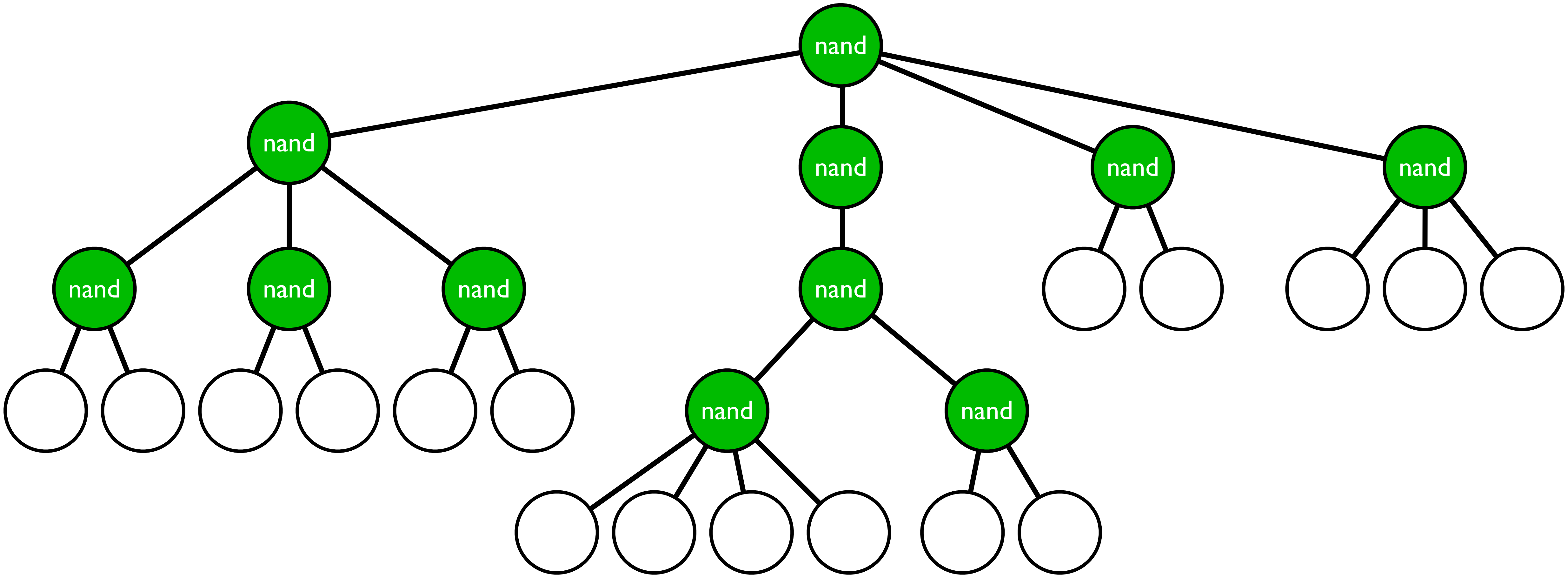

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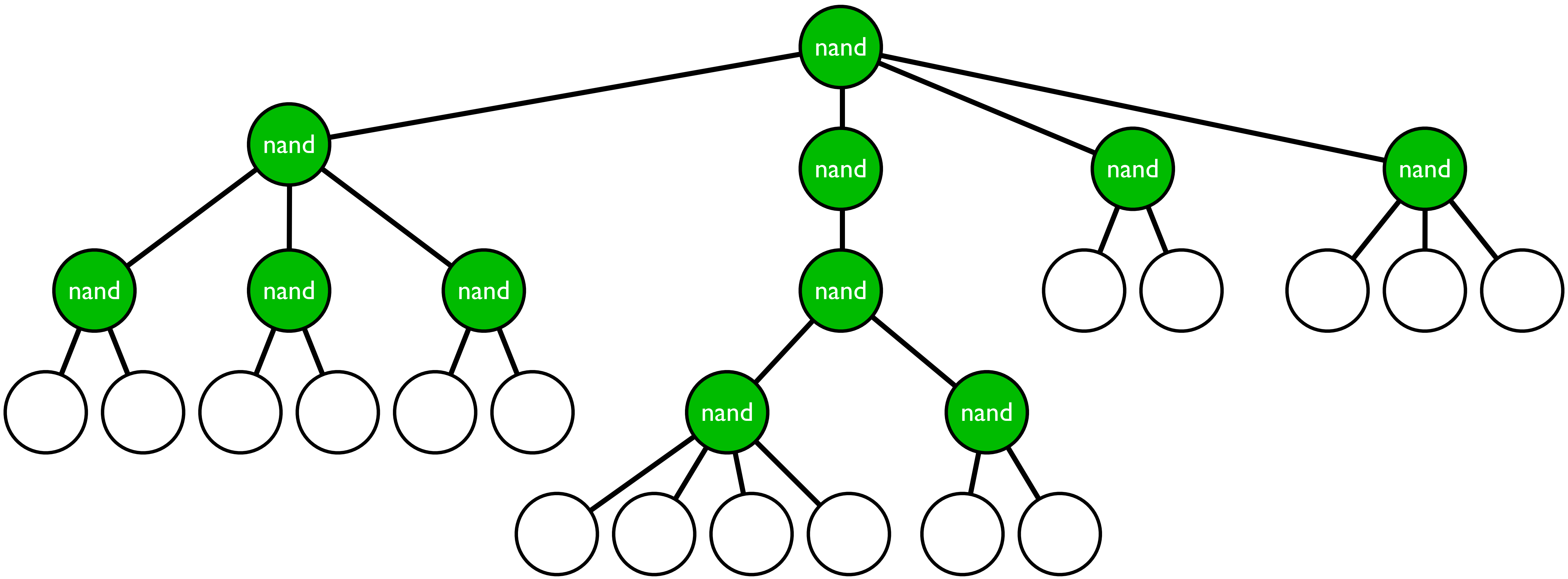

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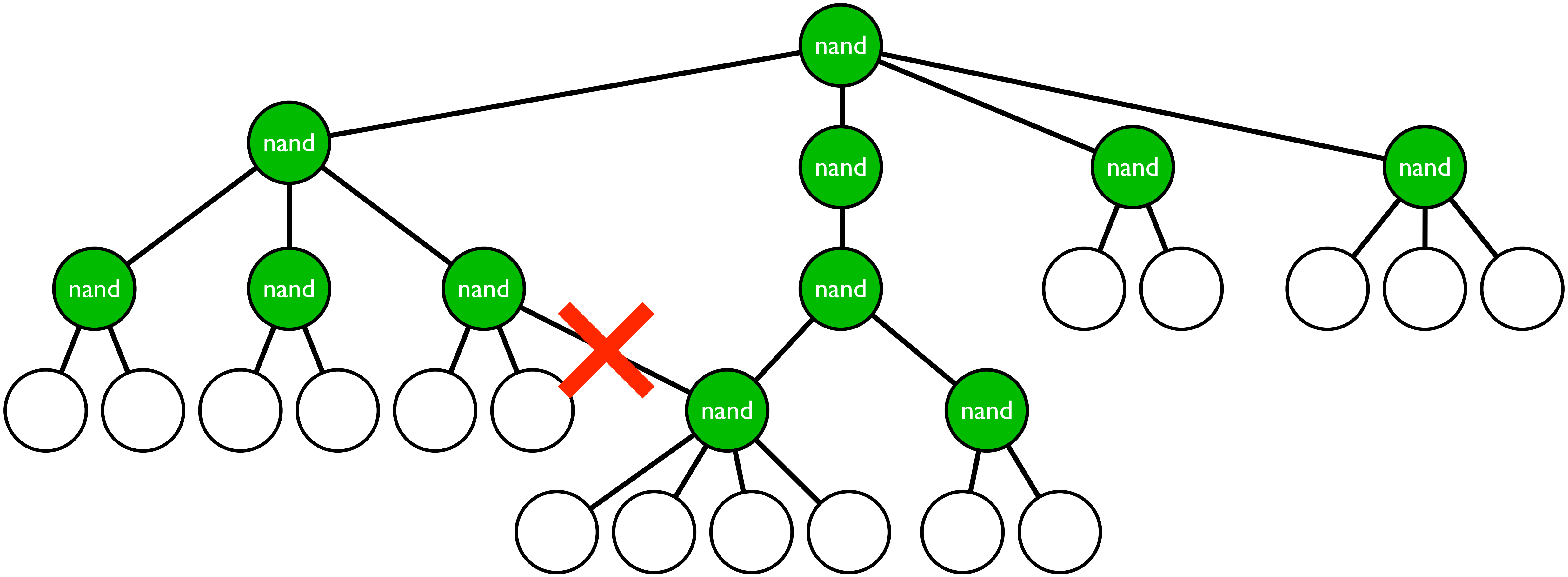

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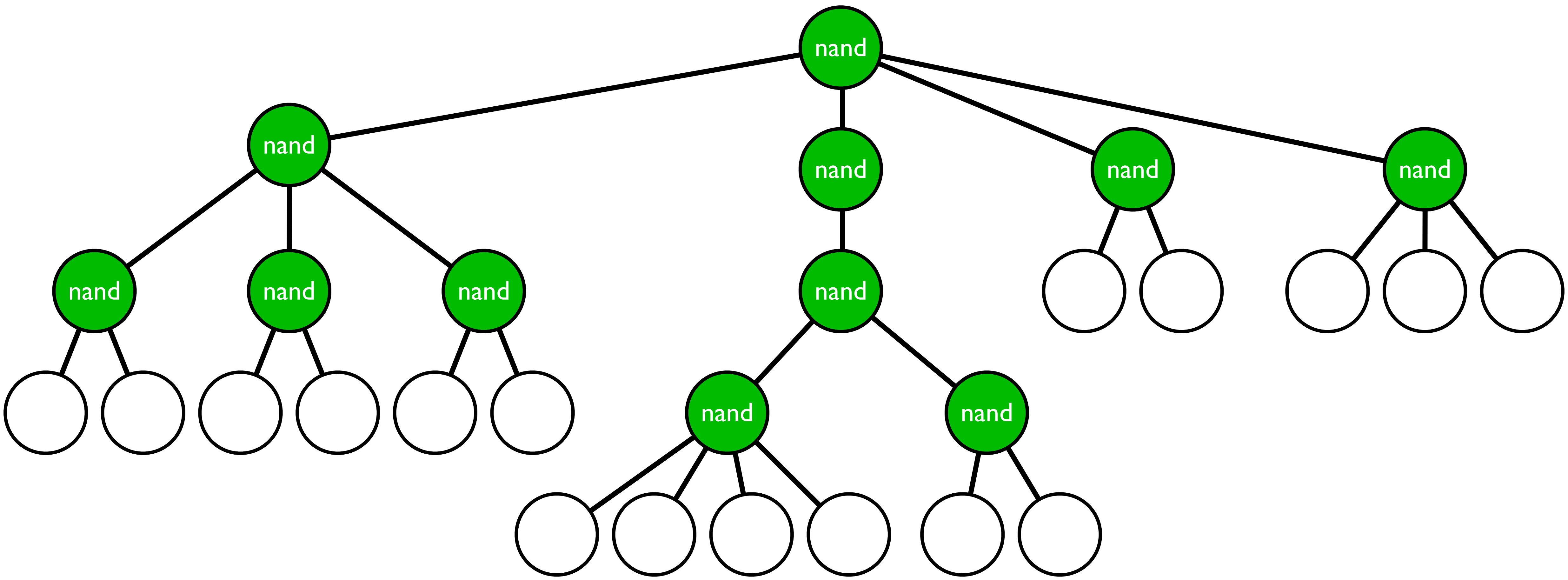

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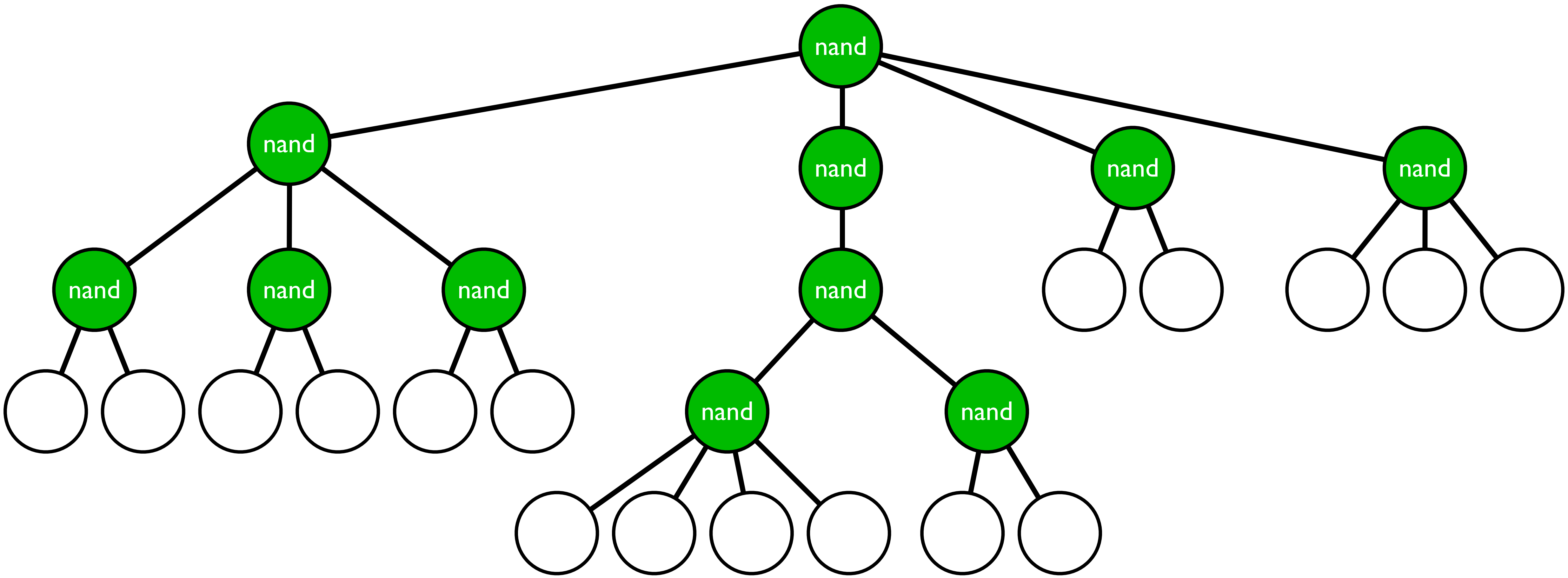

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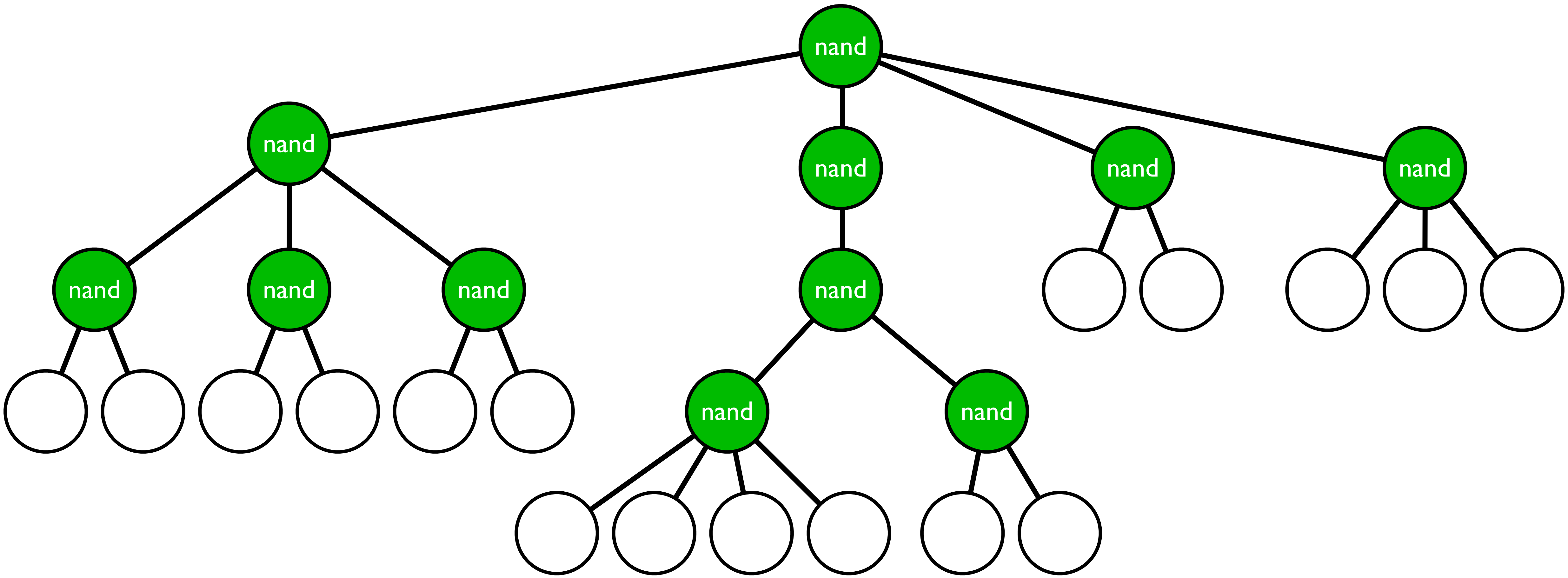

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Quantum lower bound [Barnum-Saks 02]: $\Omega(\sqrt{n})$

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This talk:

- $O(\sqrt{n})$ query quantum algorithm for evaluating "approximately balanced" NAND formulas (optimal!)
- $\sqrt{n^{1+o(1)}}$ time (and query) quantum algorithm for evaluating arbitrary NAND formulas

The algorithm

1. Start at the root of the tree
2. Perform phase estimation with precision $\approx 1/\sqrt{n}$ on a discrete-time quantum walk on the tree
3. If the estimated phase is 0 or π, then output 1; otherwise output 0

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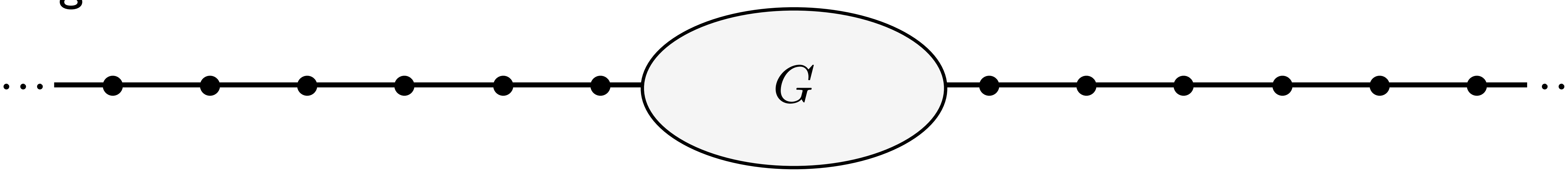
Outline

- Scattering → phase estimation
- Hamiltonian for a continuous-time quantum walk (with non-uniform edge weights)
- Low-energy eigenstates "compute NAND"
- Continuous time → discrete time (gives a small speedup)
- Formula rebalancing

From scattering to phase estimation

To do scattering calculations, we compute a complete basis of eigenstates:



Left:	$e^{ikx} + R(k)\, e^{-ikx}$	$T(k)\, e^{ikx}$
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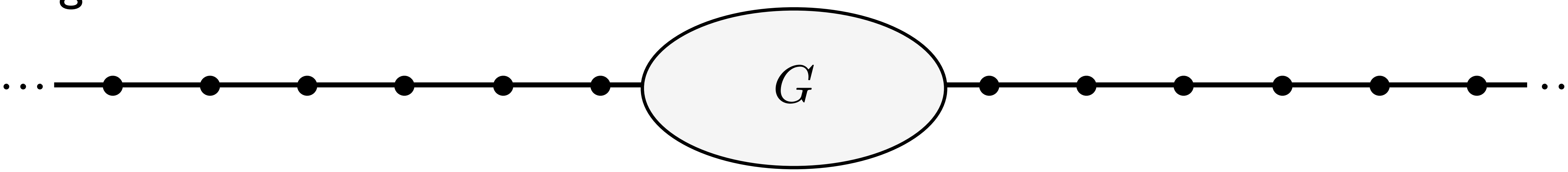

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Instead, we can just look at eigenstates of the graph itself.

Phase estimation: Given U and an eigenstate $|\varphi\rangle$ with $U|\varphi\rangle = e^{i\varphi}|\varphi\rangle$, we can estimate φ to precision δ in $O(1/\delta)$ steps.
(Equivalent to measuring $H = i \log U$.)

The Hamiltonian

Graph: Tree representing the NAND formula, with edges added to 1 inputs (so that all leaves evaluate to 0).

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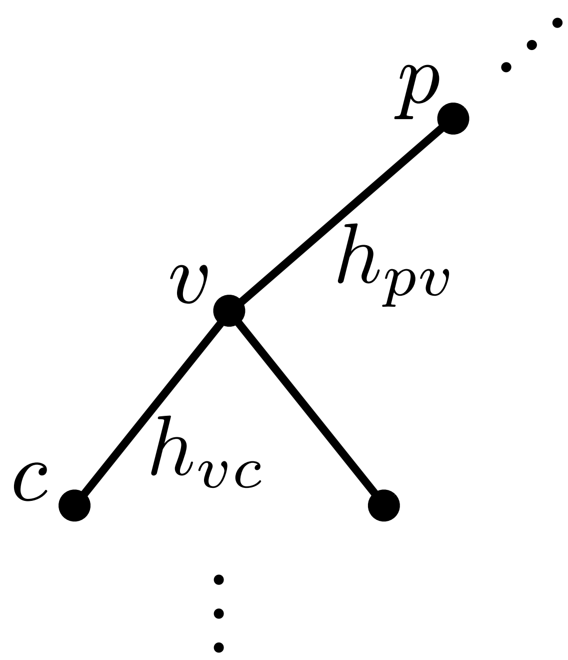
$$H|v\rangle = h_{pv}|p\rangle + \sum_c h_{vc}|c\rangle$$



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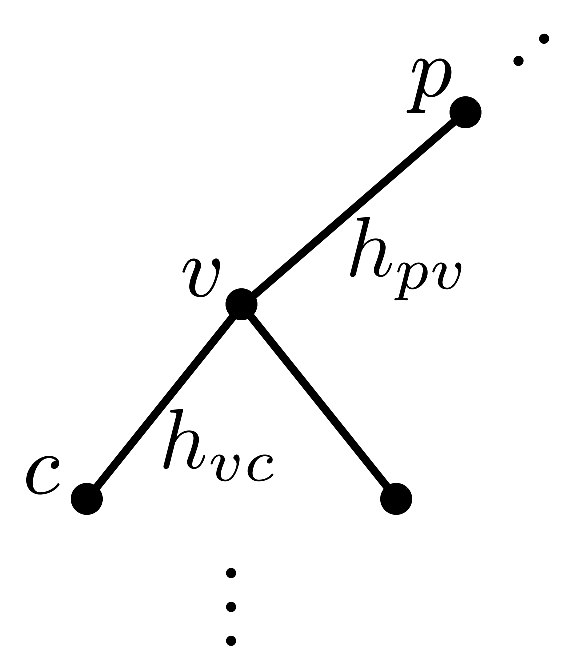

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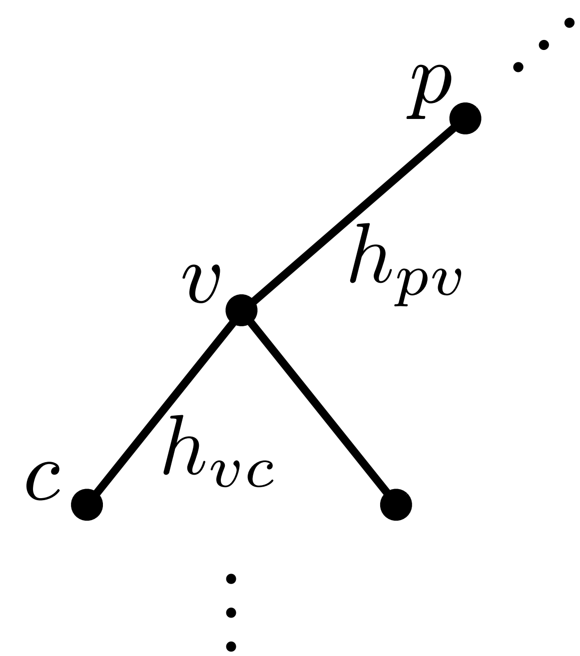

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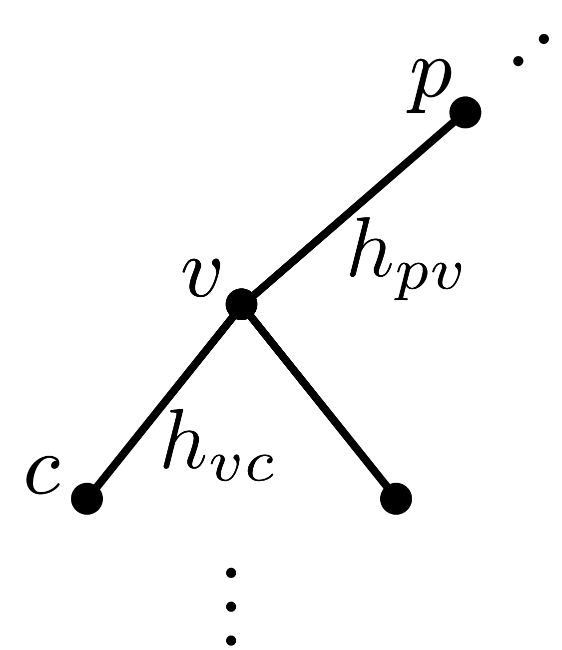

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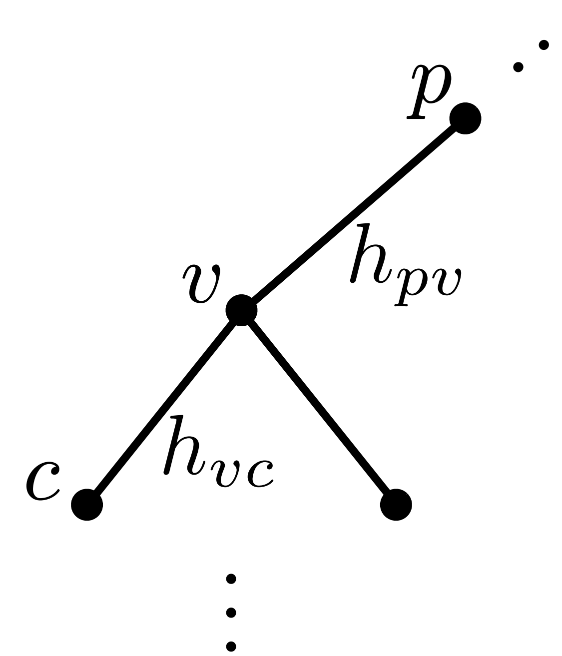

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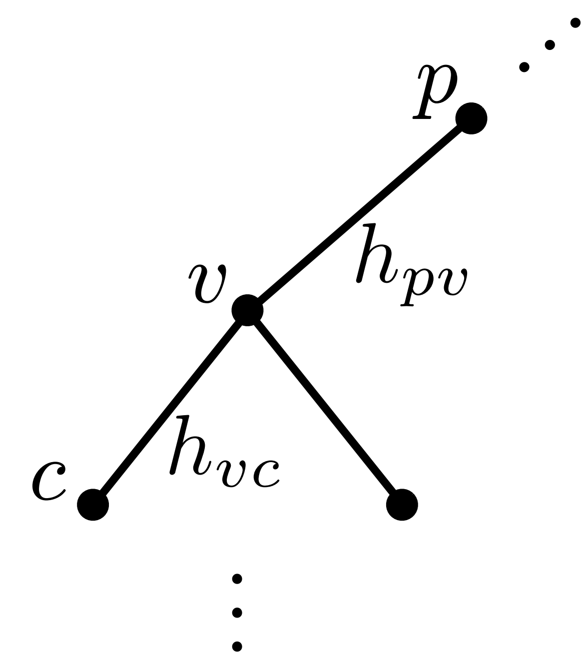

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For $E = 0$: $\quad \langle p|\Psi\rangle = -\sum_c \frac{h_{vc}}{h_{pv}}\langle c|\Psi\rangle$

Zero-energy eigenstates evaluate NAND: Qualitative version

Let $\mathrm{NAND}(p)$ denote the value of the NAND subformula under p.

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Theorem.

If $\mathrm{NAND}(p) = 1$, then $\langle p|\Psi\rangle = 0$ for any $|\Psi\rangle$ with $H|\Psi\rangle = 0$.

If $\mathrm{NAND}(r) = 0$, then $|\langle r|\Psi\rangle| > 0$ for some $|\Psi\rangle$ with $H|\Psi\rangle = 0$.

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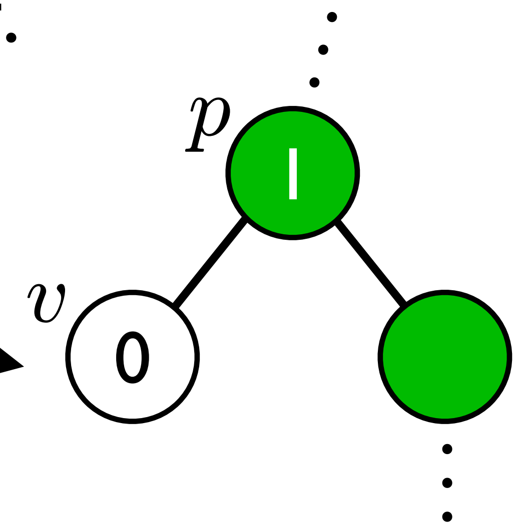

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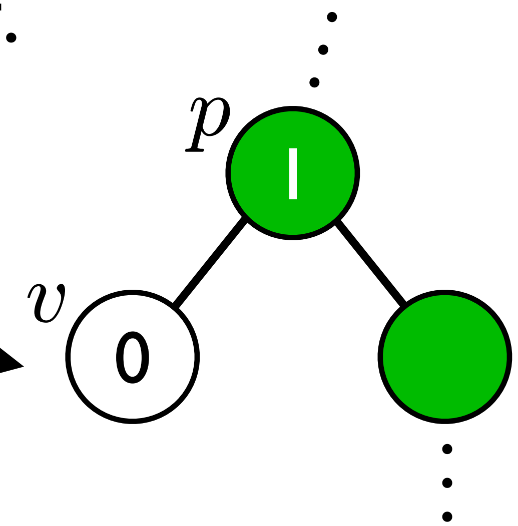

Induction: Some child v of p has $\mathrm{NAND}(v) = 0$;
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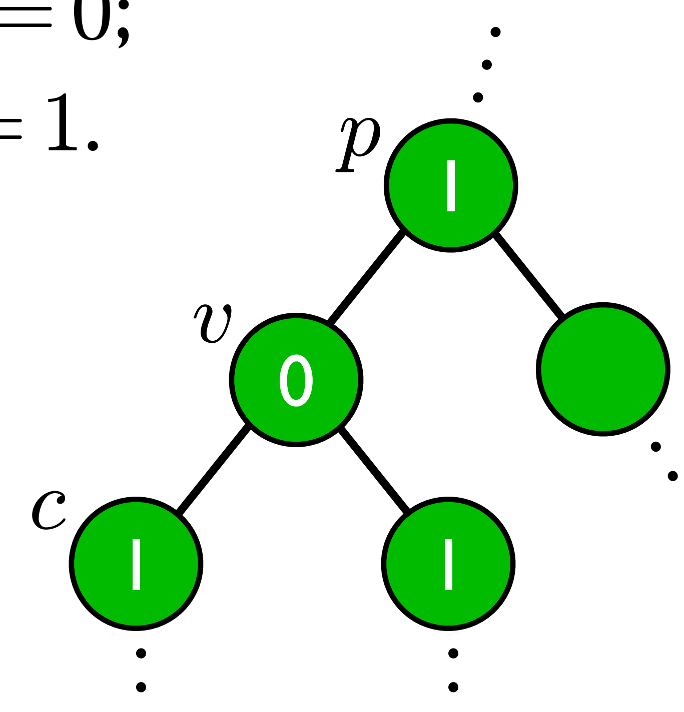

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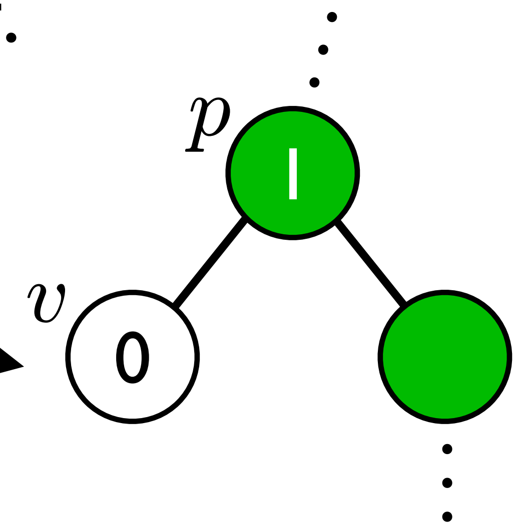

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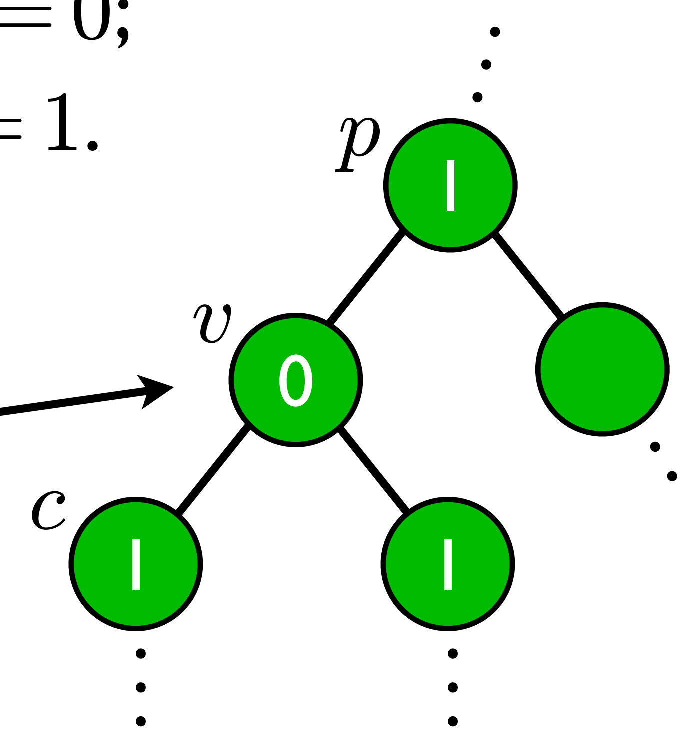

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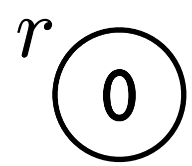

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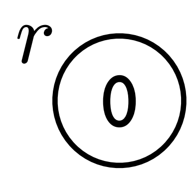

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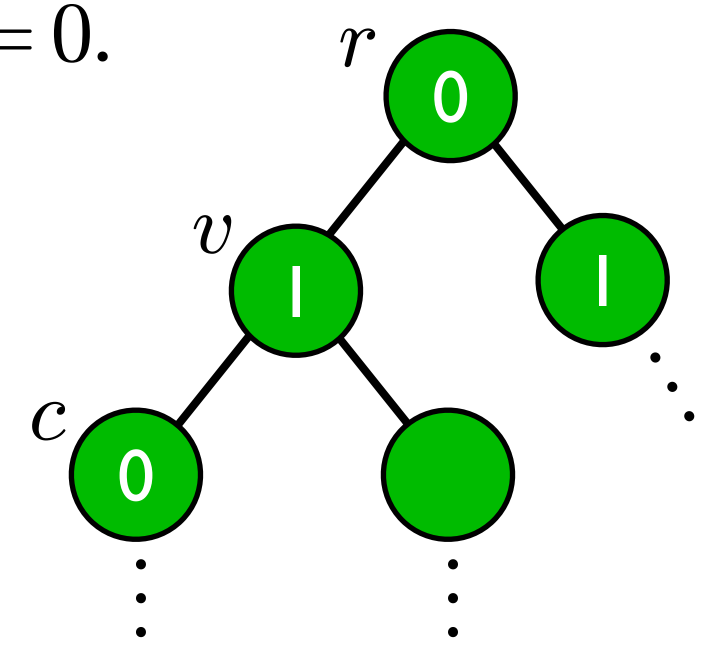

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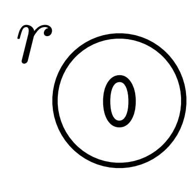

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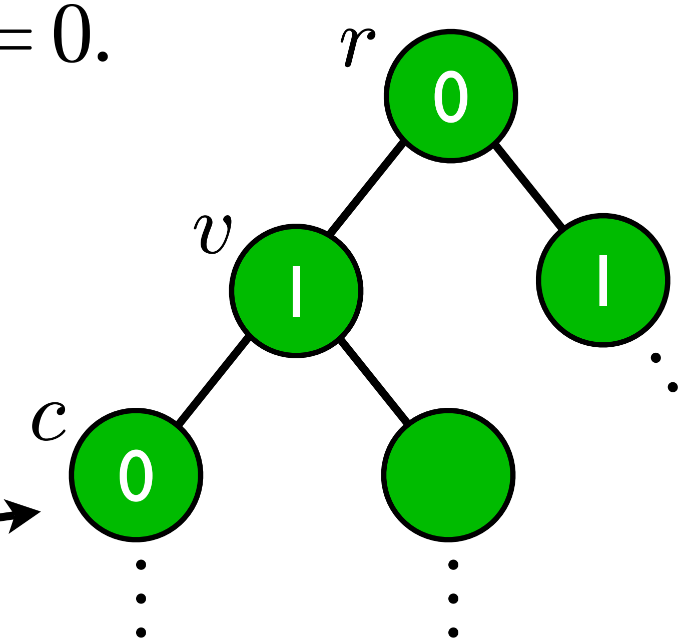

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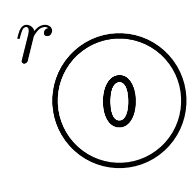

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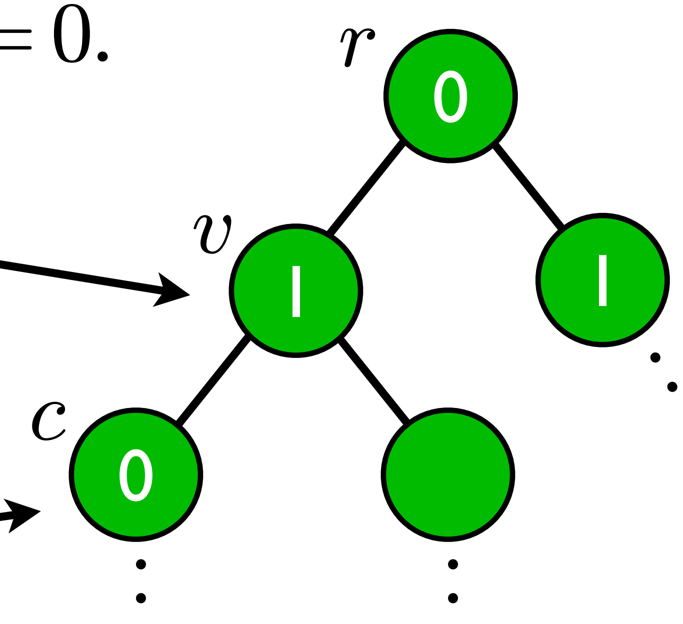

Zero-energy eigenstates evaluate NAND: Quantitative version

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Theorem (quantitative). For approximately balanced formulas:

If $\mathrm{NAND}(r) = 1$, then eigenstates $|E\rangle$ with $|E| < O(\frac{1}{\sqrt{n}})$ have $\langle r|E\rangle = 0$.

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Simulating the quantum walk

[C., Cleve, Jordan, Yeung 07]

We could perform phase estimation directly on the dynamics of this Hamiltonian (i.e., measure the energy).

But this would require *simulating* the dynamics by a sequence of quantum gates, using the black box to simulate the walk near the leaves, and combining that simulation with the input-independent part.

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$$

Instead, we can avoid the $o(1)$ by using a discrete-time quantum walk.

Continuous time → discrete time

Szegedy quantization of classical Markov chains:

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Classical random walk

Stochastic matrix P

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Claim: Any symmetric matrix H with positive entries can be factorized as $H = \sqrt{P \circ P^T}$ for some stochastic matrix P. (use Perron vector)

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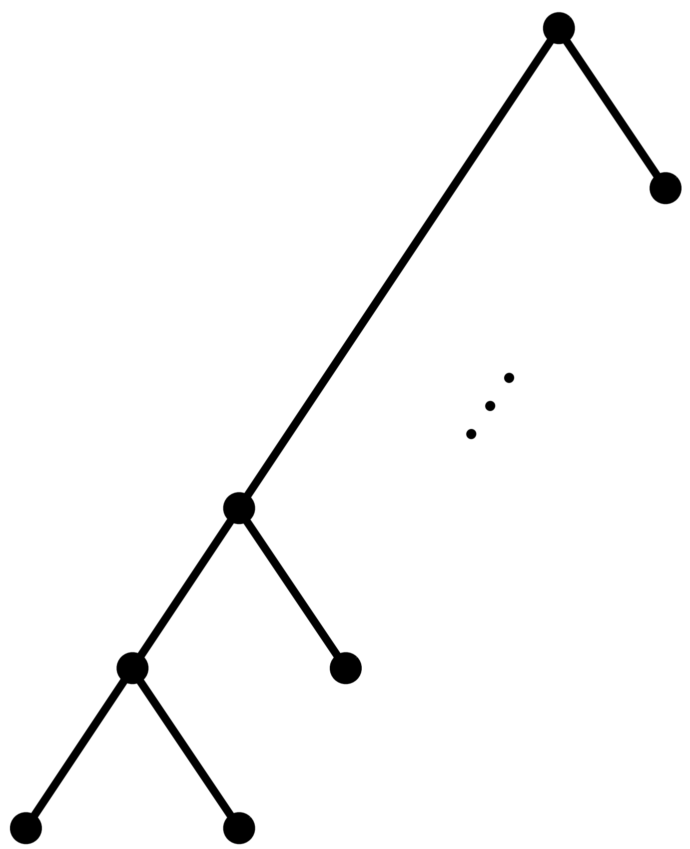
This gives a general way to relate continuous- and discrete-time quantum walk. Small eigenphases of e^{-iH} and U are equal up to third order.

Formula rebalancing

A quantum walk algorithm clearly cannot work for highly unbalanced formulas:



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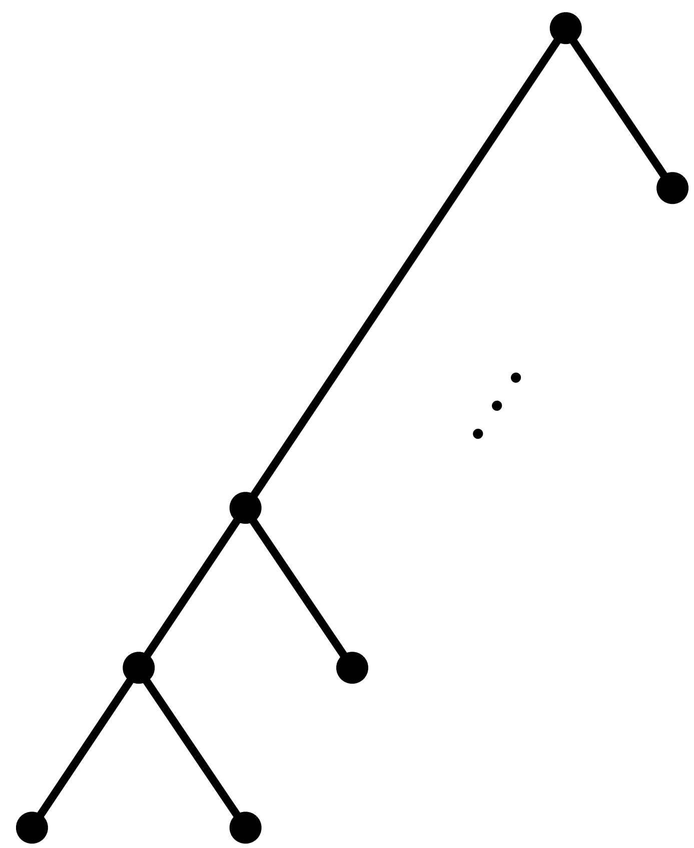

But we can apply

Theorem [Bshouty, Cleve, Eberly 91]: Any NAND formula of size n can be rewritten as an equivalent NAND formula of depth $O(\log n)$ and size $n^{1+o(1)}$.

Applications to recursive functions

Recursive "all equal" function [Ambainis 03]

$$f(x,y,z) = \begin{cases} 1 & x = y = z \\ 0 & \text{otherwise} \end{cases}$$

recurse k times

Polynomial degree: 2^k

Q. query complexity: $\Omega((\frac{3}{\sqrt{2}})^k) = \Omega(2.12^k)$ (adversary method)

$O(\sqrt{6^k}) = O(2.45^k)$ (NAND of 6)

Recursive majority function [Boppana 86]

$$f(x,y,z) = \begin{cases} 1 & x + y + z \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

recurse k times

C. query complexity [JKS 03]: $\Omega((\frac{7}{3})^k) = \Omega(2.33^k)$

$o((\frac{8}{3})^k) = o(2.67^k)$

Q. query complexity: $\Omega(2^k)$ (adversary method)

$O(\sqrt{5^k}) = O(2.24^k)$ (NAND of 5)

Closed problems

This also resolves a conjecture of [O'Donnell-Servedio 03]:

Any NAND formula of size n can be approximated by a polynomial of degree $\sqrt{n^{1+o(1)}}$.

Hence formulas are (classically!) PAC learnable in time $2^{\sqrt{n^{1+o(1)}}}$.

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[Reichardt, Špalek 07]: Generalization to formulas built from other gates, using new gate widgets, as well as the concept of *span programs*.

Gives optimal (or nearly optimal) algorithms for many other functions, including an optimal algorithm ($O(2^k)$) for recursive ternary majority.

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Any NAND formula of size n can be approximated by a polynomial of degree $\sqrt{n^{1+o(1)}}$.

Hence formulas are (classically!) PAC learnable in time $2^{\sqrt{n^{1+o(1)}}}$.

[Reichardt, Špalek 07]: Generalization to formulas built from other gates, using new gate widgets, as well as the concept of *span programs*.

Gives optimal (or nearly optimal) algorithms for many other functions, including an optimal algorithm ($O(2^k)$) for recursive ternary majority.

Open problems

- Formulas with yet more general gates?
- Similar algorithm for *circuits*?
- Can we compute a certificate for the value of a formula?
- Improved formula rebalancing?