Factoring integers with a quantum computer

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The fundamental theorem of arithmetic

Every positive integer larger than 1 can be uniquely* factored as a product of prime numbers.

$$N = 2^{n_2} \times 3^{n_3} \times 5^{n_5} \times \cdots$$

* Up to the order of the factors.

Examples

$$15 =$$

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$$15 = 3 \times 5$$

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$$91 =$$

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$$91 = 7 \times 13$$

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$$\begin{array}{l}
3107418240490043721350750 \\
0358885679300373460228427 \\
2754572016194882320644051 \\
8081504556346829671723286 \\
7824379162728380334154710 \\
7310850191954852900733772 \\
4822783525742386454014691 \\
7366024776523466 09
\end{array} =$$

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=
\begin{array}{l}
1634733645809253848 4 \\
4313388386509085984 1 \\
7836700330923121811 1 \\
0852389333100104508 1 \\
5121211816751157 9 \\
\times \\
1900871281664822113 1 \\
2685157393541397547 1 \\
8967899685154936666 3 \\
8539088027103802104 4 \\
989571912614655 71
\end{array}$$

"The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic. It has engaged the industry and wisdom of ancient and modern geometers to such an extent that it would be superfluous to discuss the problem at length... Further, the dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated."

– Carl Friedrich Gauss, *Disquisitiones Arithmeticæ* (1801)

RSA public-key cryptosystem

M

message

Rivest, Shamir, Adleman 1977

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primes p, q

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$n = pq$

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$$n = pq$$

$$e \in \mathbb{Z}^{\times}_{(p-1)(q-1)}$$

encryption key

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$C := M^e \bmod n$

cyphertext

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cyphertext | C | $C^d = M^{ed} \bmod n$

Rivest, Shamir, Adleman 1977

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cyphertext

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$C^d = M^{ed} \bmod n$
$= M$

Rivest, Shamir, Adleman 1977

Outline

- Integers
- The quantum Fourier transform
- Period finding over $\mathbb{Z}_N$
- Period finding over $\mathbb{Z}$
- Reduction of factoring to period finding
- Beyond factoring

$$\mathbb{Z}$$

All else is the work of man

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

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$$\begin{aligned}
\mathbb{Z}_N = \{ \quad & \bar{0} := \{\ldots, -N, 0, N, 2N, \ldots\} \\
& \bar{1} := \{\ldots, -N+1, 1, N+1, 2N+1, \ldots\} \\
& \bar{2} := \{\ldots, -N+2, 2, N+2, 2N+2, \ldots\} \\
& \quad \vdots \\
& \overline{N-1} := \{\ldots, -N-1, -1, N-1, 2N-1, \ldots\}\}
\end{aligned}$$

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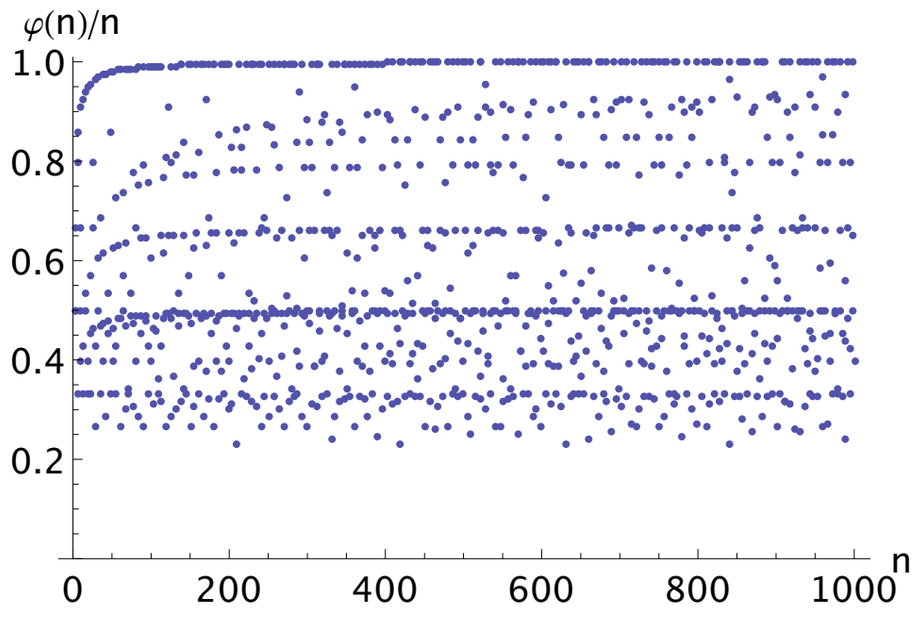
Fact: $$\frac{\varphi(N)}{N} = \Omega\left(\frac{1}{\log\log N}\right)$$



The quantum Fourier transform

Quantum Fourier transform over $\mathbb{Z}_N$

$$\omega_N := e^{2\pi \mathrm{i}/N}$$

$$|x\rangle \overset{F_{\mathbb{Z}_N}}{\longmapsto} \frac{1}{\sqrt{N}} \sum_{y \in \mathbb{Z}_N} \omega_N^{xy} |y\rangle$$

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$$= \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2N-2} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix}$$

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$$\cong F_{\mathbb{Z}_2} \otimes F_{\mathbb{Z}_3}$$

by the isomorphism $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$

QFT implementation ($N = 2^n$)

Write $x = \sum_{j=0}^{n-1} 2^j x_j,\ y = \sum_{j=0}^{n-1} 2^j y_j.$

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QFT circuit

We have $F_{\mathbb{Z}_{2^n}} |x\rangle = \bigotimes_{j=0}^{n-1} |z_j\rangle$ where

$$|z_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi \mathrm{i}(2^{j-n}x_0 + 2^{j+1-n}x_1 + \cdots + 2^{-1}x_{n-1-j})}|1\rangle)$$

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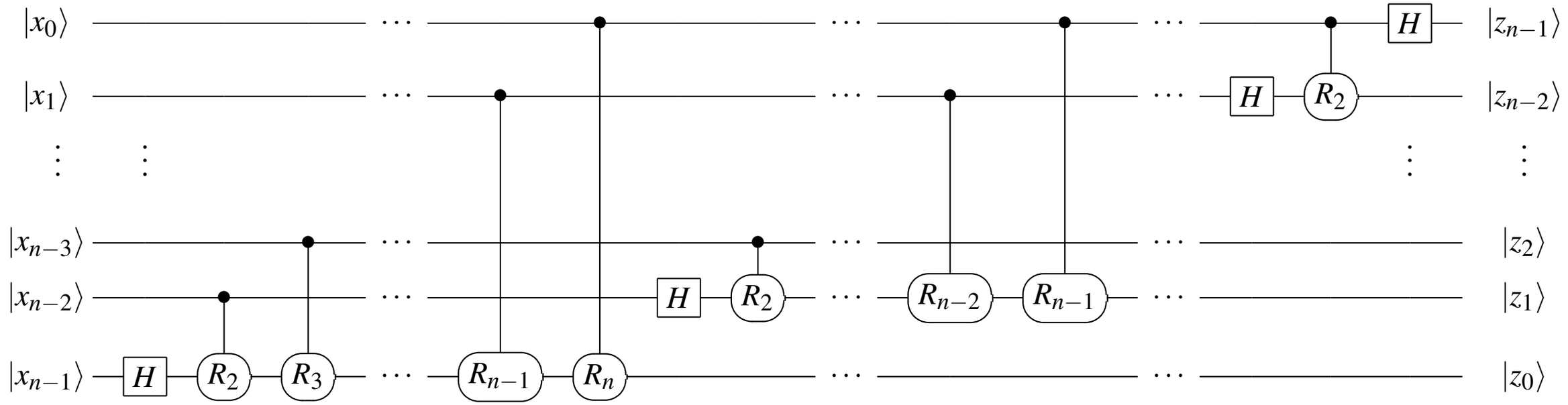
Quantum circuit:



$$R_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi \mathrm{i}/2^r} \end{pmatrix}$$

note reversal of order

More general QFTs

There are efficient quantum circuits for $F_{\mathbb{Z}_N}$ for general N; see

- Kitaev, *Quantum measurements and the abelian stabilizer problem*, quant-ph/9511026
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We can also define a Fourier transform for a nonabelian group; many of these can be implemented efficiently as well.

Period finding over $\mathbb{Z}_N$

Periodic functions

A function $f : \mathbb{Z}_N \to S$ is *periodic with period* $r \in \mathbb{Z}_N$ if

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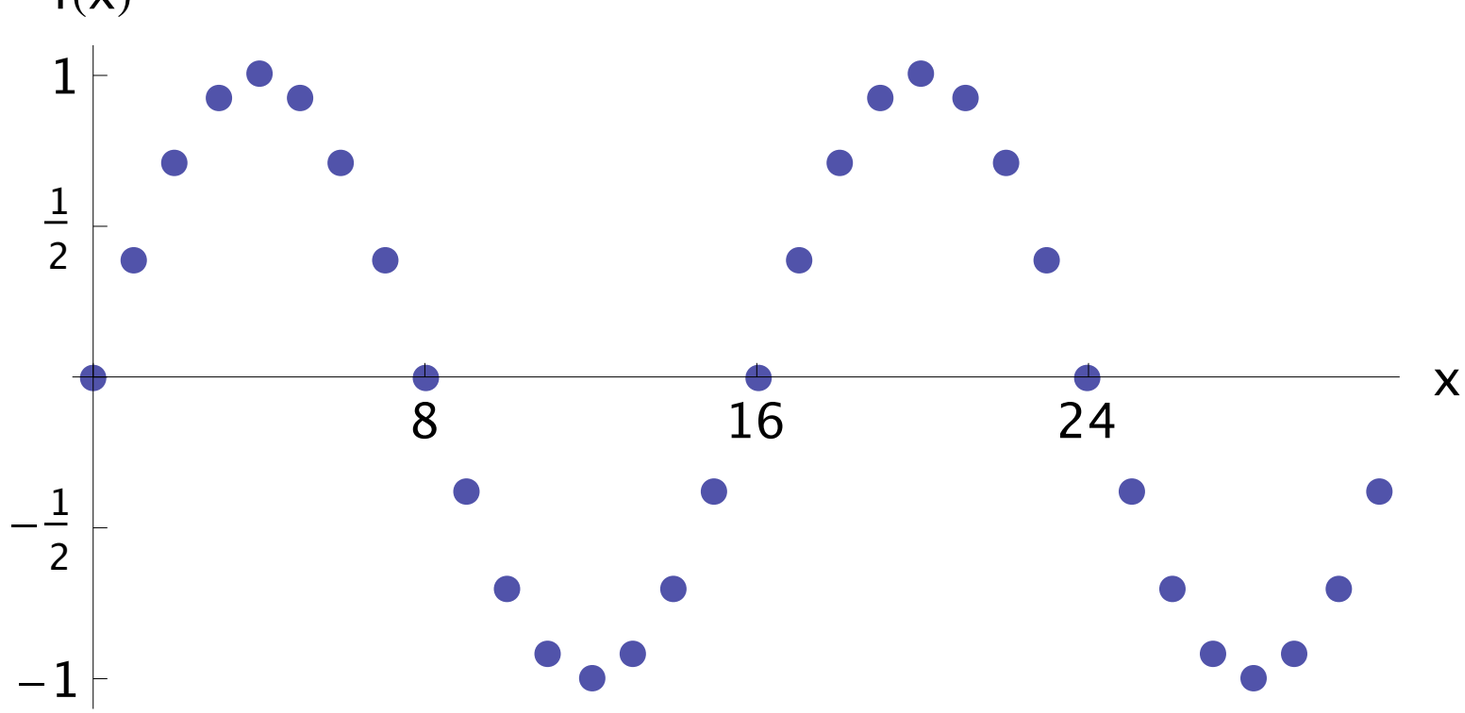

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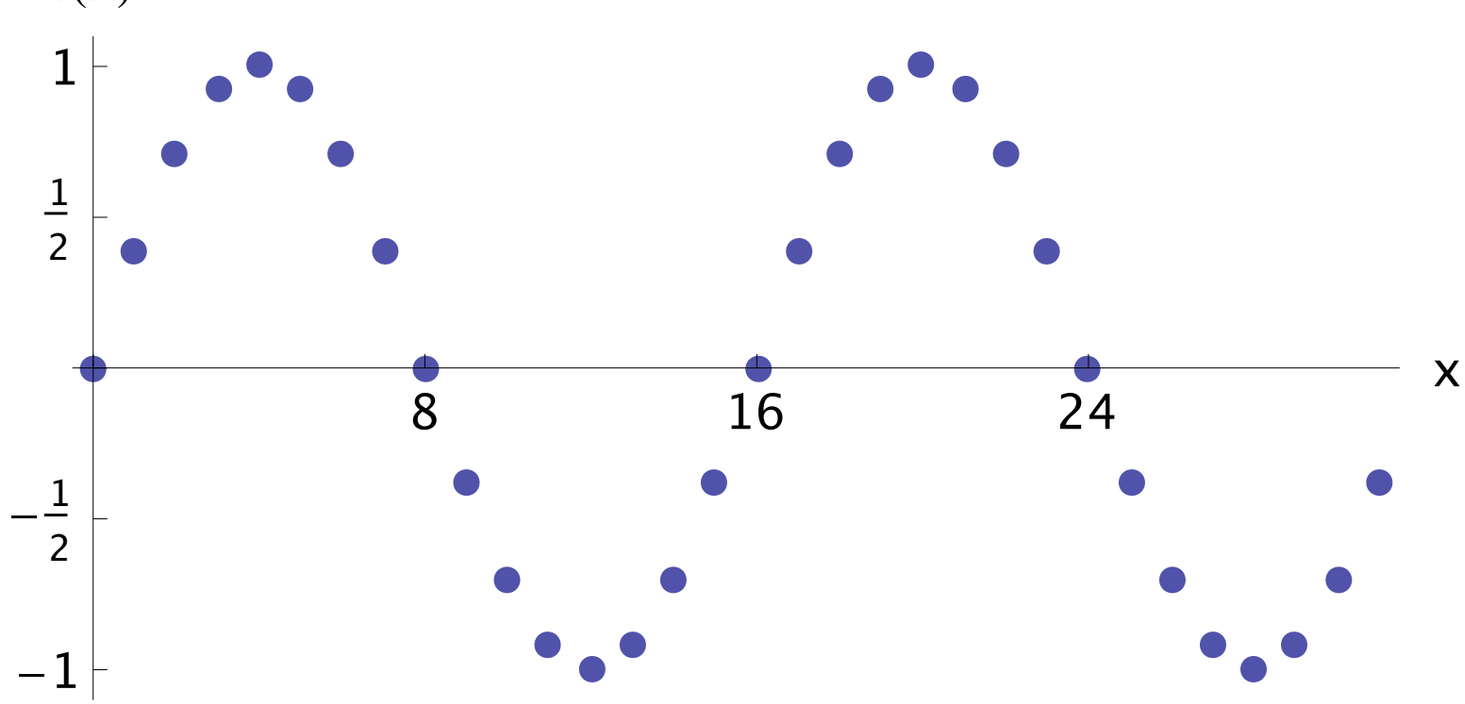

(Notice that r must divide N.)

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A function $f : \mathbb{Z}_N \to S$ is *injectively periodic with period* $r \in \mathbb{Z}_N$ if

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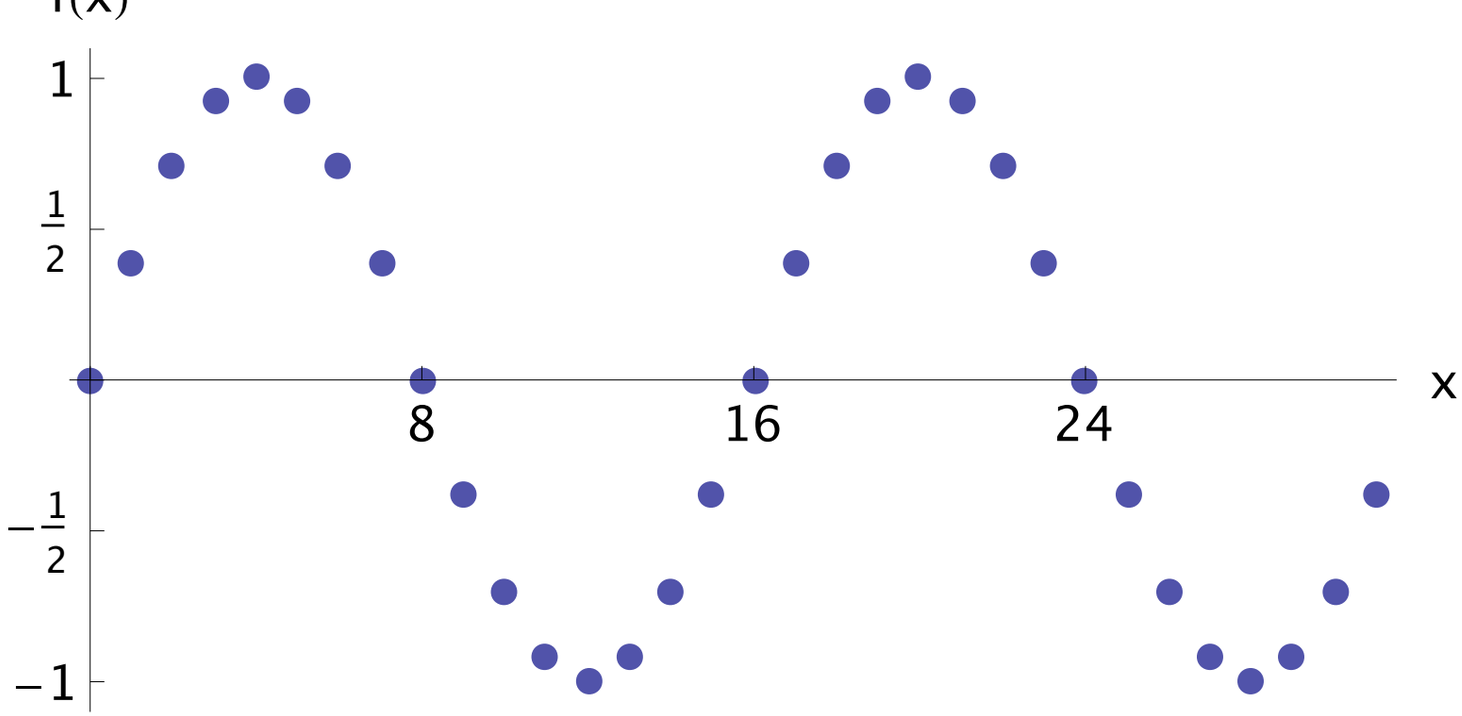

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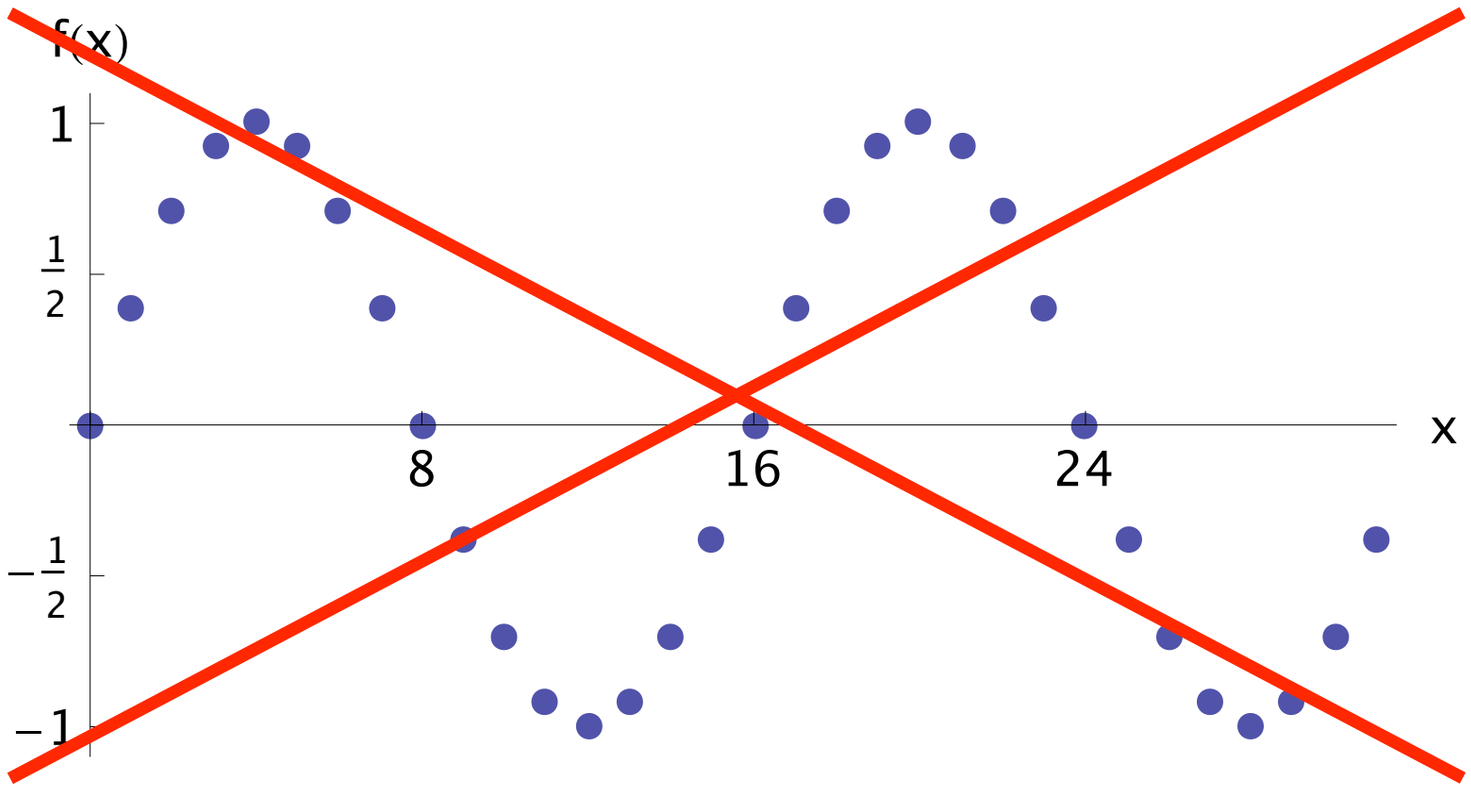
$$f(x) = f(y) \text{ if and only if } \frac{x-y}{r} \in \mathbb{Z}$$



periodic, but not *injectively* periodic

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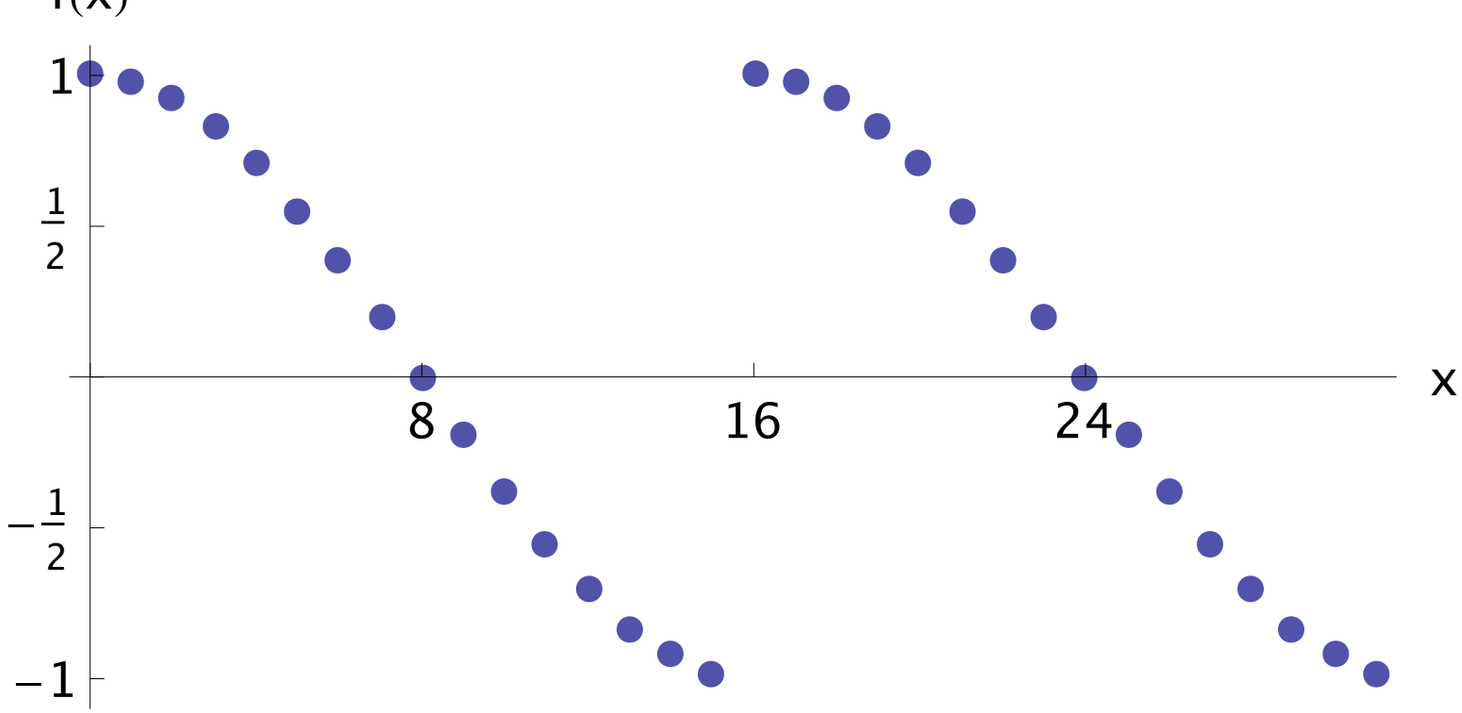

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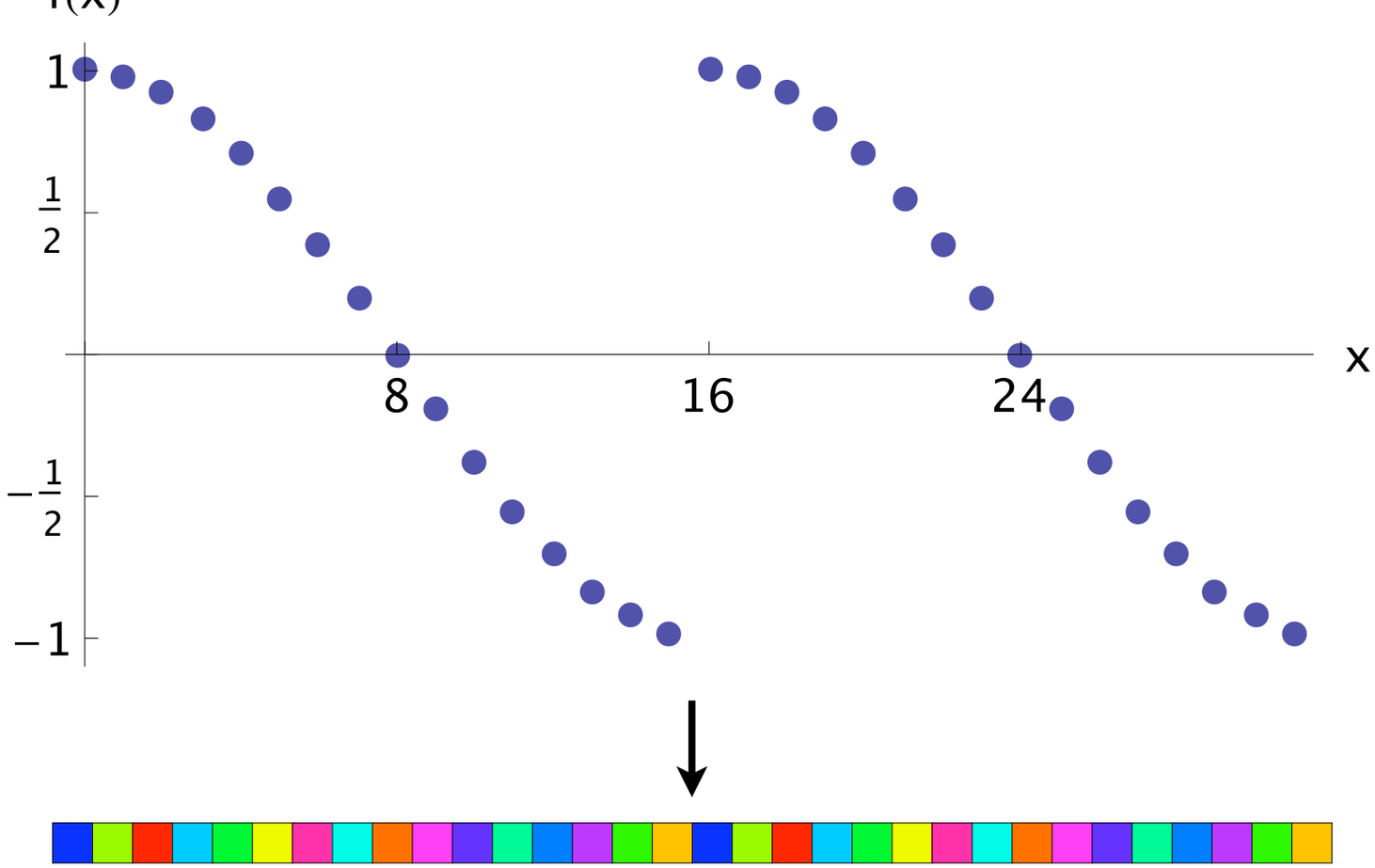

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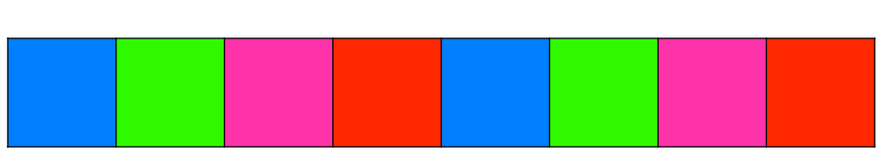

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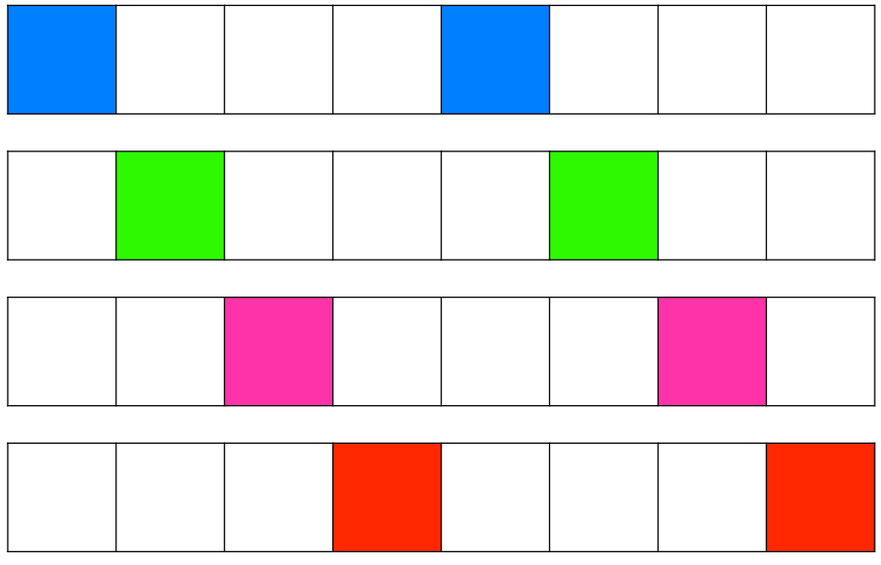

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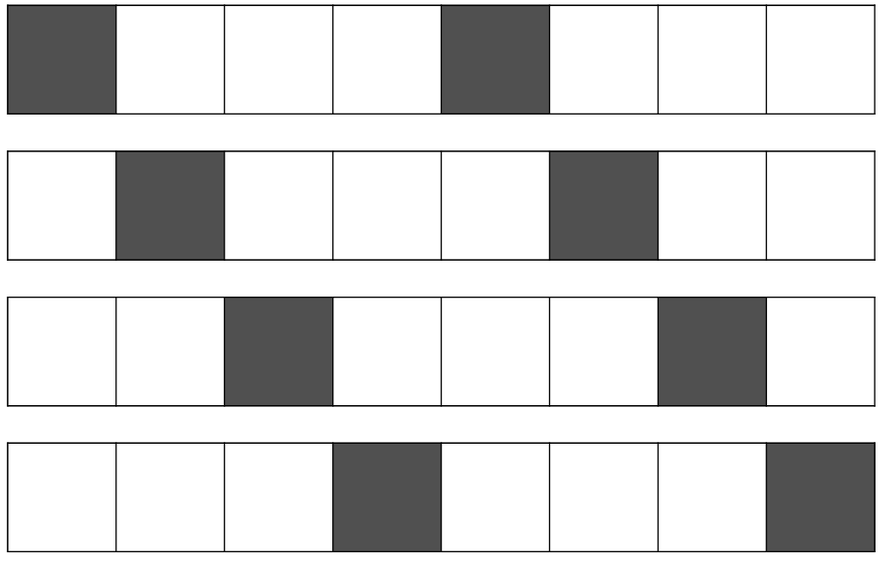

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Recall $$|x\rangle \overset{F_{\mathbb{Z}_N}}{\longmapsto} \frac{1}{\sqrt{N}} \sum_{y \in \mathbb{Z}_N} \omega_N^{xy} |y\rangle$$

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Claim: $$\frac{1}{M}\sum_{j=0}^{M-1}\omega_M^{jk} = \delta_{k \bmod M,0}$$

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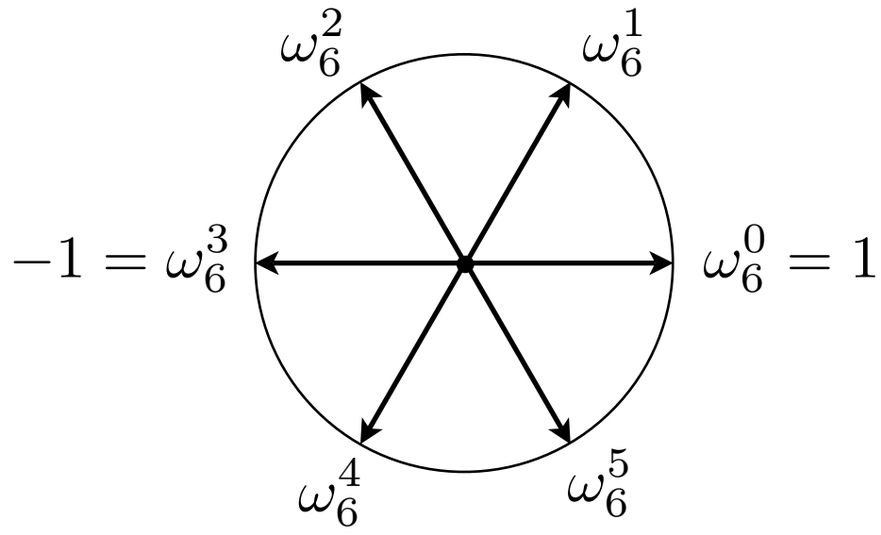
For $k \neq 0$,

$$\sum_{j=0}^{M-1}\omega_M^{jk} = \frac{1-\omega_M^{kM}}{1-\omega_M^k} \quad \text{(geometric series)}$$

$$= \frac{1-e^{2\pi \mathrm{i} k}}{1-\omega_M^k}$$

$$= 0$$



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Effect of the QFT

$$\sqrt{\frac{r}{N}} \sum_{j=0}^{\frac{N}{r}-1} |s + jr\rangle$$

periodic, period r

unknown offset s

$$\downarrow F_{\mathbb{Z}_N}$$

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Measure $\frac{1}{\sqrt{r}} \sum_{z \in \mathbb{Z}_r} \omega_{N/r}^{sz} |z \tfrac{N}{r}\rangle$

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Probability of success is

$$\frac{\varphi(r)}{r} = \Omega\left(\frac{1}{\log\log r}\right) = \Omega\left(\frac{1}{\log\log N}\right)$$

An equivalent formulation

Prepare $\frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}_N} |x, f(x)\rangle$ and discard the second register:

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$$\rho_r = \frac{r}{N^2} \sum_{s \in \mathbb{Z}_N} \sum_{j,k=0}^{\frac{N}{r}-1} |s + jr\rangle\langle s + kr| = \frac{1}{N} \sum_{s \in \mathbb{Z}_N} \sum_{j=0}^{\frac{N}{r}-1} |s + jr\rangle\langle s|$$

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Period finding over $\mathbb{Z}$

Unknown domain of periodicity

Suppose we have a function $f : \mathbb{Z} \to S$ satisfying

$$f(x) = f(y) \text{ if and only if } \frac{x-y}{r} \in \mathbb{Z}$$

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Strategy: Choose a large N, and hope that the procedure still works, even though r will probably not divide r.

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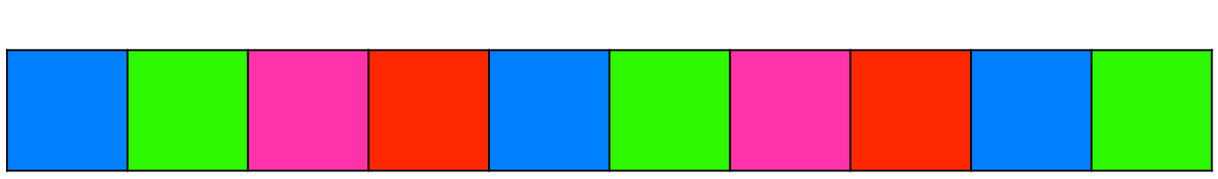

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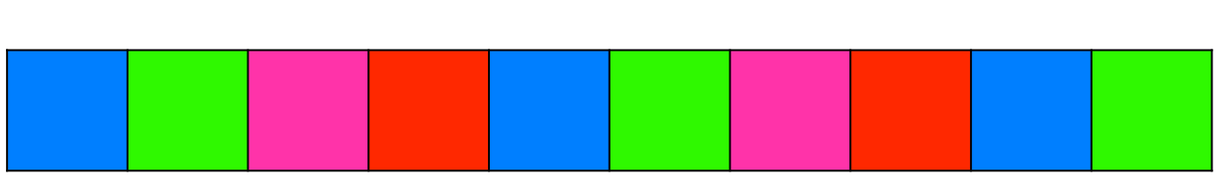

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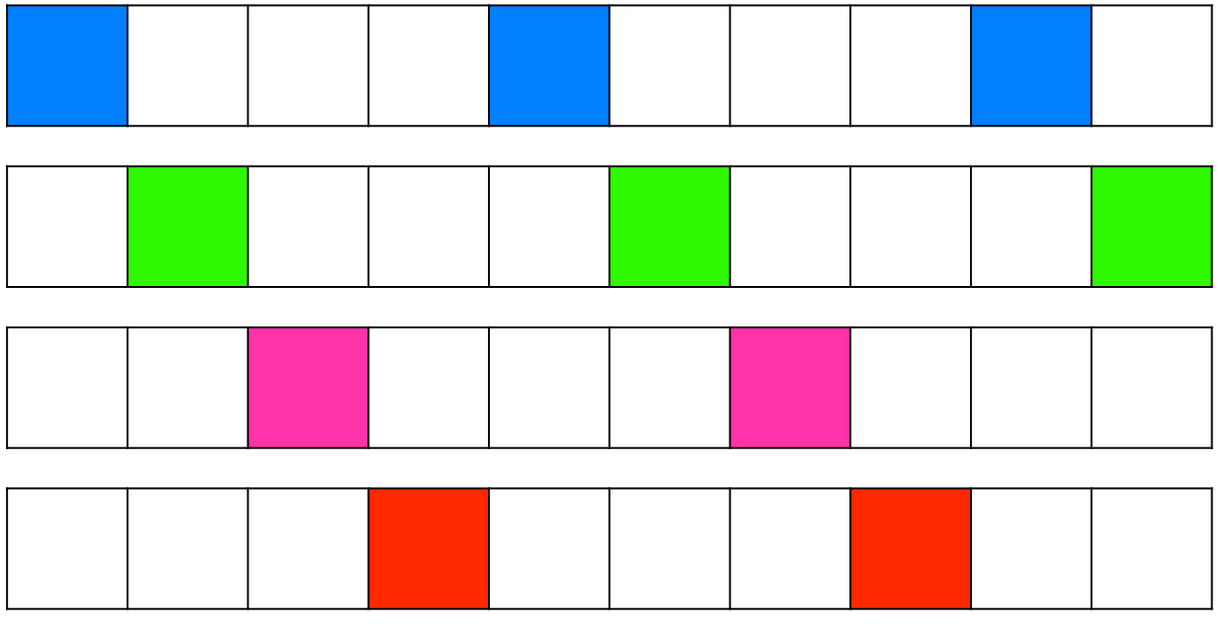

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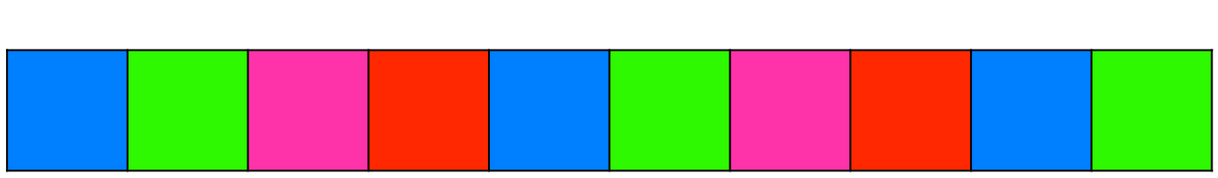

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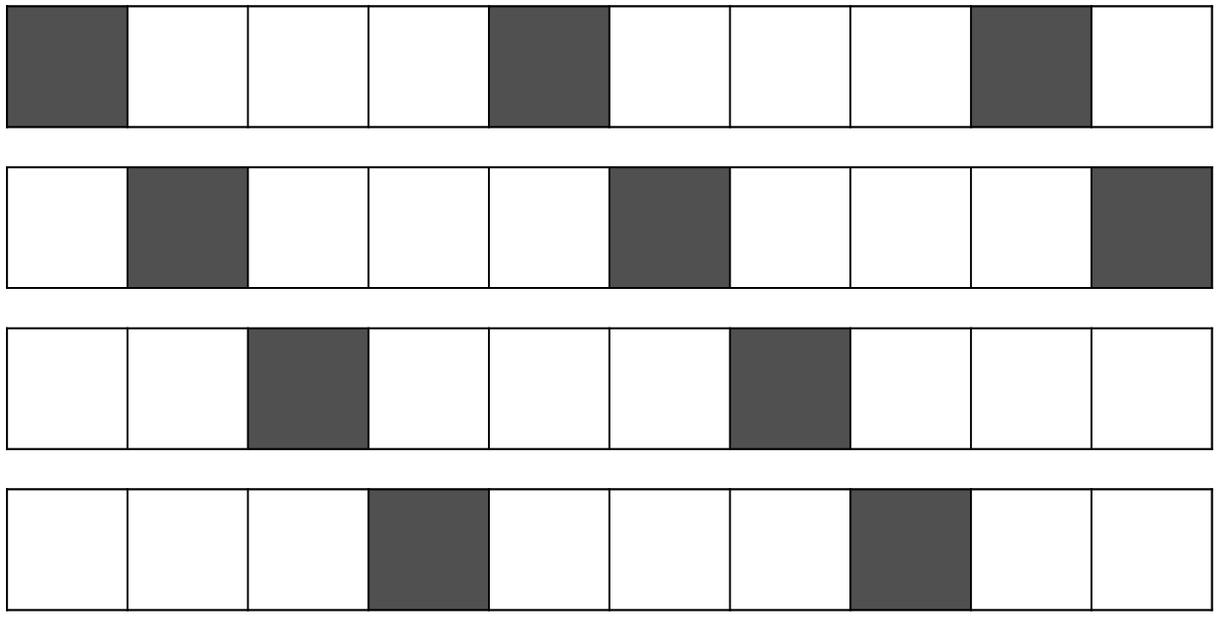

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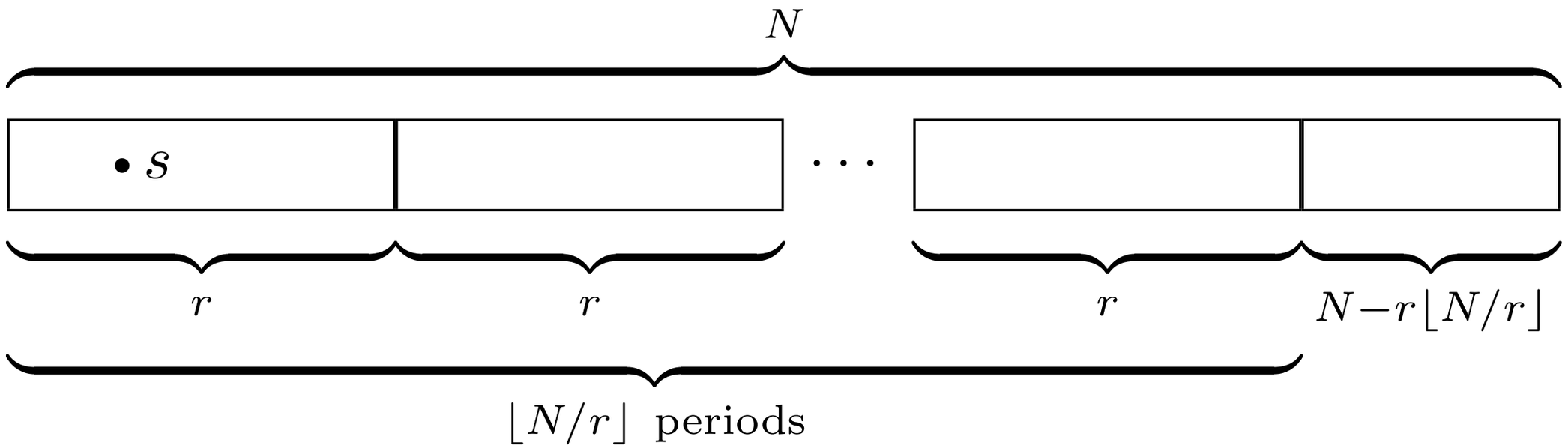

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |s + jr\rangle \qquad n = \begin{cases} \lfloor N/r \rfloor + 1 & s < N - r\lfloor N/r \rfloor \\ \lfloor N/r \rfloor & \text{otherwise.} \end{cases}$$

where s occurs with probability n/N

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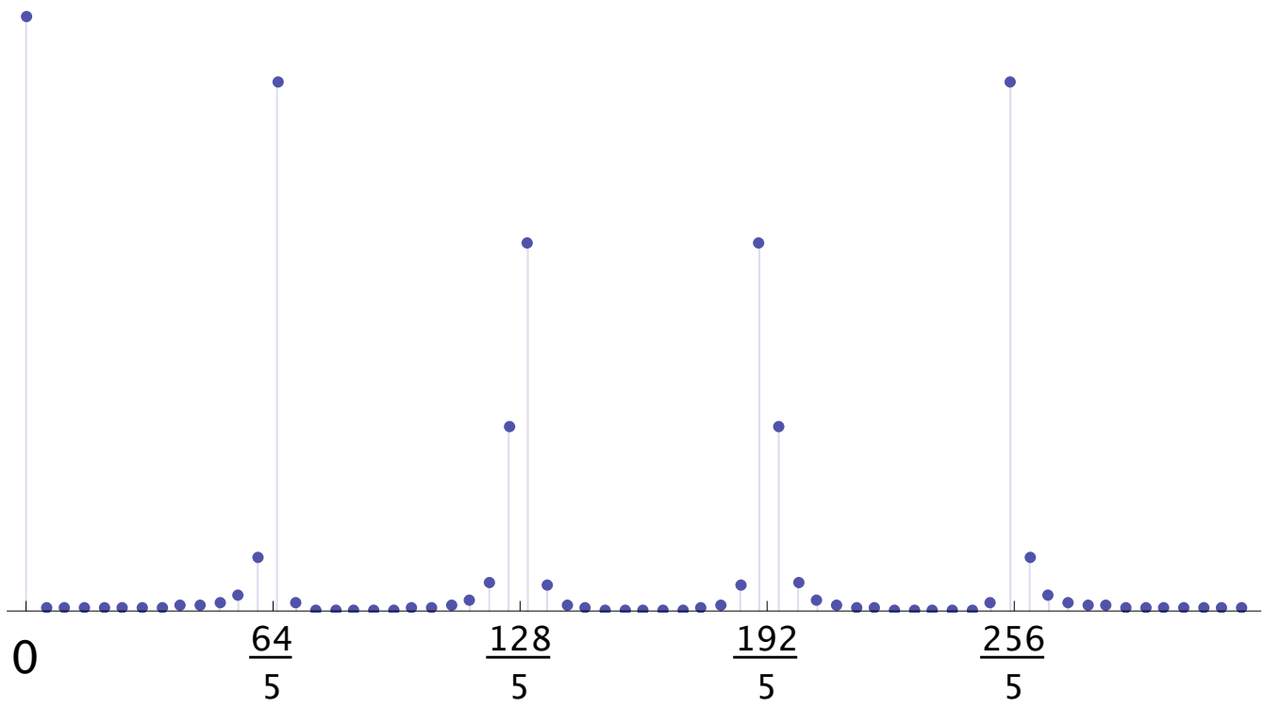
In general, $\Pr(y)$ is sharply peaked around multiples of N/r.

Example ($N = 64$, $r = 5$):



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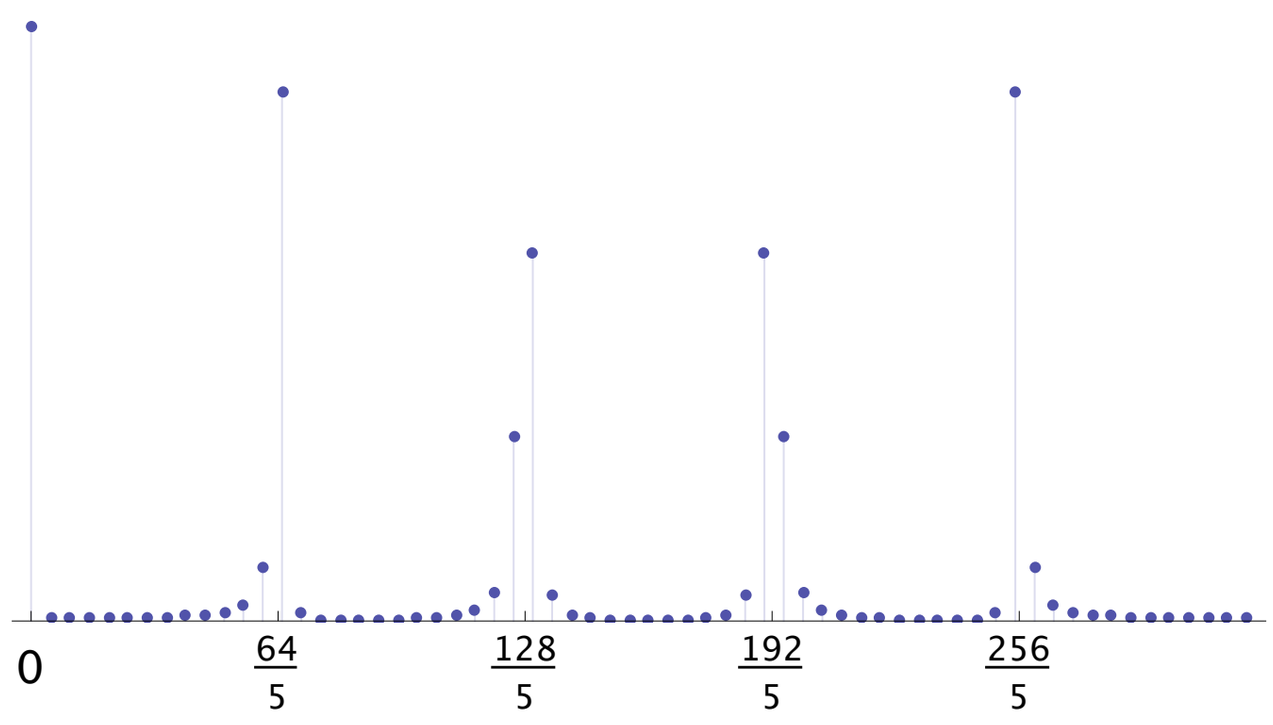

Exercise: Show $\Pr\left(y \in \{0, \lfloor \frac{N}{r} \rceil, \lfloor 2\frac{N}{r} \rceil, \ldots, \lfloor (r-1)\frac{N}{r} \rceil\}\right) = \Omega(1)$.

Continued fractions

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Suppose we know that $r < r_{\max}$. Then the closest continued fraction approximation with denominator less than $r_{\max}$ is guaranteed to be r provided $N > (r_{\max})^2$.

Continued fractions

Problem: Given a sample from $\{0, \lfloor \frac{N}{r} \rceil, \lfloor 2\frac{N}{r} \rceil, \ldots, \lfloor (r-1)\frac{N}{r} \rceil\}$, determine r.

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If we are not given an a priori upper bound $r_{\max}$, we can start with $r_{\max} = 2$ and successively double it until we find r; then the running time will be $\mathrm{poly}(\log r)$.

Application: Order finding

Let G be a group. The *order* of $g \in G$ is the smallest $r \in \{1, 2, 3, ...\}$ such that $g^r = 1$.

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So there is an efficient (running time $\mathrm{poly}(\log |G|)$) quantum algorithm for order finding.

Reduction of factoring to period finding

Factoring → Finding a nontrivial factor

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We can also assume that N contains at least two distinct prime powers, since it is easy to check if it is a power of some integer.

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so we might hope for $\gcd(\mathrm{a}^{r/2} - 1, N)$ to be a nontrivial factor of N.

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Lemma: Suppose $a \in \mathbb{Z}_N^\times$ is chosen uniformly at random, where N is an odd integer with at least two distinct prime factors. Then with probability at least $1/2$, the multiplicative order r of a is even and $a^{r/2} \neq -1 \bmod N$.

Proof (part 1 of 2)

Let $N = p_1^{m_1} \times \cdots \times p_k^{m_k}$ (p_i distinct primes, $k \geq 2$)

$a = a_i \bmod p_i^{m_i}$

$r_i =$ order of a_i modulo $p_i^{m_i}$

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If r is even and $a^{r/2} = -1 \bmod N$,

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so r_i does not divide $r/2$; but notice that r_i does divide r.

Hence r/r_i is an odd integer for each i, and every r_i must contain the same number of powers of 2 as r.

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Since $\mathbb{Z}_{p_i^{m_i}}^\times$ is cyclic and of even order, exactly half its elements have the maximal value of c_i, so in particular the probability of any particular c_i is at most $1/2$.

Shor's factoring algorithm

Input: Integer N Output: A nontrivial factor of N

1. Choose a random $a \in \{2, 3, \ldots, N-1\}$
2. Compute $\gcd(a, N)$; if it is not 1 then it is a nontrivial factor, and otherwise we continue
3. Prepare the uniform superposition $|\mathbb{Z}_{N^2}\rangle$ (or replace N^2 by the next largest power of 2)
4. Append an ancilla register, and conditioned on the value x in the first register, compute $f(x) = a^x \bmod N$ in the ancilla register; discard the ancilla
5. Perform the quantum Fourier transform over $\mathbb{Z}_{N^2}$
6. Measure in the computational basis
7. Compute the continued fraction expansion of the result divided by N^2, obtaining the best approximation with denominator less than N; call this denominator r
8. Compute $\gcd(a^{r/2} - 1, N)$. If it is 1 or N then we have failed, and we start over; otherwise the result is a nontrivial factor of N.

Performance

Most expensive step: Modular exponentiation.
Using repeated squaring, this can be done in time $O((\log N)^3)$.

$\Rightarrow$ Running time of Shor's algorithm is $O((\log N)^3)$

In contrast, the best known classical algorithm for factoring N (the *number field sieve*) takes time $2^{O((\log N)^{1/3}(\log\log N)^{2/3})}$.

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Note that the quantum part of Shor's algorithm can be highly parallelized: with polynomial-time classical pre- and post-processing, can achieve depth $O((\log\log N)^2)$ and size $O((\log N)^3)$ (Cleve and Watrous 2000).

Beyond factoring

So much more than Shor and Grover

- Quantum simulation
- Algebraic problems

  Factoring, discrete log, decomposing abelian groups, Pell's equation, principal ideal problem, computing unit/class groups of number fields, shifted Legendre symbol, approximating Gauss sums, counting points on curves, some nonabelian hidden subgroup problems, ...

- Search and its applications (Peter Høyer's lectures yesterday)

  Unstructured search (decision, finding, counting), collision, median finding, graph connectivity, minimum spanning trees, single source shortest paths, matchings, network flows, ...

- Quantum walk algorithms (Eddie Farhi's lectures on Friday)

  Black box graph traversal, spatial search, element distinctness, triangle finding, checking matrix multiplication, testing group commutativity, formula evaluation, ...

- Approximation of #P-hard problems

  Jones polynomial, HOMFLYPT polynomial, Tutte polynomial/Potts model partition function, ...

- Miscellanea

  Oracle interrogation, ordered search, gradient estimation, ...

Discrete log

Let $G = \{g^\alpha : \alpha \in \mathbb{Z}_N\}$ be a cyclic group generated by g.

Given $x \in G$, define $\log_g x := \min\{\ell \in \mathbb{Z}^+ : g^\ell = x\}$.

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Common groups:

	best classical algorithm
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But there is an efficient quantum algorithm for discrete log in general groups (Shor 1994).

The abelian hidden subgroup problem

Let G be an abelian group.

We say that $f : G \to S$ *hides* the subgroup $H \leq G$ if

$$f(x) = f(y) \text{ if and only if } x - y \in H$$

Abelian HSP: Given the ability to query f, find a generating set for H.

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There is an efficient quantum algorithm for the hidden subgroup problem in any finitely generated abelian group.

Counting points on curves

Problem: How many solutions are there to the polynomial equation

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Approach: Use the algorithm for the abelian HSP to learn the structure of the *class group* of the curve, which determines the number of points.

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Similar ideas lead to efficient quantum algorithms for computing properties of number fields.

The nonabelian hidden subgroup problem

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Potential applications:

- Graph isomorphism
- Lattice problems

Graph automorphism

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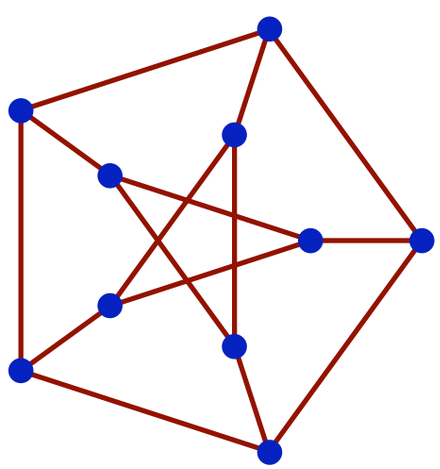
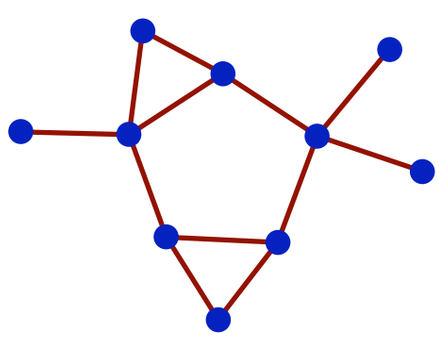
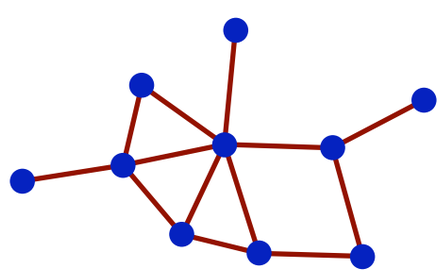
$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

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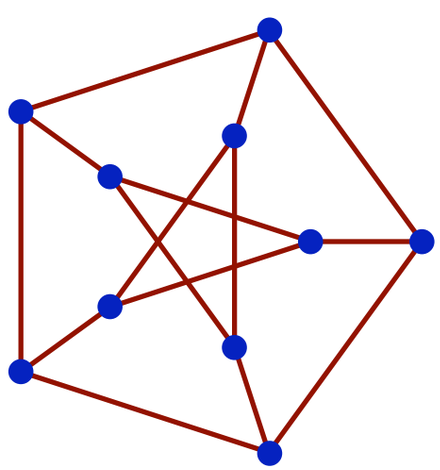
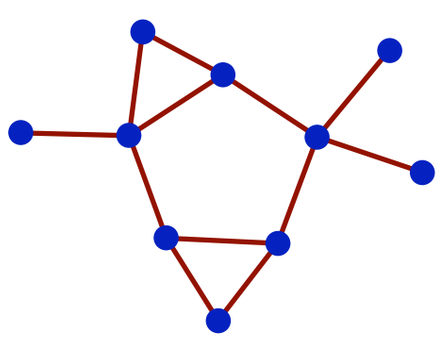
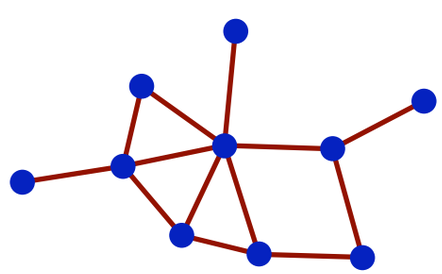
$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

$\mathrm{Aut}(\Gamma_1) \cong S_5$

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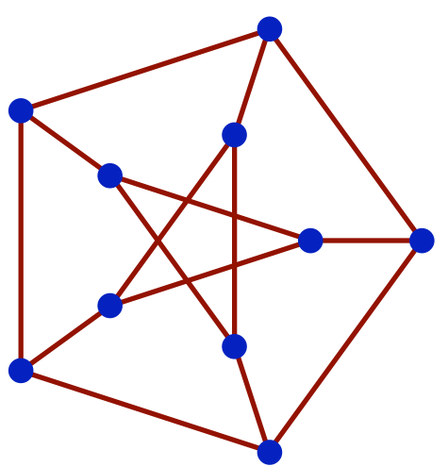
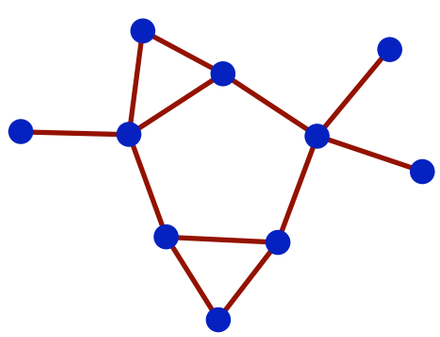
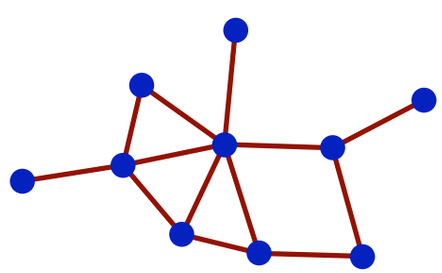
$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

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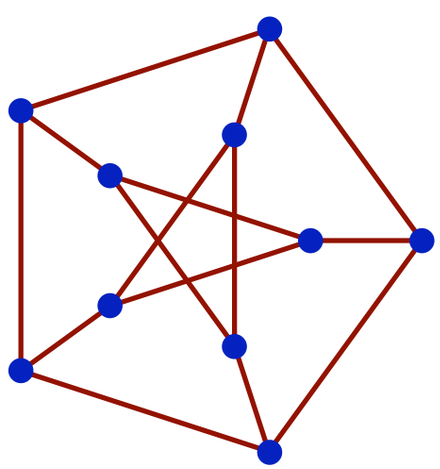
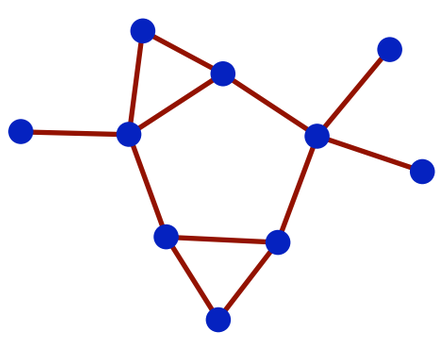
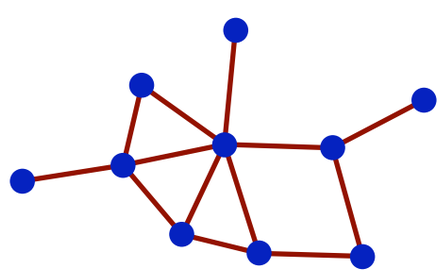
$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

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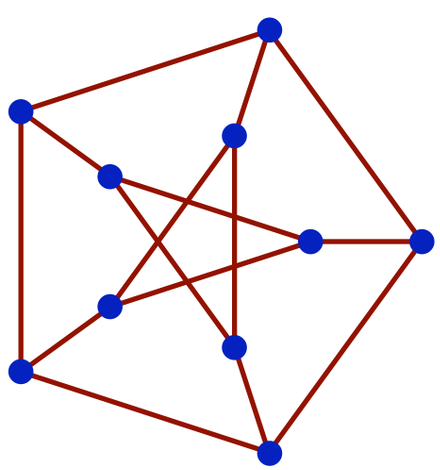
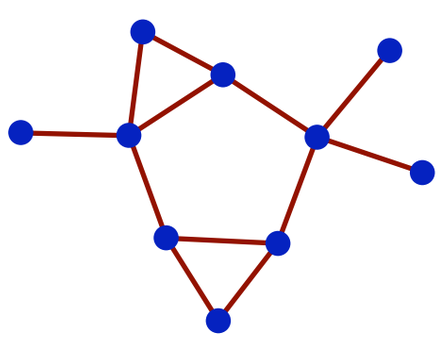
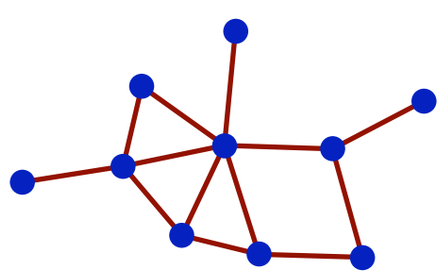
$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

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This is an HSP in $G = S_n$

with $f(\pi) = \pi(\Gamma)$

having hidden subgroup $H = \mathrm{Aut}(\Gamma)$

Shortest lattice vector problem

Basis vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ define a lattice $L := \{\sum_i c_i \vec{v}_i : c_i \in \mathbb{Z}\}$

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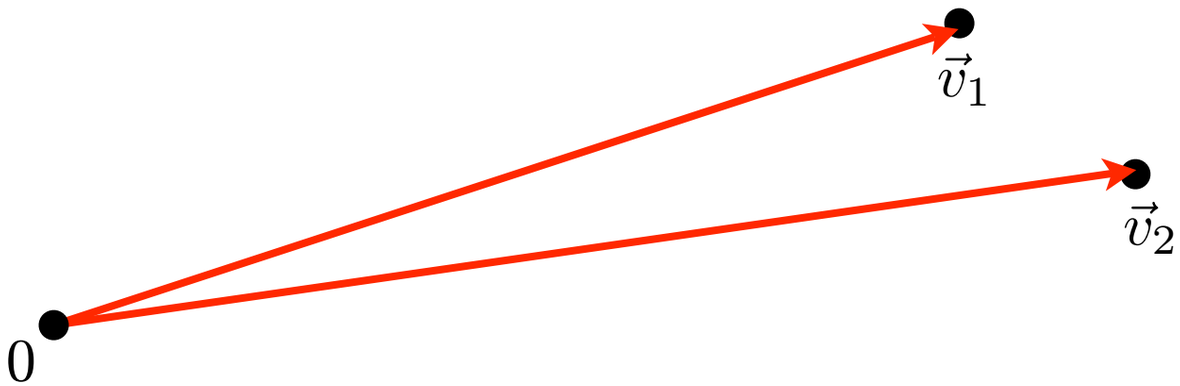

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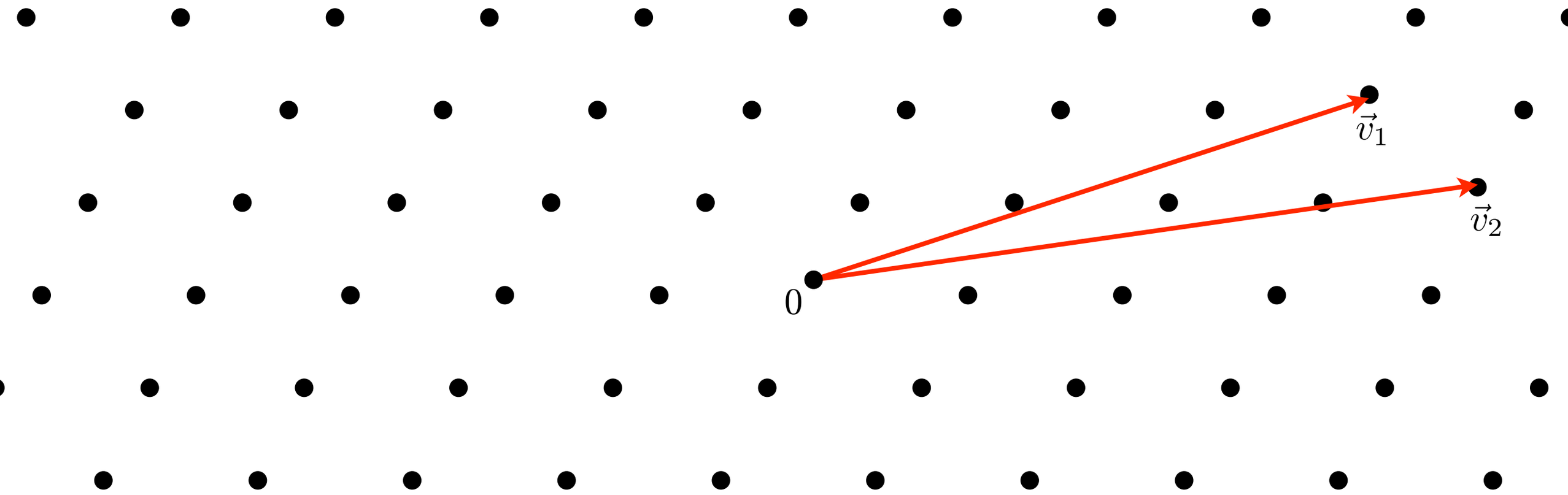

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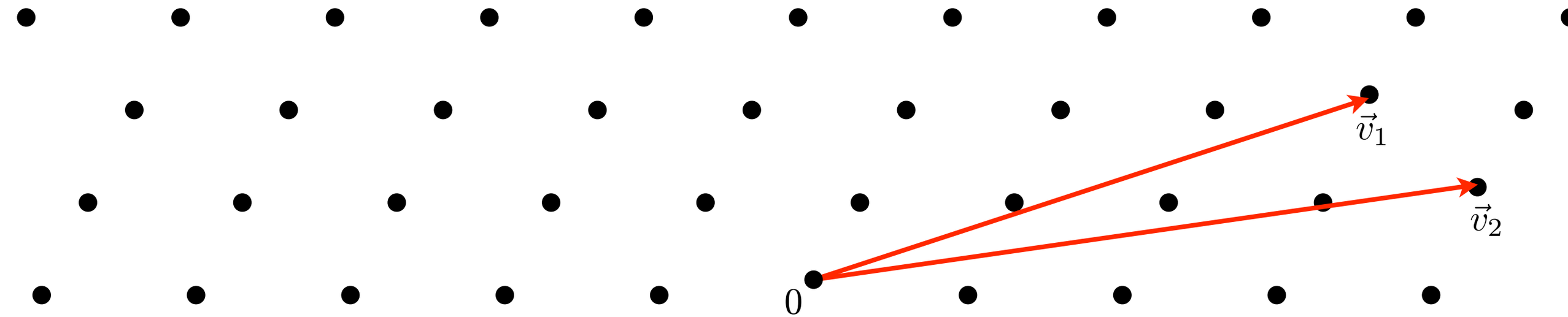

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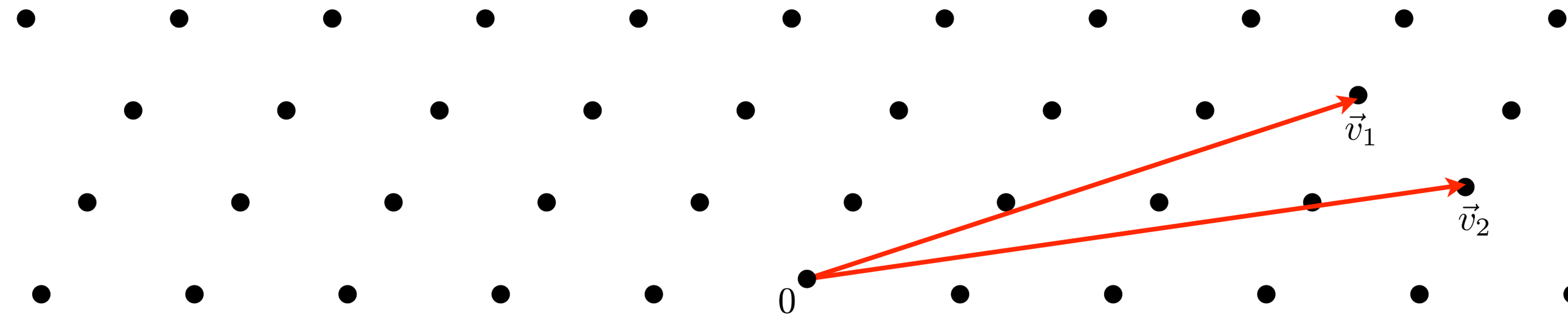

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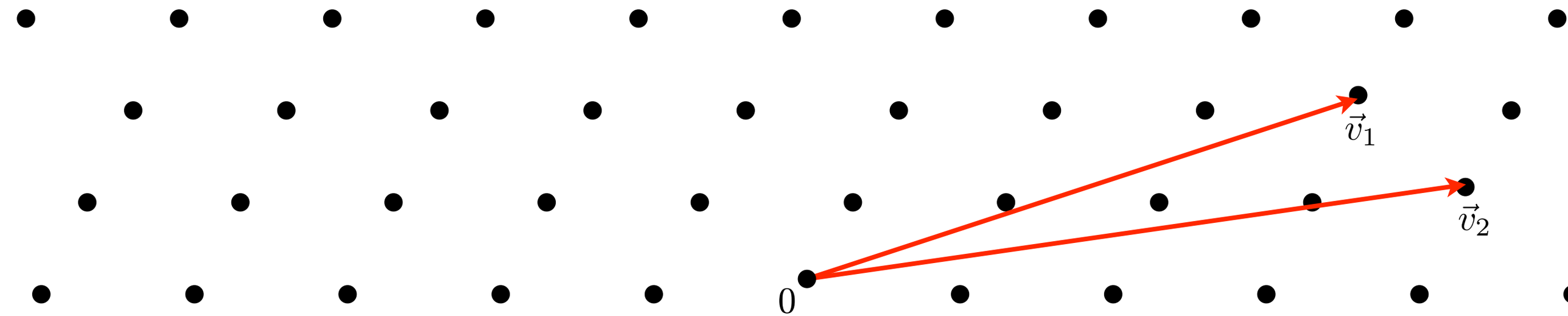

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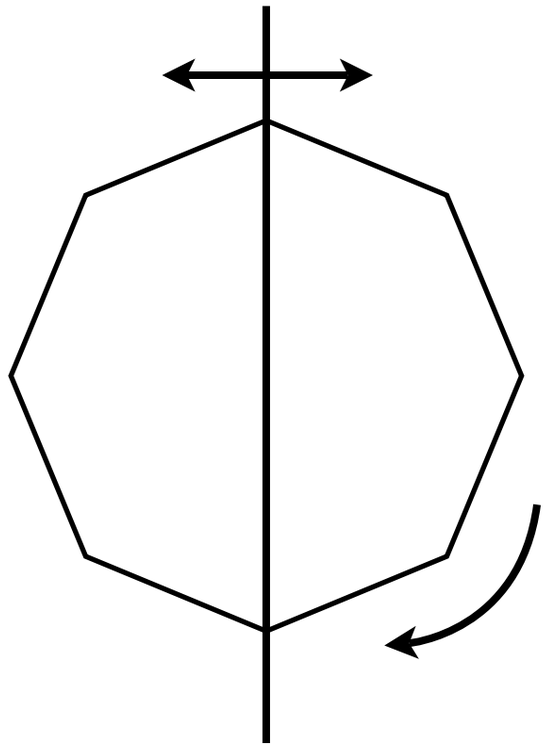
Lattice cryptosystems assume this is hard with a factor $\mathrm{poly}(n)$.

Dihedral hidden subgroup problem

The *dihedral group* of order $2N$ is the group of symmetries of an N-sided regular polygon.



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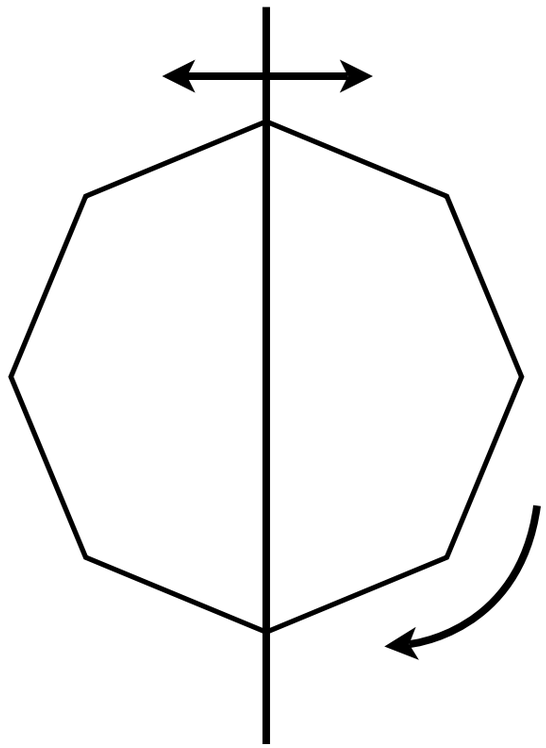

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Note: It suffices to consider hidden subgroups of order 2 (hidden reflections).

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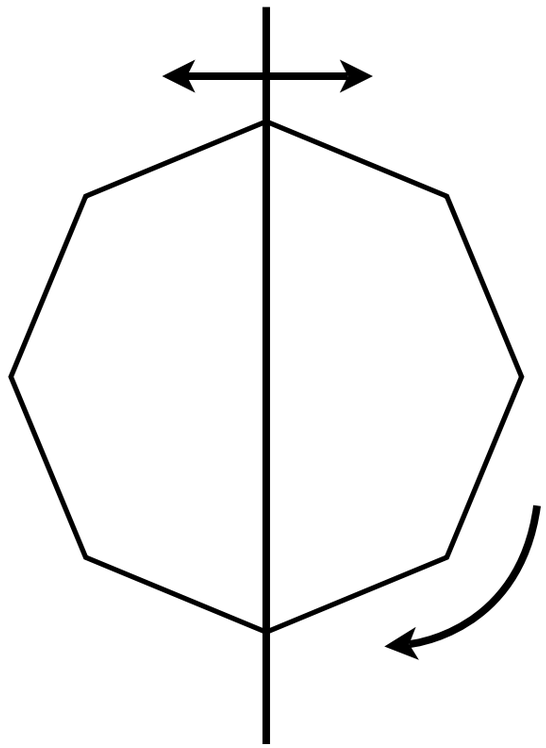

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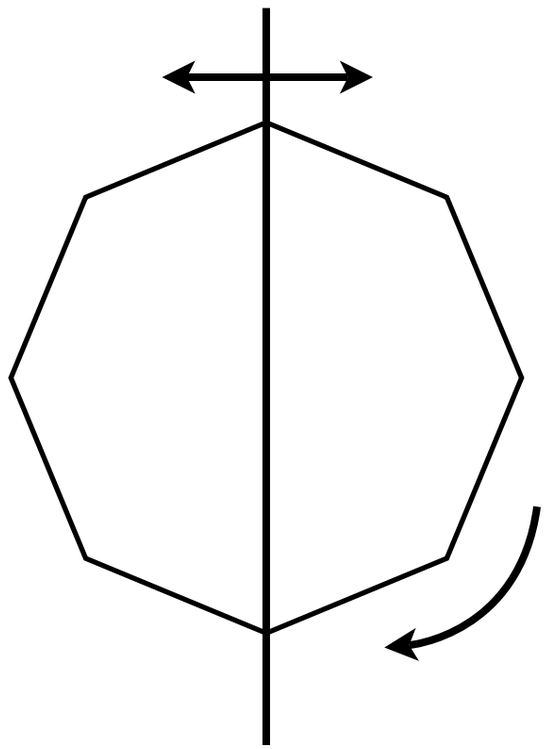

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Solving this problem would break lattice-based cryptography, one of the few types of cryptosystems not yet known to be broken by quantum computers!