Quantum algorithms for hidden nonlinear structures

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Key idea: The Fourier transform of a linear structure exhibits sharp constructive interference that reveals the answer.

Are there other ways to create sharp constructive interference over a high-dimensional space?

Beyond Shor: The hidden subgroup problem

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Symmetric group	Graph automorphism, graph isomorphism
Dihedral group	Finding short lattice vectors [Regev 03]

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... but these cases appear hard.

courtesy of the NAIC Arecibo Observatory, a facility of the NSF


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$(\mathbb{F}_q)^d$
d fixed
degree fixed
$q \to \infty$
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Quantum computers can find hidden nonlinear structures

Shifted subset problems

Two examples:

- Hidden radius problem (partial solution, by *Fourier sampling*)
- Hidden flat of centers problem (complete solution for d odd, by *quantum walk*)

Both have:

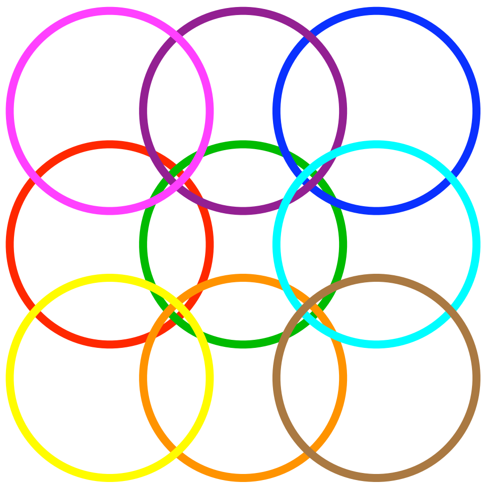
- Polynomial-time quantum algorithms
- A black-box formulation with exponential classical query complexity

Hidden polynomial problem

- Naturally formulated as a black-box problem with exponential classical query complexity
- Quantum query complexity is polynomial

Hidden radius problem

Quantum formulation: Suppose we can sample a quantum state that is uniform over points on a sphere of radius r, with the center chosen uniformly at random.



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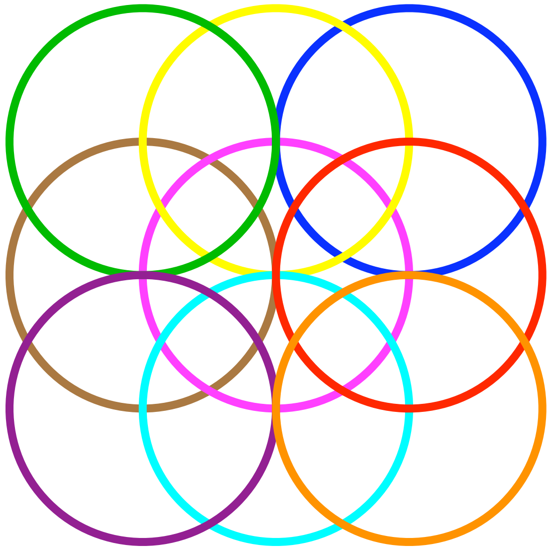

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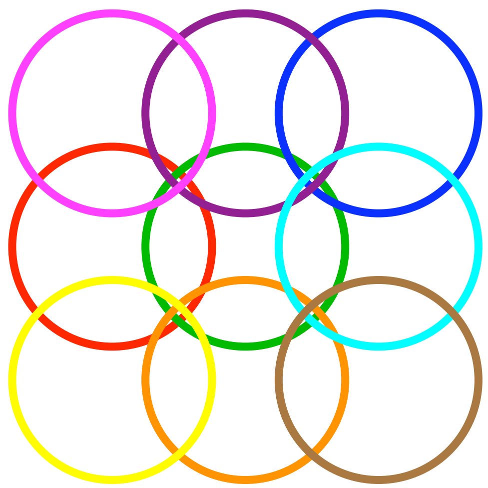

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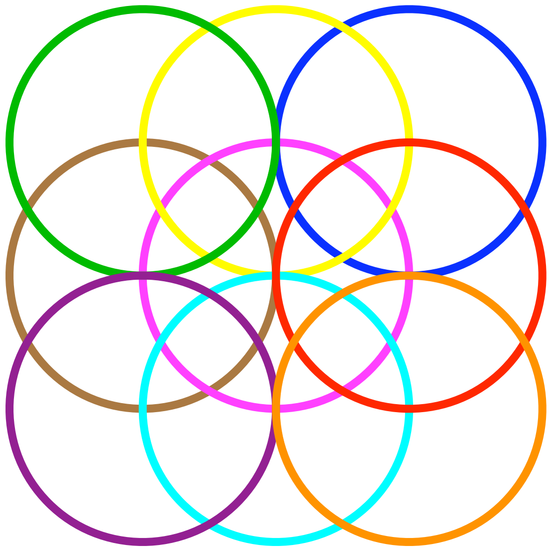

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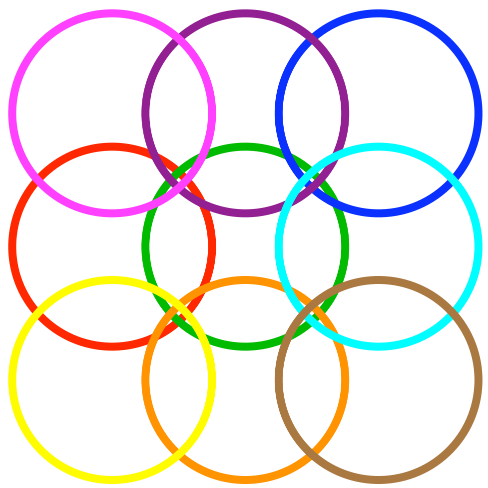

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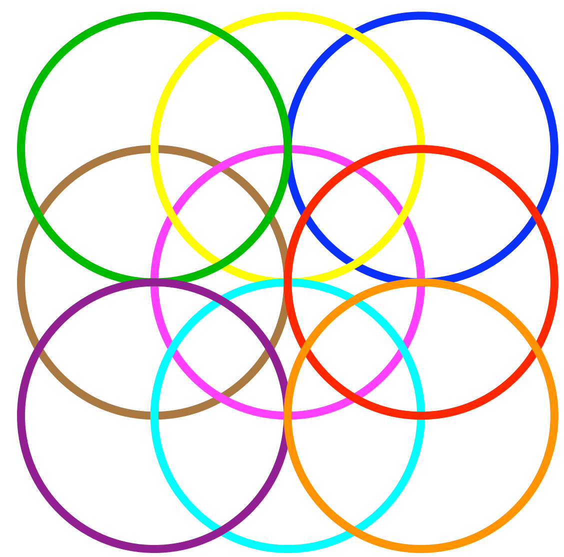

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Theorem. There is quantum algorithm that determines $\chi(r)$ in time $\mathrm{poly}(\log q)$, provided $d = O(1)$ is odd.

quadratic character

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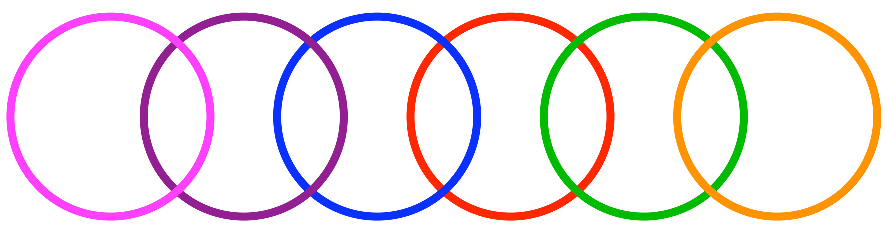
Odd d: Salié sum ($\eta = \chi$).

$$K_\chi(a) = e^{i\phi}\sqrt{q} \begin{cases} 1 & a = 0 \\ 2\cos \frac{4\pi \operatorname{tr}(\sqrt{a})}{p} & \chi(a) = +1 \\ 0 & \chi(a) = -1 \end{cases}$$

Hidden flat of centers problem

Quantum formulation: Suppose we can sample a quantum state that is a uniform superposition over points on a sphere of radius 1, with the center chosen uniformly at random from an unknown flat.



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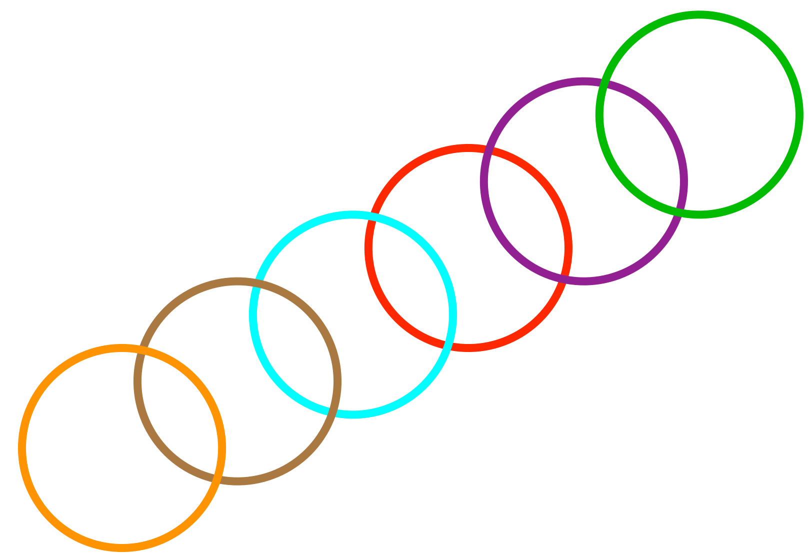

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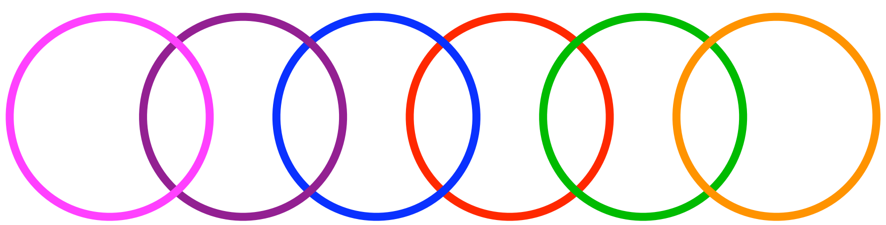

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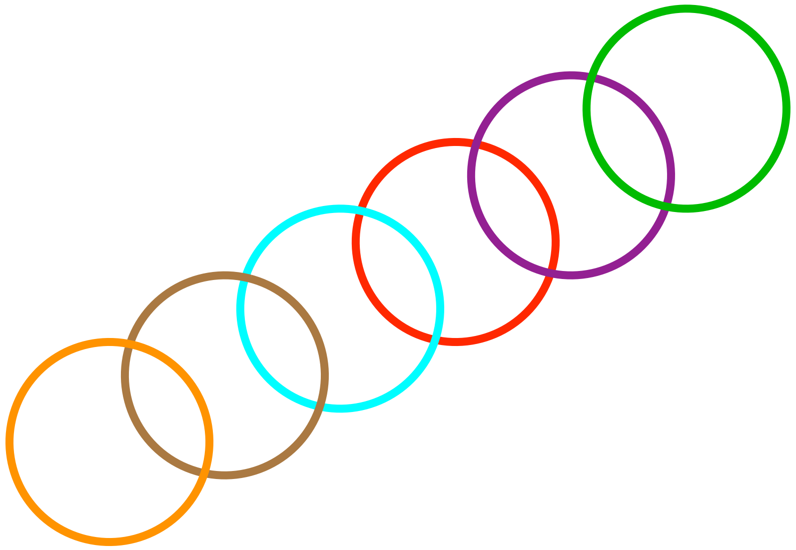

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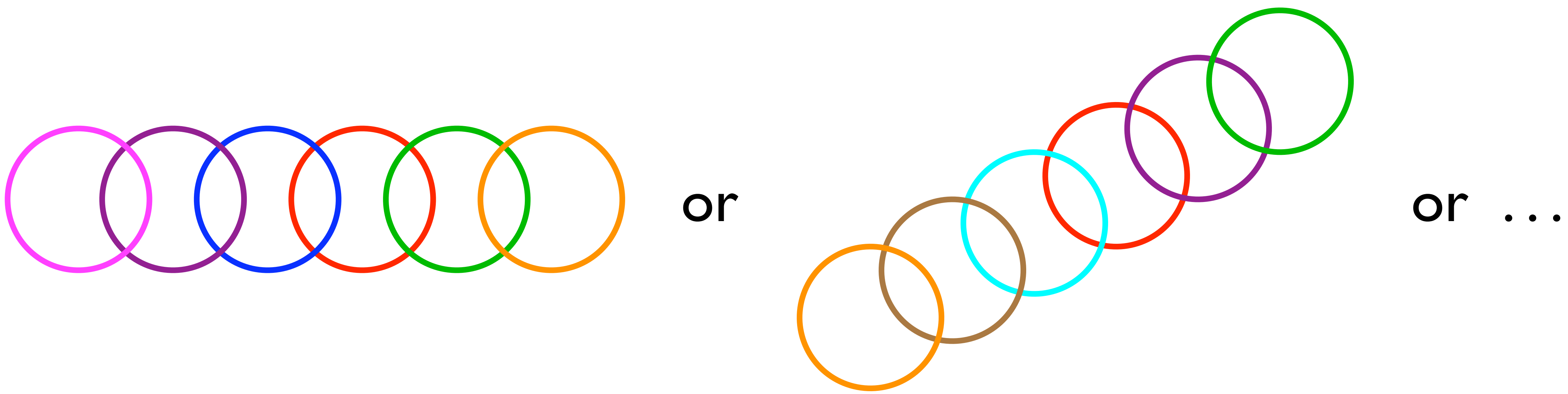

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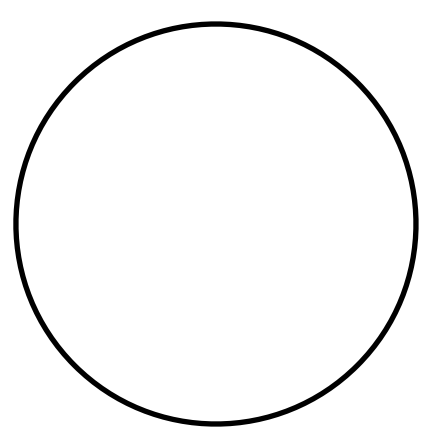

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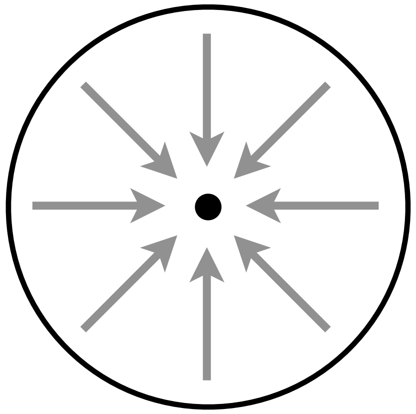

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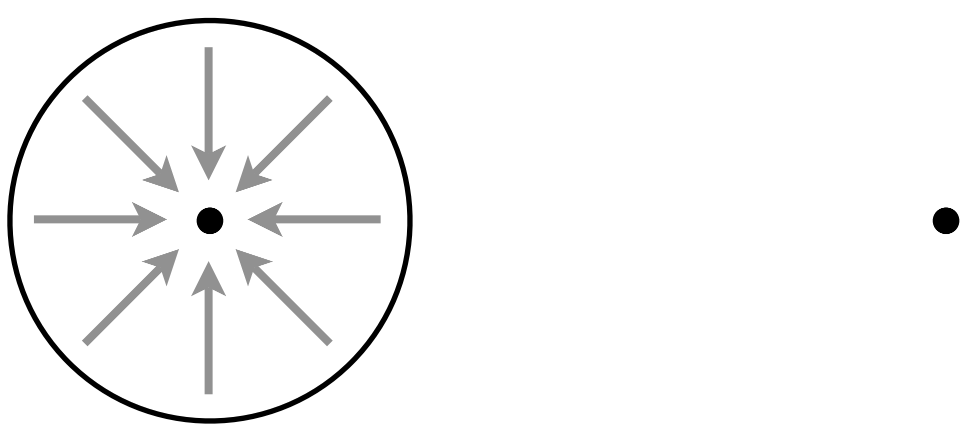

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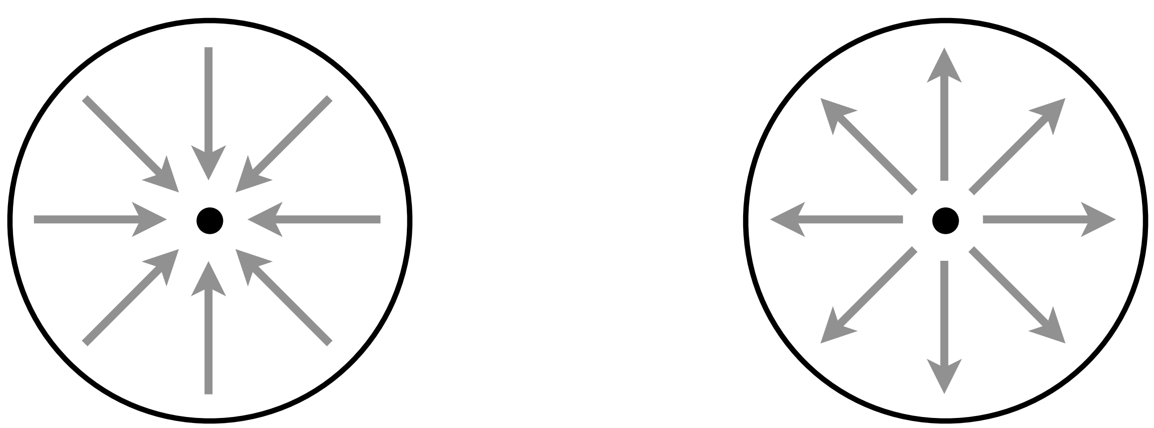

Reconstructing a noisy flat

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Note: It is crucial here that $d = O(1)$.

Exponential speedups by quantum walk

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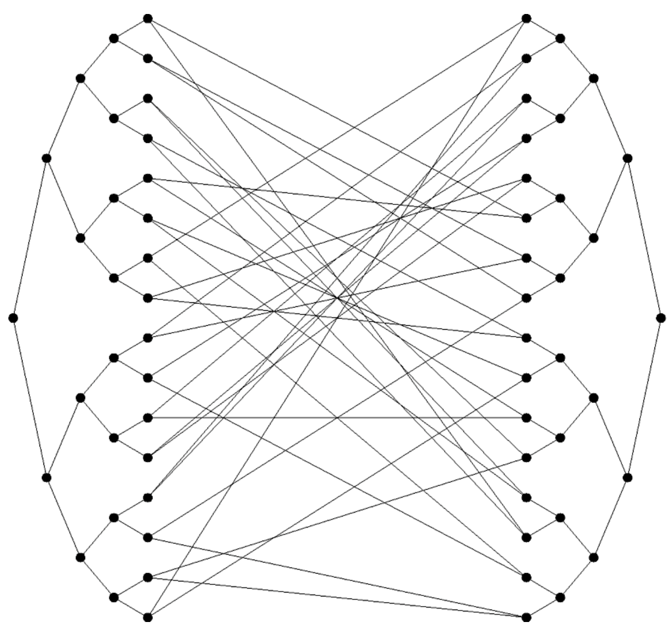

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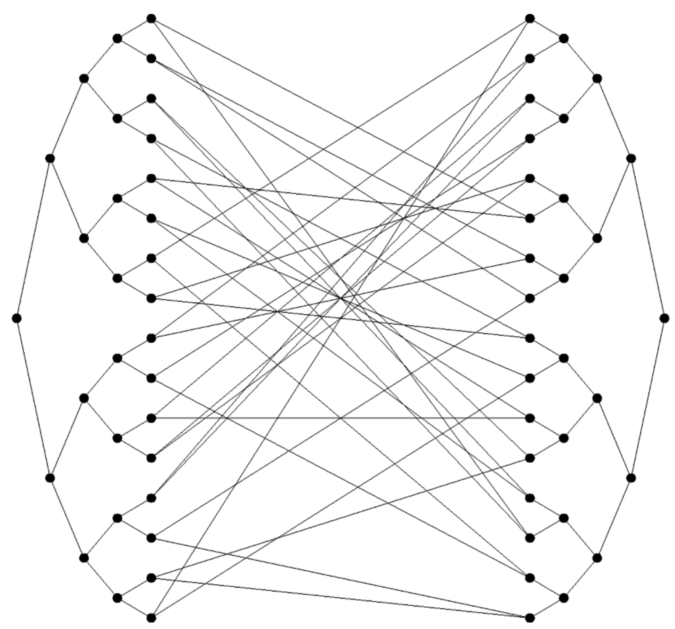

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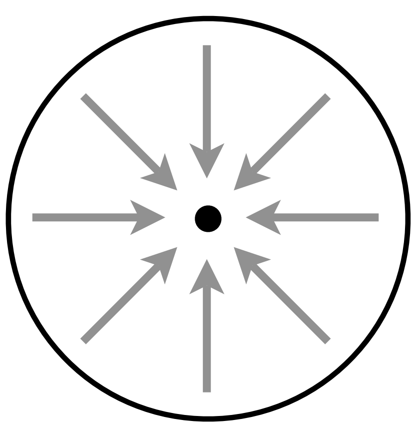

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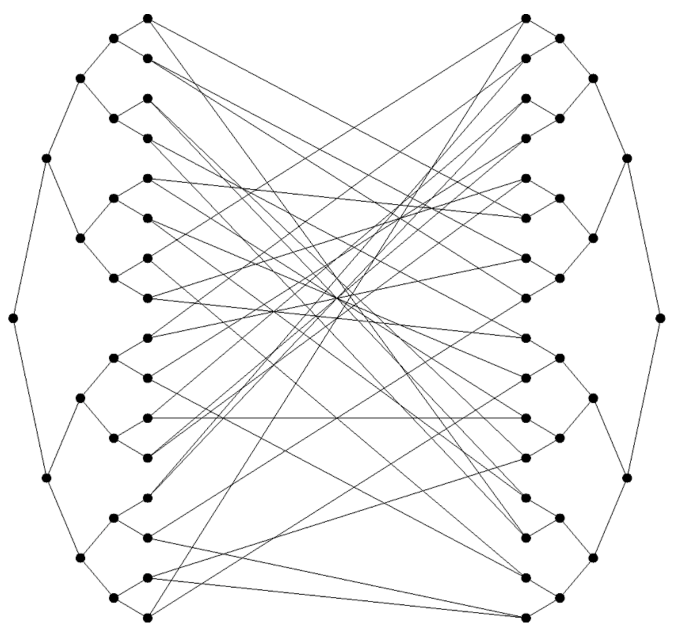

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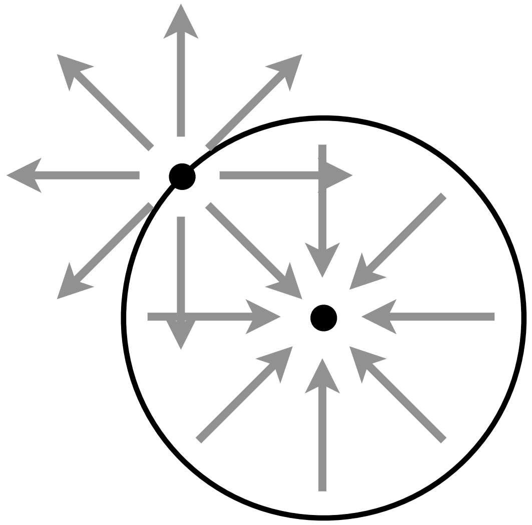

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The hidden polynomial problem

Problem: Given a black-box function that is constant on the level sets of $f \in \mathbb{F}_q[x_1, \ldots, x_d]$ (of constant total degree), and distinct on different level sets, determine f (projectively).

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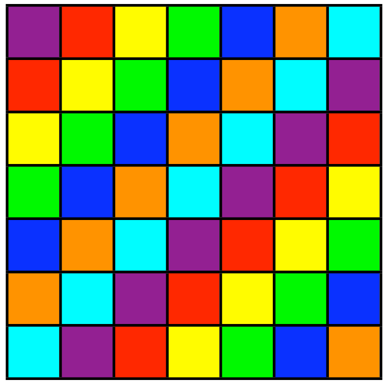

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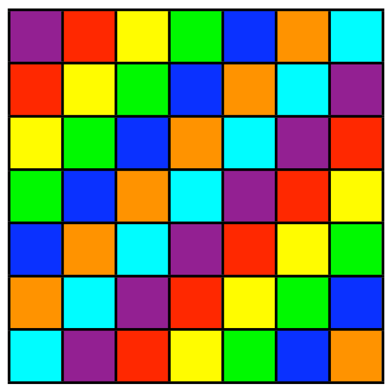

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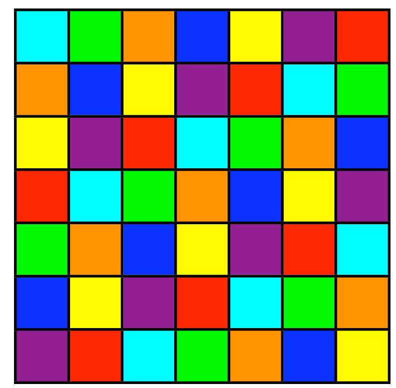

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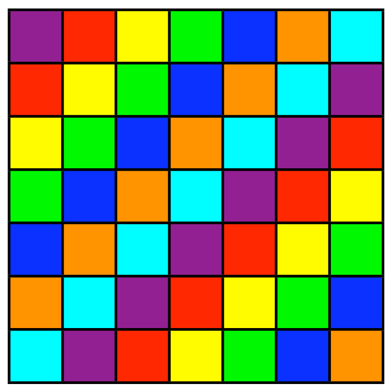

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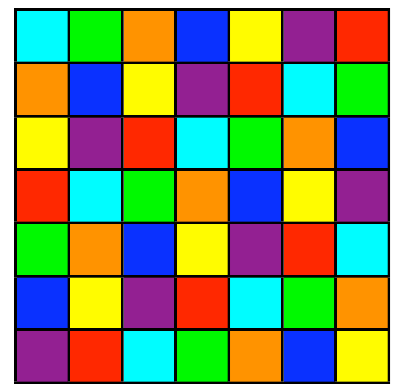

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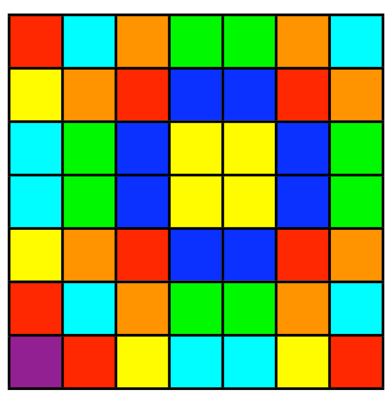
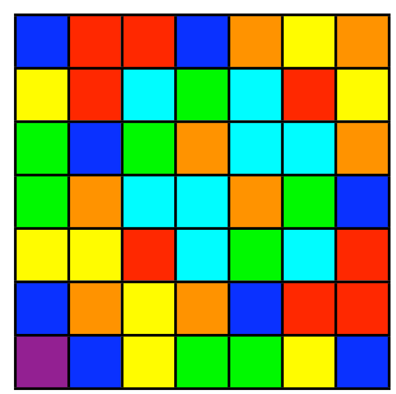
Quadratic f:



$x^2 + y^2$

,



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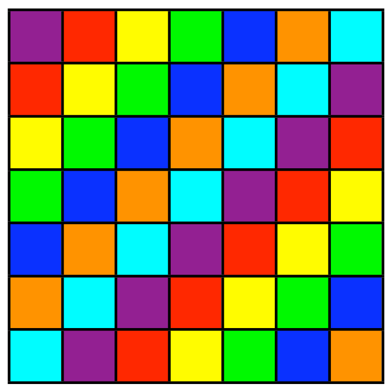

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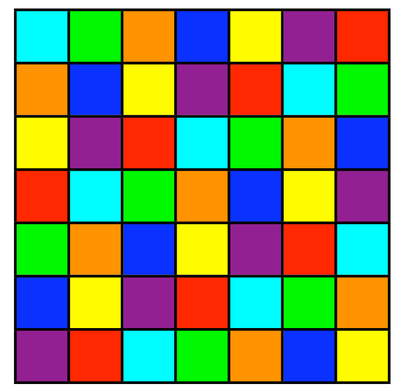

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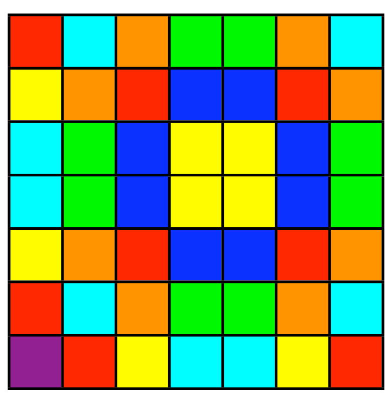

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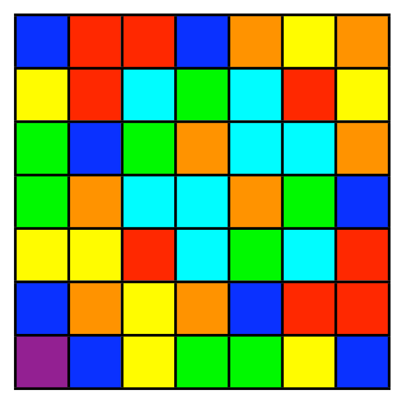

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Classical query complexity is exponential in $\log q$ (because it's hard to even find a collision).

Quantum query complexity of the HPP

Theorem. The quantum query complexity of the hidden polynomial problem is $\mathrm{poly}(\log q)$ for almost all polynomials.

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Proof idea:

- By standard techniques, reduce to a problem of distinguishing quantum states
- States are distinguishable if the level sets of the polynomials have small intersection
- Typical size of a level set: $c\, q^{d-1}$ [Schwartz-Zippel]
- Typical size of the intersection of two level sets: $c'\, q^{d-2}$ [Weil]
- Almost all polynomials are absolutely irreducible

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  - Gauss sums: [van Dam, Seroussi 02]
  - Small characteristic: Apply quantum point-counting algorithm of [Kedlaya 06] (as suggested by Shparlinski)
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- Applications?