Universal computation by multi-particle quantum walk

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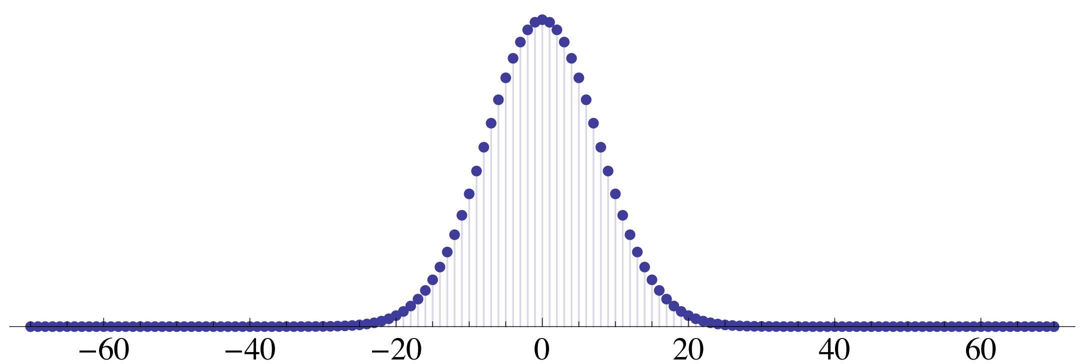
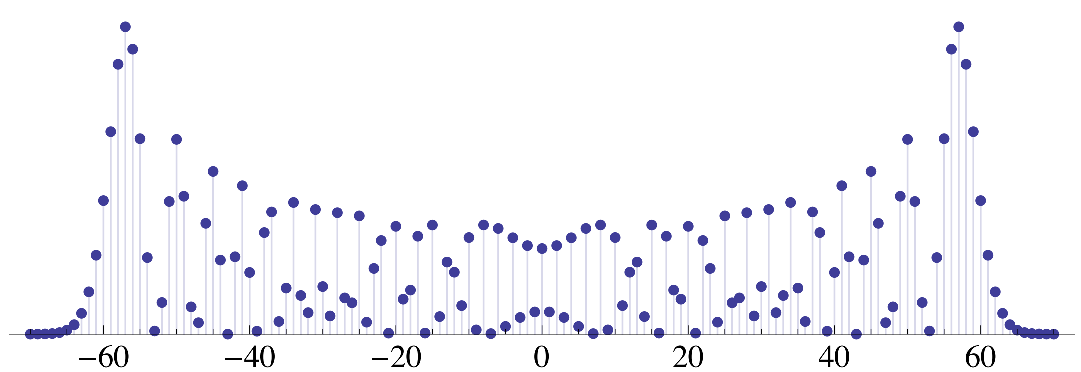
Quantum walk

Quantum analog of a random walk on a graph.

Idea: Replace probabilities by quantum amplitudes.
Interference can produce radically different behavior!



classical



quantum

Quantum walk algorithms

Quantum walk is a major tool for quantum algorithms (especially query algorithms with polynomial speedup).

- Exponential speedup for black-box graph traversal [Childs, Cleve, Deotto, Farhi, Gutmann, Spielman 02]
- Quantum walk search framework [Szegedy 05], [Magniez et al. 06]
  - Spatial search [Shenvi, Kempe, Whaley 02], [Childs, Goldstone 03, 04], [Ambainis, Kempe, Rivosh 04]
  - Element distinctness [Ambainis 03]
  - Subgraph finding [Magniez, Santha, Szegedy 03], [Childs, Kothari 10]
  - Matrix/group problems [Buhrman, Špalek 04], [Magniez, Nayak 05]
- Evaluating formulas/span programs
  - AND-OR formula evaluation [Farhi, Goldstone, Gutmann 07], [ACRŠZ 07]
  - Span programs for general query problems [Reichardt 09]
  - Learning graphs [Belovs 11] → new upper bounds (implicitly, quantum walk algorithms), new kinds of quantum walk search

Universality of quantum walk

A quantum walk can be efficiently simulated by a universal quantum computer.

N-vertex graph max degree $\mathrm{poly}(\log N)$ efficiently computable neighbors	$\Rightarrow$	circuit with $\mathrm{poly}(\log N)$ qubits $\mathrm{poly}(\log N)$ gates

Conversely, quantum walk is a *universal computational primitive*: any quantum circuit can be simulated by a quantum walk. [Childs 09]

N dimensions g gates	$\Rightarrow$	graph with $\mathrm{poly}(N, g)$ vertices max degree 3 walk for time $\mathrm{poly}(g)$

Note: The graph is necessarily exponentially large in the number of qubits! Vertices represent basis states.

Quantum walk experiments

Multi-particle quantum walk

With many walkers, the Hilbert space can be much bigger.

m distinguishable particles on an n-vertex graph: n^m dimensions

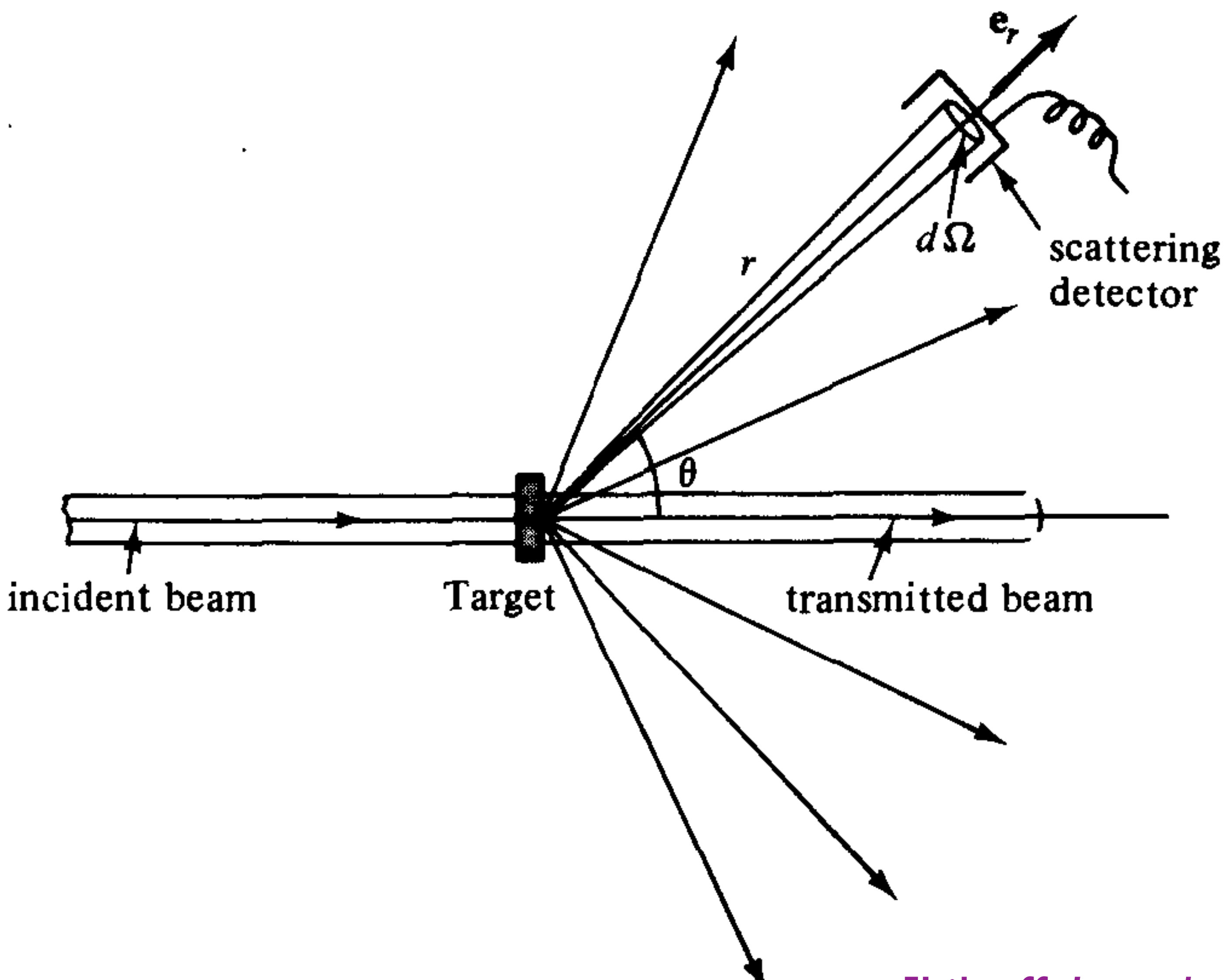
(similar scaling for indistinguishable bosons/fermions)

Main result: Any n-qubit, g-gate quantum circuit can be simulated by a multi-particle quantum walk of $n+1$ particles interacting for time $\mathrm{poly}(n,g)$ on a graph with $\mathrm{poly}(n,g)$ vertices.

Consequences:

- Architecture for a quantum computer with no time-dependent control
- Simulating interacting many-body systems is BQP-hard (e.g., Bose-Hubbard model on a sparse, unweighted, planar graph)

Scattering theory on graphs



[Liboff, *Introductory Quantum Mechanics*]

Quantum walk

Quantum analog of a random walk on a graph $G = (V, E)$.

Idea: Replace probabilities by quantum amplitudes.

$$|\psi(t)\rangle = \sum_{v \in V} a_v(t) |v\rangle$$

amplitude for vertex v at time t

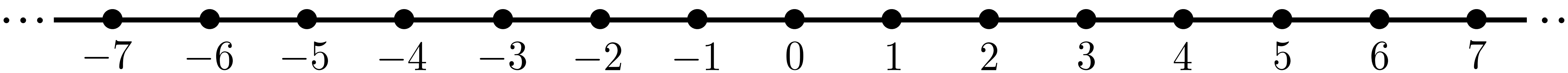
Define time-homogeneous, local dynamics on G.

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle$$

Adjacency matrix: $H = \sum_{(u,v) \in E(G)} |u\rangle\langle v|$

Momentum states

Consider an infinite path:



Hilbert space: $\mathrm{span}\{|x\rangle : x \in \mathbb{Z}\}$

Eigenstates of the adjacency matrix: $|\tilde{k}\rangle$ with

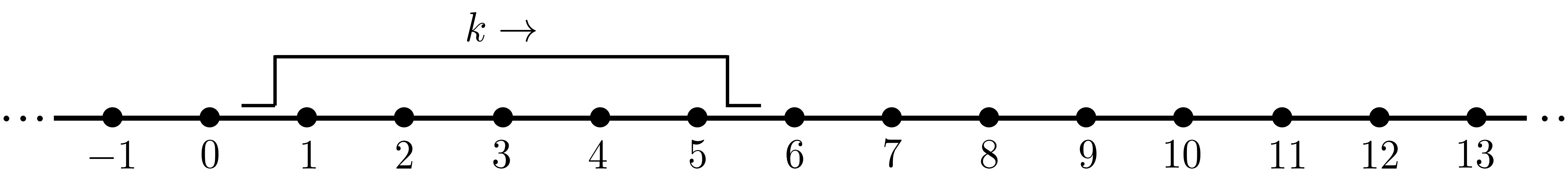
$$\langle x|\tilde{k}\rangle := e^{\mathrm{i}kx} \qquad k \in [-\pi, \pi)$$

Eigenvalue: $2\cos k$

Wave packets

A *wave packet* is a normalized state with momentum concentrated near a particular value k.

Example: $\displaystyle \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{-\mathrm{i}kx} |x\rangle$ (large L)

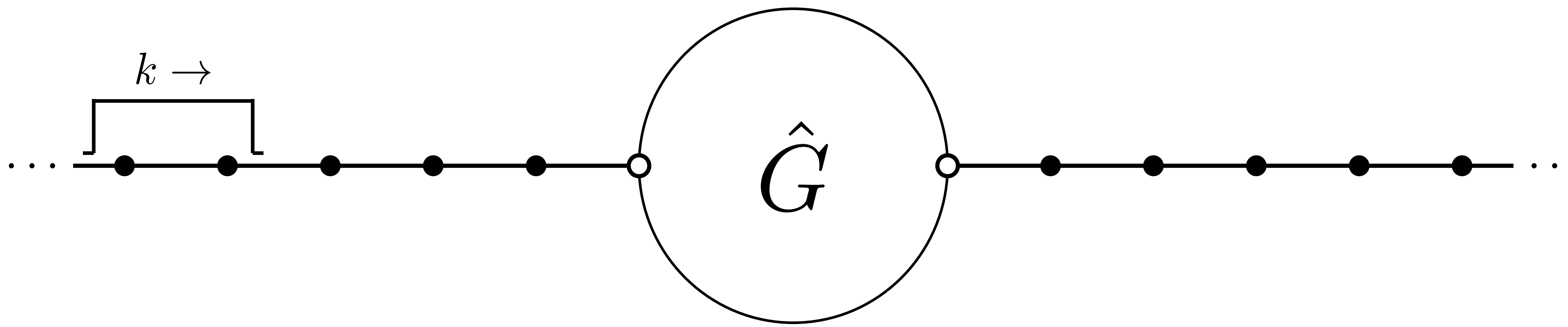


Propagation speed: $\displaystyle \left| \frac{\mathrm{d}E}{\mathrm{d}k} \right| = 2|\sin k|$

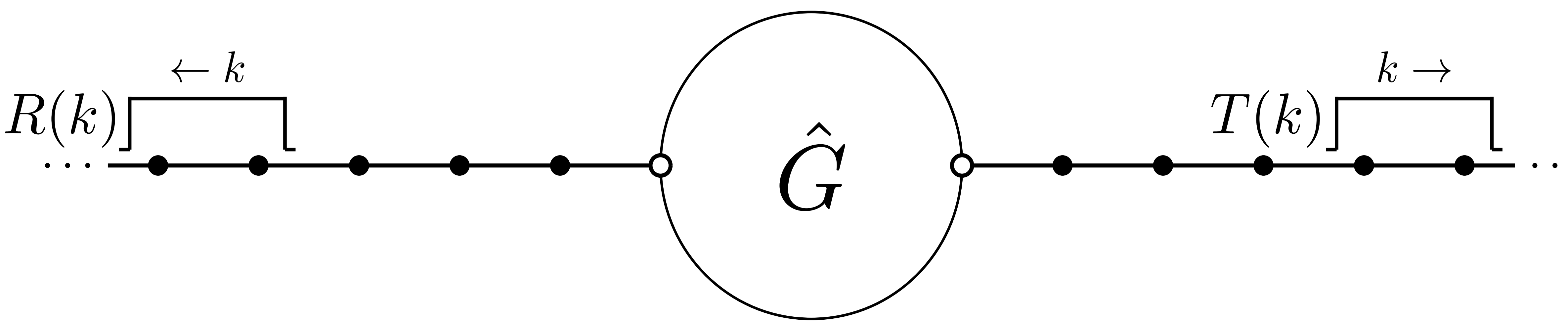
Scattering on graphs

Now consider adding semi-infinite lines to two vertices of an arbitrary finite graph.

Before:

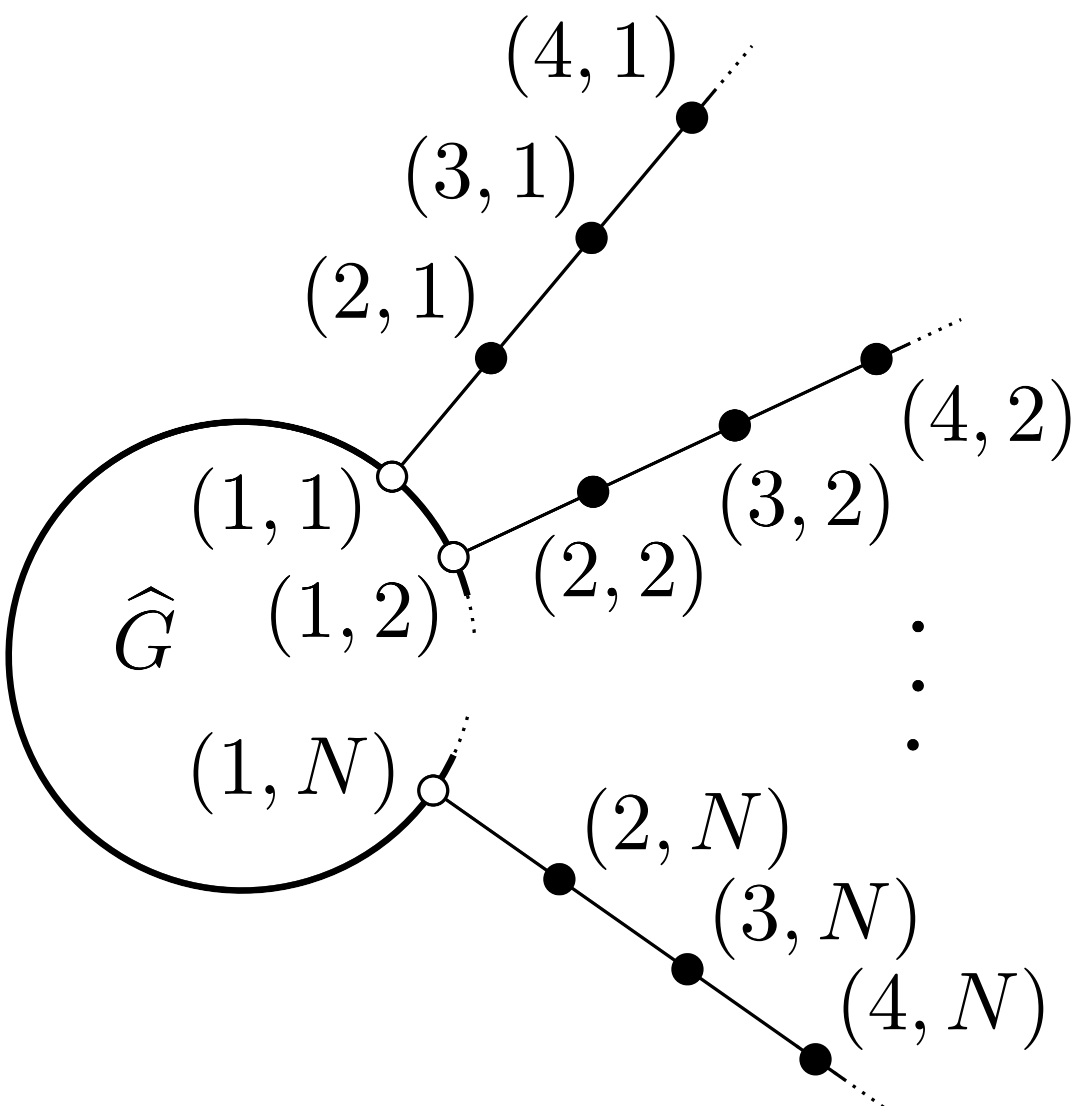


After:



The S-matrix

This generalizes to any number N of semi-infinite paths attached to any finite graph.



Incoming wave packets of momentum near k are mapped to outgoing wave packets (of the same momentum) with amplitudes corresponding to entries of an $N \times N$ unitary matrix $S(k)$, called the S-matrix.

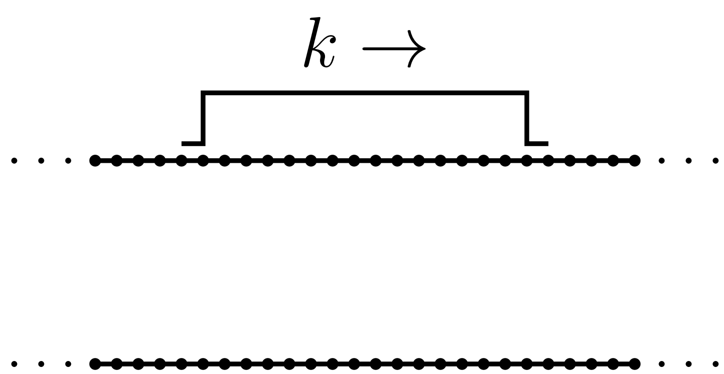
Universal computation

Encoding a qubit

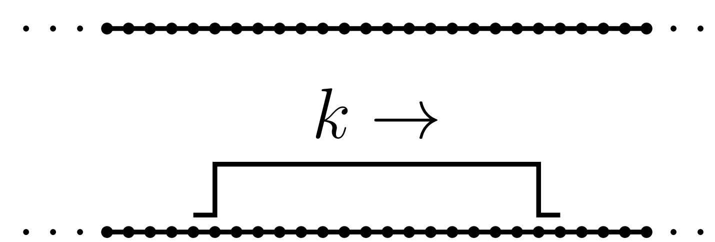
Encode quantum circuits into graphs.

Computational basis states correspond to paths (“quantum wires”).

For one qubit, use two wires (“dual-rail encoding”):



encoded $|0\rangle$



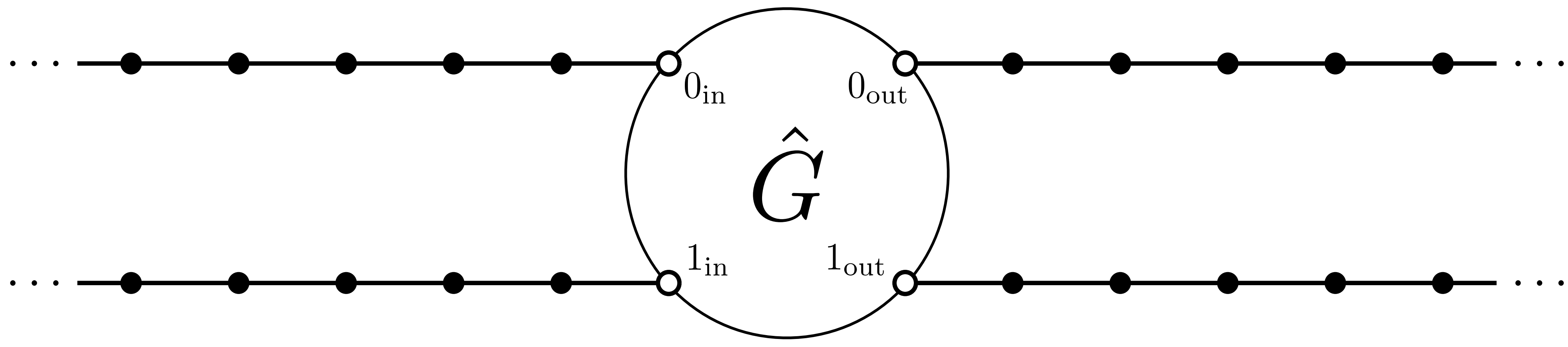
encoded $|1\rangle$

Fix some value of the momentum (e.g., $k = \pi/4$).

Quantum information propagates from left to right at constant speed.

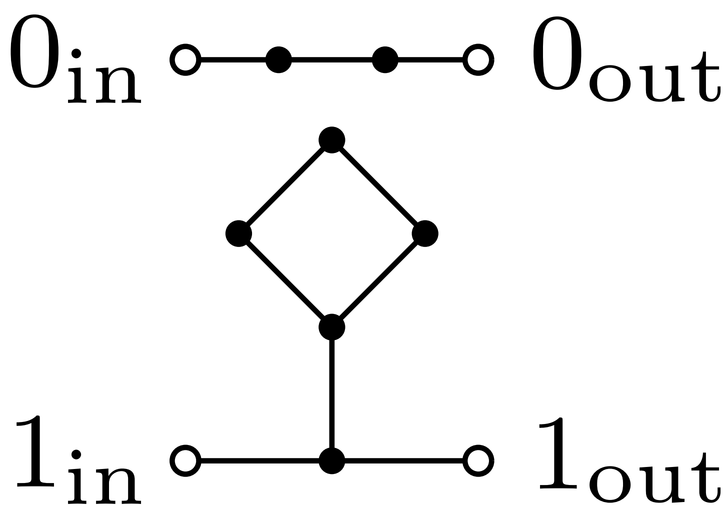
Implementing a gate

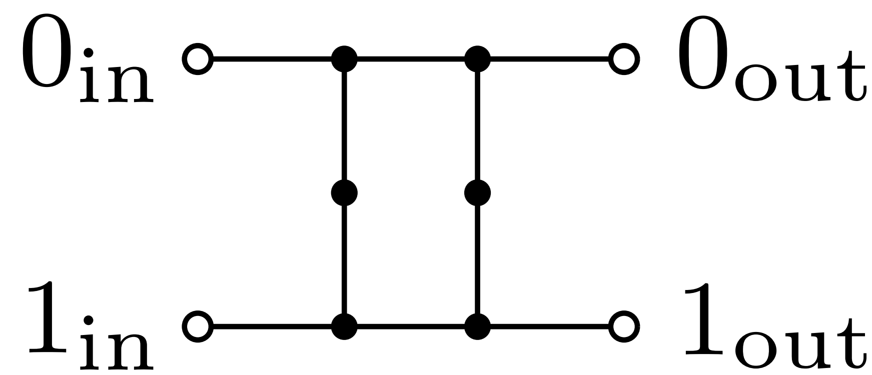
To perform a gate, design a graph whose S-matrix implements the desired transformation U at the momentum used for the encoding.



$$S(k) = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$$

Universal set of single-qubit gates





$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\mathrm{i}} \end{pmatrix} \qquad -\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix}$$

momentum for logical states: $k = \pi/4$

[Childs, Phys. Rev. Lett. **102**, 180501 (2009)]

Multi-particle quantum walk

With m distinguishable particles:

states: $|v_1, \ldots, v_m\rangle \qquad v_i \in V(G)$

Hamiltonian: $$H_G^{(m)} = \sum_{i=1}^{m} \sum_{(u,v)\in E(G)} |u\rangle\langle v|_i + \mathcal{U}$$

Indistinguishable particles:

bosons: symmetric subspace

fermions: antisymmetric subspace

Many possible interactions:

on-site: $$\mathcal{U} = J \sum_{v\in V(G)} \hat{n}_v(\hat{n}_v - 1) \qquad \hat{n}_v = \sum_{i=1}^{m} |v\rangle\langle v|_i$$

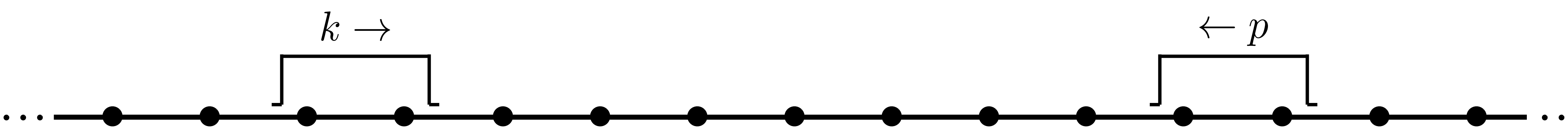
nearest-neighbor: $$\mathcal{U} = J \sum_{(u,v)\in E(G)} \hat{n}_u \hat{n}_v$$

Two-particle scattering

In general, multi-particle scattering is complicated.

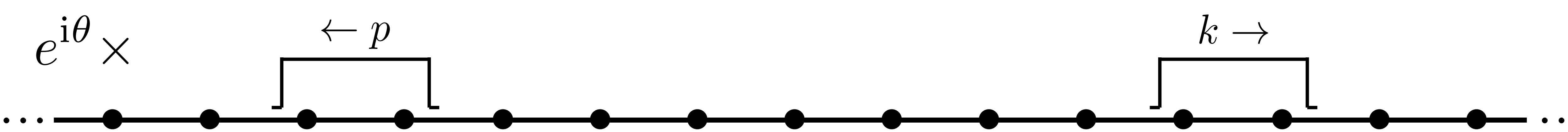
But scattering of indistinguishable particles on an infinite path is simple.

Before:



After:

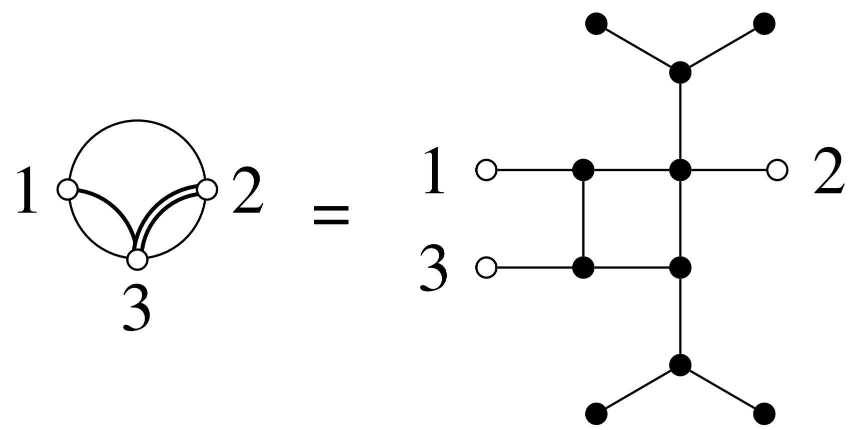
$e^{\mathrm{i}\theta} \times$



Phase θ depends on momenta and interaction details.

Momentum switch

To selectively induce the two-particle scattering phase, we route particles depending on their momentum.

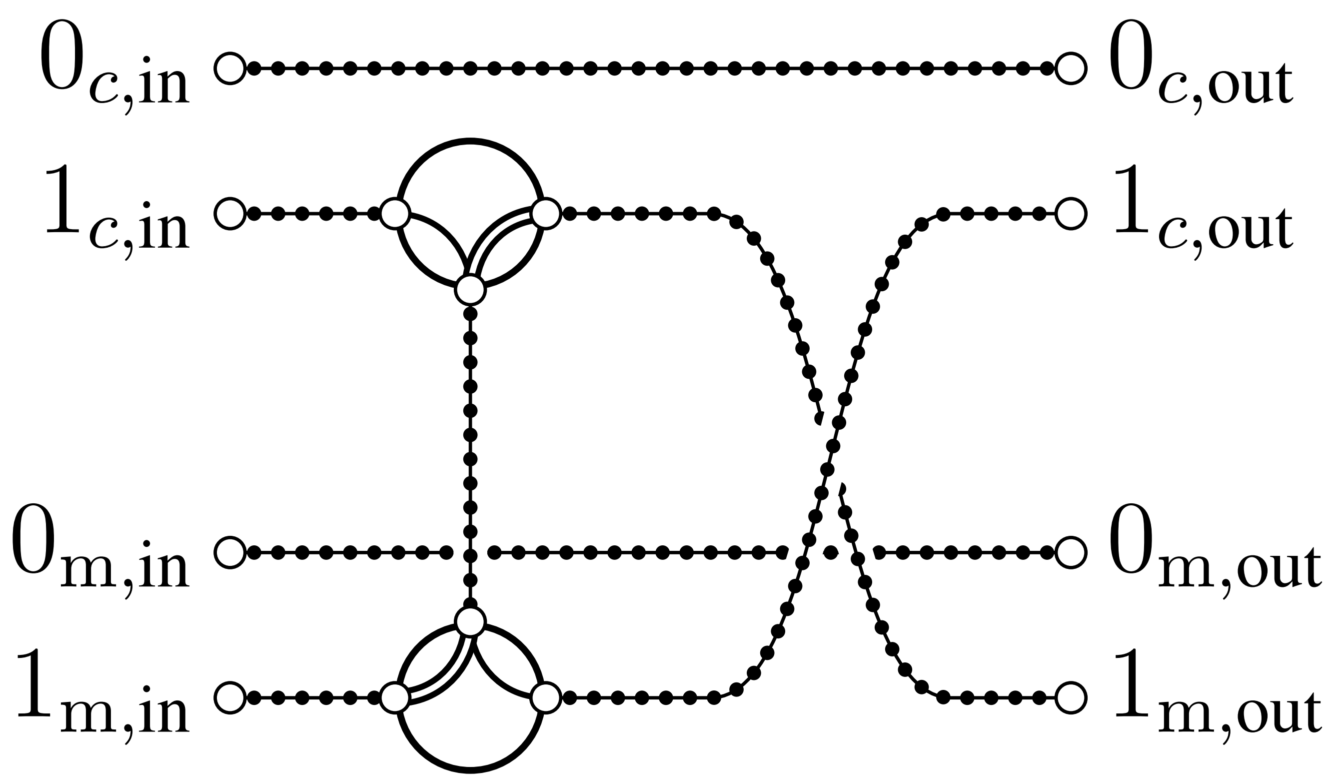


Particles with momentum $\pi/4$ follow the single line.

Particles with momentum $\pi/2$ follow the double line.

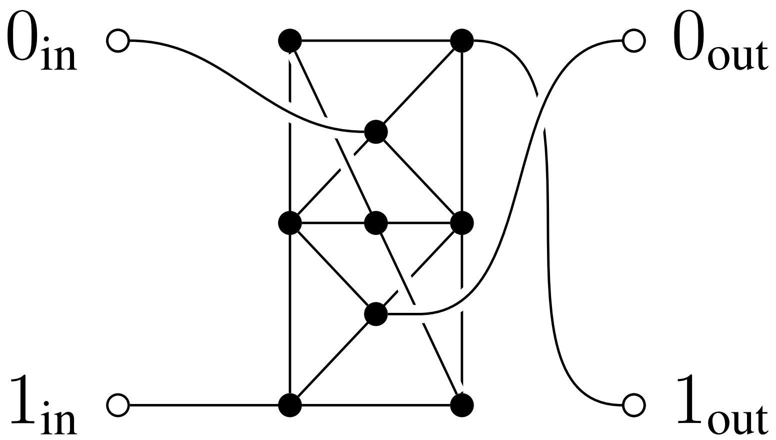
Controlled phase gate

Computational qubits have momentum $\pi/4$. Introduce a "mediator qubit" with momentum $\pi/2$. We can perform an entangling gate with the mediator qubit.



$$\mathrm{C}\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}\theta} \end{pmatrix}$$

Hadamard on mediator qubit

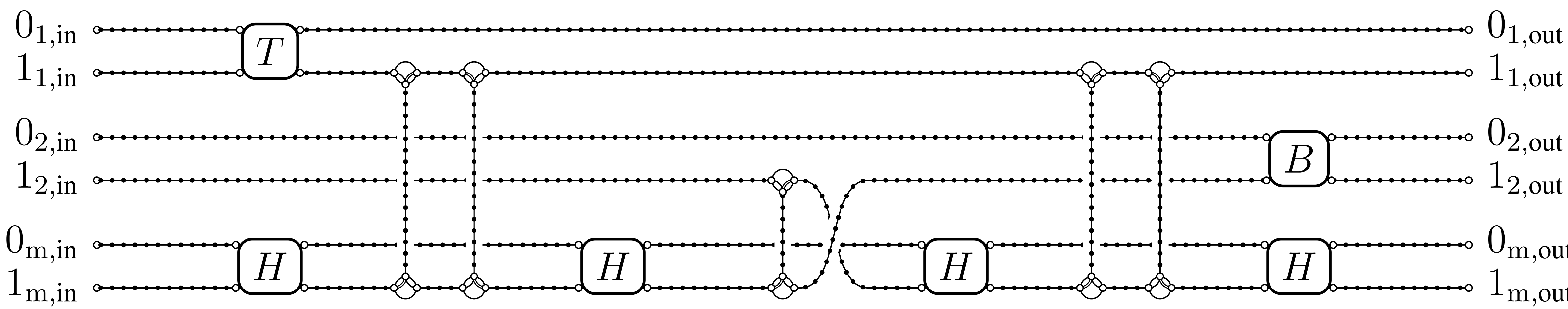


[Blumer, Underwood, Feder 11]

Error bound

Initial state: each particle is a square wave packet of length L

Consider a g-gate, n-qubit circuit:



$2(n+1)$ paths, $O(gL)$ vertices on each path

Evolution time $O(gL)$

Total # of vertices $O(ngL)$

Theorem: The error can be made arbitrarily small with $L = \mathrm{poly}(n, g)$.

Example: For Bose-Hubbard model, $L = O(n^{12} g^4)$ suffices.

Open questions

- Improved error bounds
- Simplified initial state
- Are generic interactions universal for distinguishable particles?
- New quantum algorithms
- Experiments
- Fault tolerance