Overview of adiabatic quantum computation

Andrew Childs
Caltech Institute for Quantum Information

Outline

- Quantum mechanical computers
- Quantum computation and Hamiltonian dynamics
- The adiabatic theorem
- Adiabatic optimization
  - Examples of success
  - Example of failure
  - Random satisfiability problems
- Universal quantum computation

A brief history

- Manin, Feynman, early 1980s: Quantum computers should be good at simulating quantum systems
- Deutsch, 1985: Formal model of quantum computers
- Deutsch, Jozsa, Bernstein, Vazirani, Simon, late 1980s/early 1990s: Examples of problems where quantum computers outperform classical ones
- Shor 1994: Efficient quantum algorithms for factoring and discrete log

Quantum bits

- One qubit: $\mathcal{H} = \mathbb{C}^2$

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle\,, \quad |\alpha_0^2| + |\alpha_1^2| = 1$$

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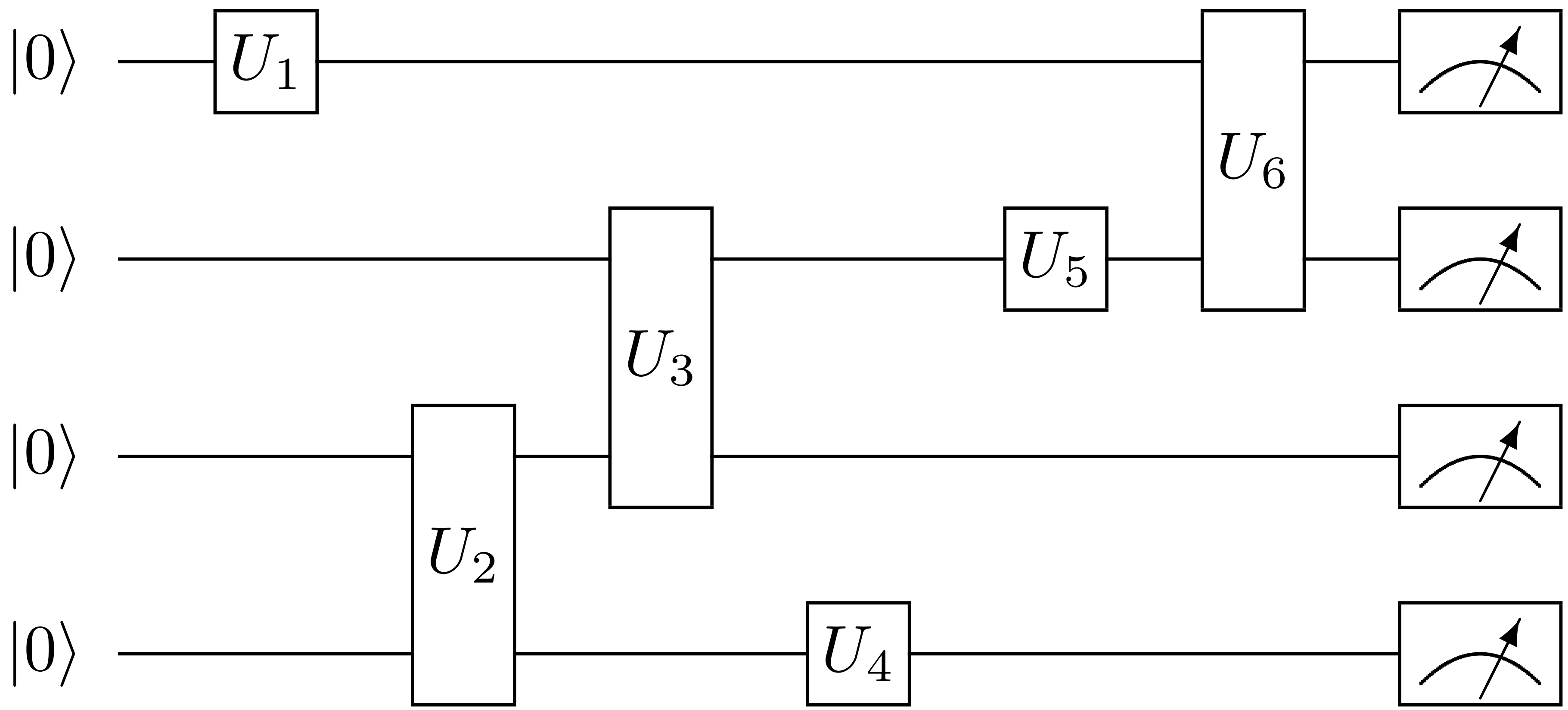
- *n* qubits: $\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n}$

$$|\psi\rangle = \sum_{z\in\{0,1\}^n} \alpha_z\,|z_1\rangle \otimes |z_2\rangle \otimes \cdots \otimes |z_n\rangle$$

$$\sum_{z\in\{0,1\}^n} |\alpha_z|^2 = 1$$

Quantum circuits

- Prepare *n* qubits in the state $|0\cdots0\rangle$
- Apply a sequence of poly(*n*) unitary operations acting on one or two qubits at a time
- Measure in the computational basis to get the result



Three major questions

- How can we build a quantum computer?
  (Implementations)
- How useful is an imperfect quantum computer?
  (Fault tolerance)
- What can we do with a perfect quantum computer?
  (Algorithms)

Hamiltonian dynamics

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

In the circuit model, we say a unitary operation can be implemented efficiently if it can be realized (approximately) by a short sequence of one- and two-qubit gates.

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- Hamiltonians we can efficiently simulate using quantum circuits

Simulating Hamiltonian dynamics

Definition. A Hamiltonian H acting on n qubits can be *efficiently simulated* if for any error $\epsilon>0$ and time $t>0$ there is a quantum circuit U consisting of poly(n, t, $1/\epsilon$) gates such that $\| U - e^{-iHt} \| < \epsilon$.

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Basic idea: Lie product formula

$$e^{-i(H_1+\cdots+H_k)t} = (e^{-iH_1t/r}\cdots e^{-iH_kt/r})^r + O(kt^2\max\{\|H_j\|^2\}/r)$$

Sparse Hamiltonians

Theorem. Suppose that for any fixed *a*, we can efficiently compute all the nonzero values of $\langle a|H|b\rangle$. (In particular, there must be only polynomially many such values.) Then *H* can be simulated efficiently. [Aharonov & Ta-Shma 2003, Childs et al. 2003, Ahokas et al. 2005]

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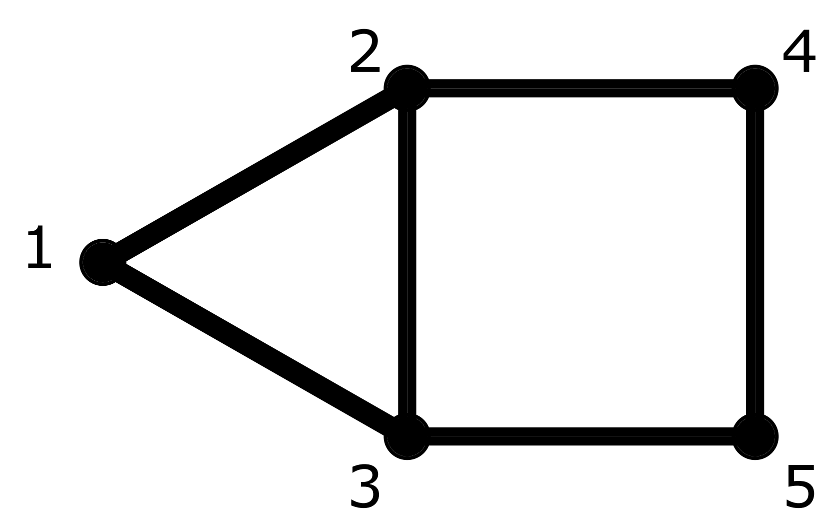

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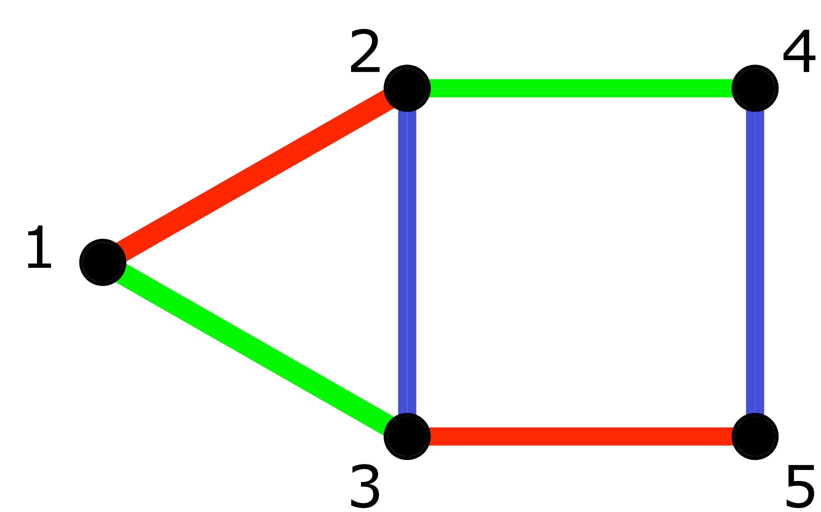

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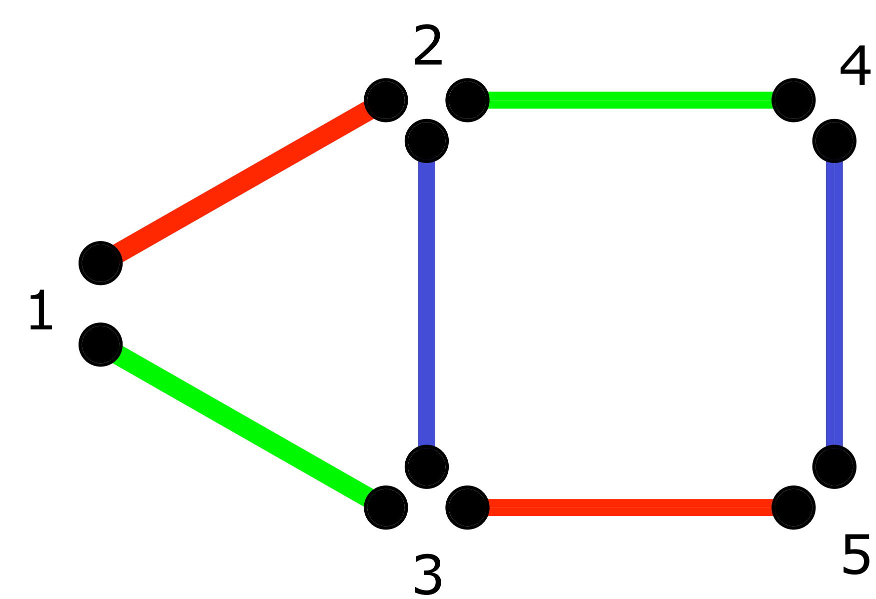
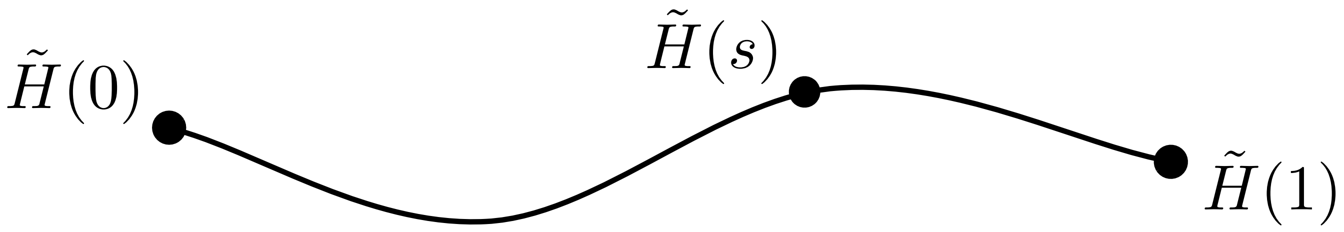


The adiabatic theorem

Let $\tilde{H}(s)$ be a smoothly varying Hamiltonian for $s \in [0,1]$



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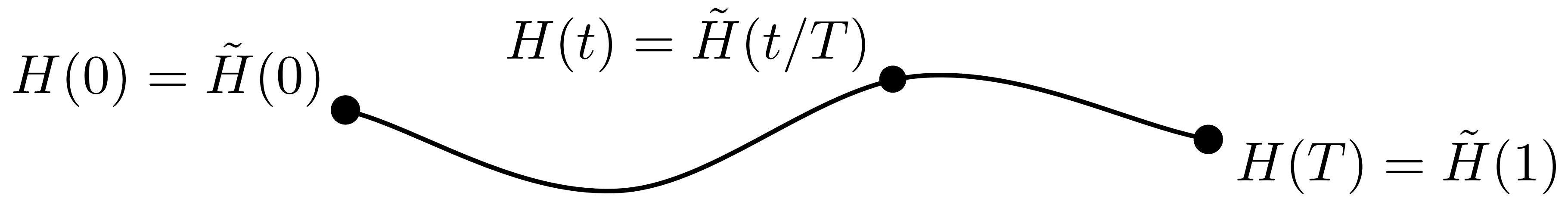

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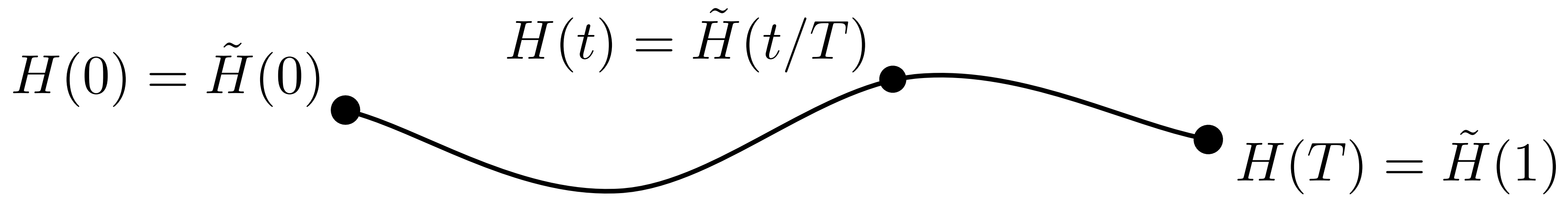

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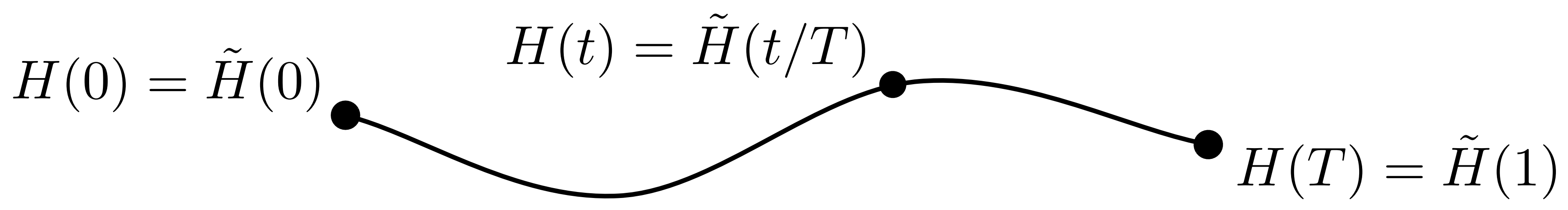

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For large T, $|\psi(T)\rangle \approx |E_0(1)\rangle$. But how large must it be?

Approximately adiabatic evolution

The total run time required for adiabaticity depends on the spectrum of the Hamiltonian.

Gap: $\Delta(s) = E_1(s) - E_0(s)\,, \quad \Delta = \min_{s\in[0,1]} \Delta(s)$

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Theorem. [Teufel 2003 + perturbation theory]

$$T \geq \frac{4}{\epsilon}\left[\frac{\|\dot{\tilde{H}}(0)\|}{\Delta(0)^2} + \frac{\|\dot{\tilde{H}}(1)\|}{\Delta(1)^2} + \int_0^1 \mathrm{d}s \left(10\frac{\|\dot{\tilde{H}}\|^2}{\Delta^3} + \frac{\|\ddot{\tilde{H}}\|}{\Delta}\right)\right]$$

implies $\left\| |\psi(T)\rangle - |E_0(1)\rangle \right\| \leq \epsilon$

Satisfiability problems

- Given *h*: $\{0,1\}^n \rightarrow \{0,1,2,\ldots\}$, is there a value of $z \in \{0,1\}^n$ such that $h(z)=0$?
- Alternatively, what z minimizes $h(z)$?
- Example: 3SAT. $(z_1 \vee z_2 \vee \bar{z}_3) \wedge \cdots \wedge (\bar{z}_{17} \vee z_{37} \vee \bar{z}_{42})$

$$h(z) = \sum_c h_c(z)$$

where $h_c(z) = \begin{cases} 0 & \text{clause } c \text{ satisfied by } z \\ 1 & \text{otherwise} \end{cases}$

Adiabatic optimization

- Define a *problem Hamiltonian* whose ground state encodes the solution:

$$H_P = \sum_{z \in \{0,1\}^n} h(z) |z\rangle\langle z|$$

- Define a beginning Hamiltonian whose ground state is easy to create, for example

$$H_B = -\sum_{j=1}^{n} \sigma_x^{(j)}$$

- Choose $\tilde{H}(s)$ to interpolate from H_B to H_P, for example

$$\tilde{H}(s) = (1-s)H_B + s\,H_P$$

- Choose total run time T so the evolution is nearly adiabatic

[Farhi et al. 2000]

Please mind the gap

Recall rough estimate:

$$T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s\in[0,1]} \left\| \left[\dot{\tilde{H}}(s)\right]^2 \right\|$$

For $\tilde{H}(s) = (1-s)H_B + s\,H_P,$

$$\begin{aligned} \|\dot{\tilde{H}}\| &= \|H_P - H_B\| \\ &\leq \|H_B\| + \|H_P\| \end{aligned}$$

Crucial question: How big is Δ?

- ≥1/poly(*n*): Efficient quantum algorithm
- 1/exp(*n*): Inefficient quantum algorithm

Unstructured search

Finding a needle in a haystack: $h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases}$

(here *h*: {0,1,...,*N*-1}→{0,1})

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Query complexity (given black box for h)

- Classically, $\Theta(N)$ queries
- Quantumly, $O(\sqrt{N})$ queries are sufficient to find w [Grover 1996] $(|z\rangle|a\rangle \mapsto |z\rangle|a \oplus h(z)\rangle)$
- This cannot be improved: $\Omega(\sqrt{N})$ queries are necessary [Bennett et al. 1997]

Example: Adiabatic unstructured search

$$h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases} \Rightarrow \; H_P = \sum_z h(z)|z\rangle\langle z| = 1 - |w\rangle\langle w|$$

Start in $|s\rangle = \frac{1}{\sqrt{N}} \sum_z |z\rangle$

$$H_B = 1 - |s\rangle\langle s|$$

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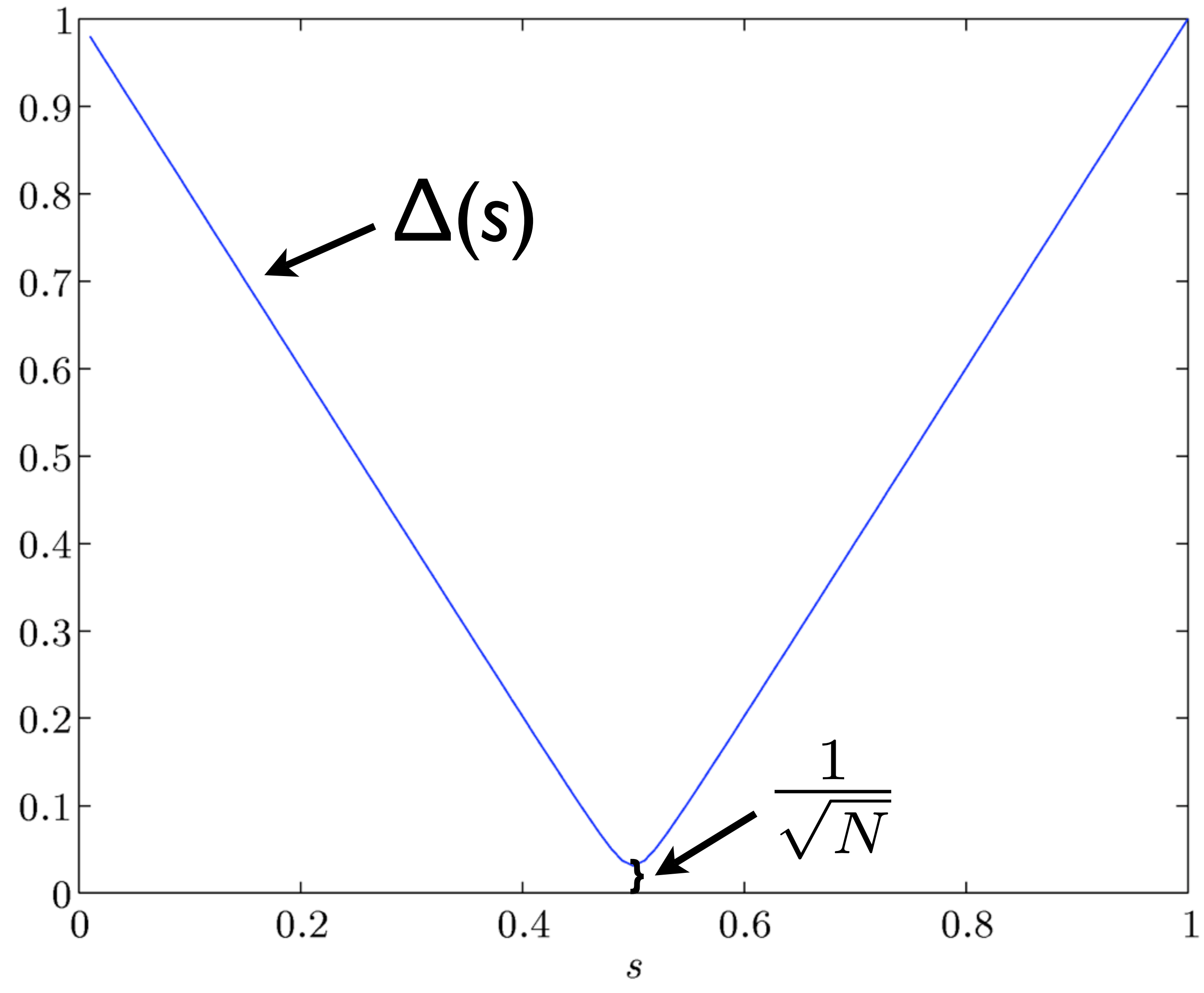

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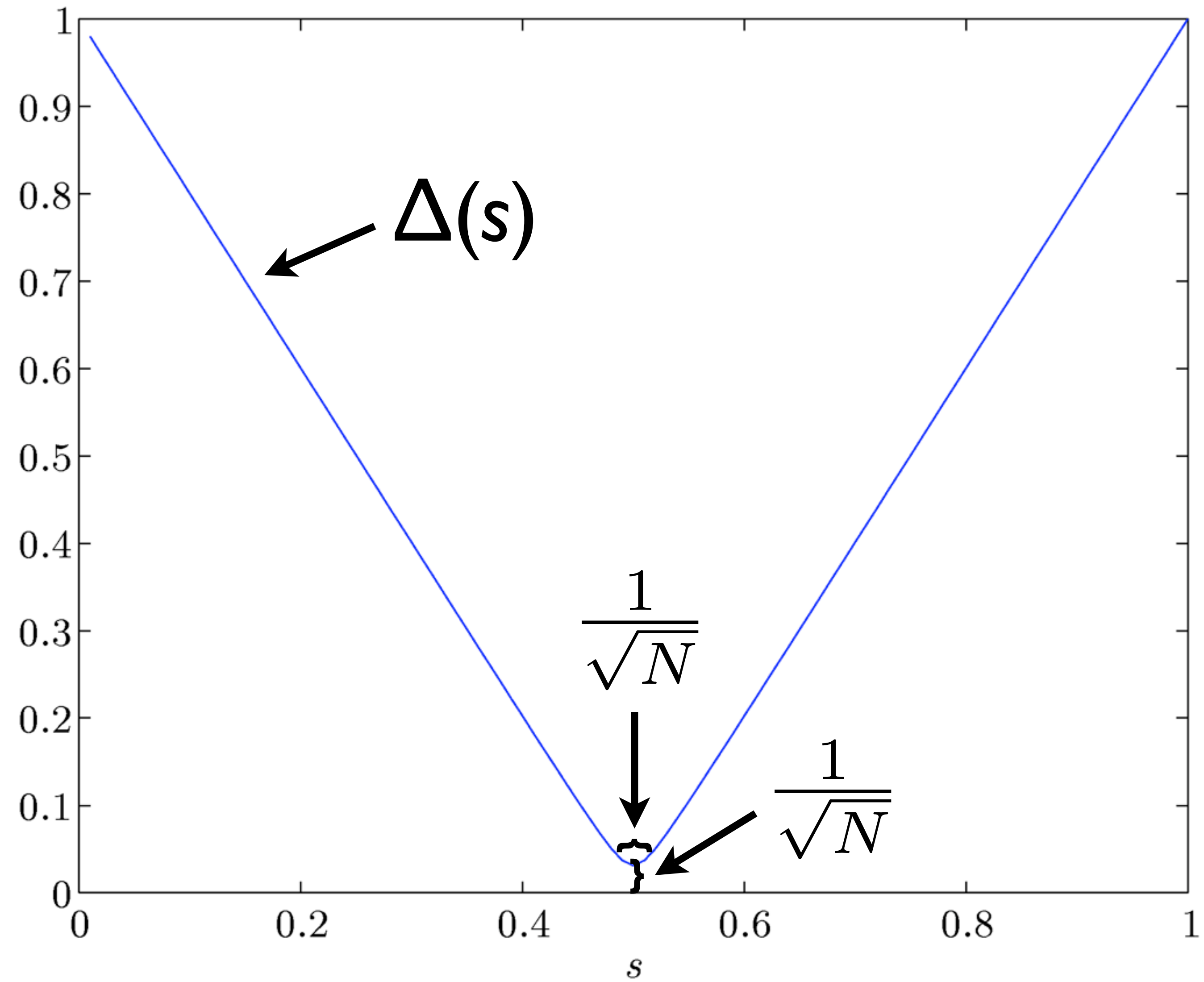

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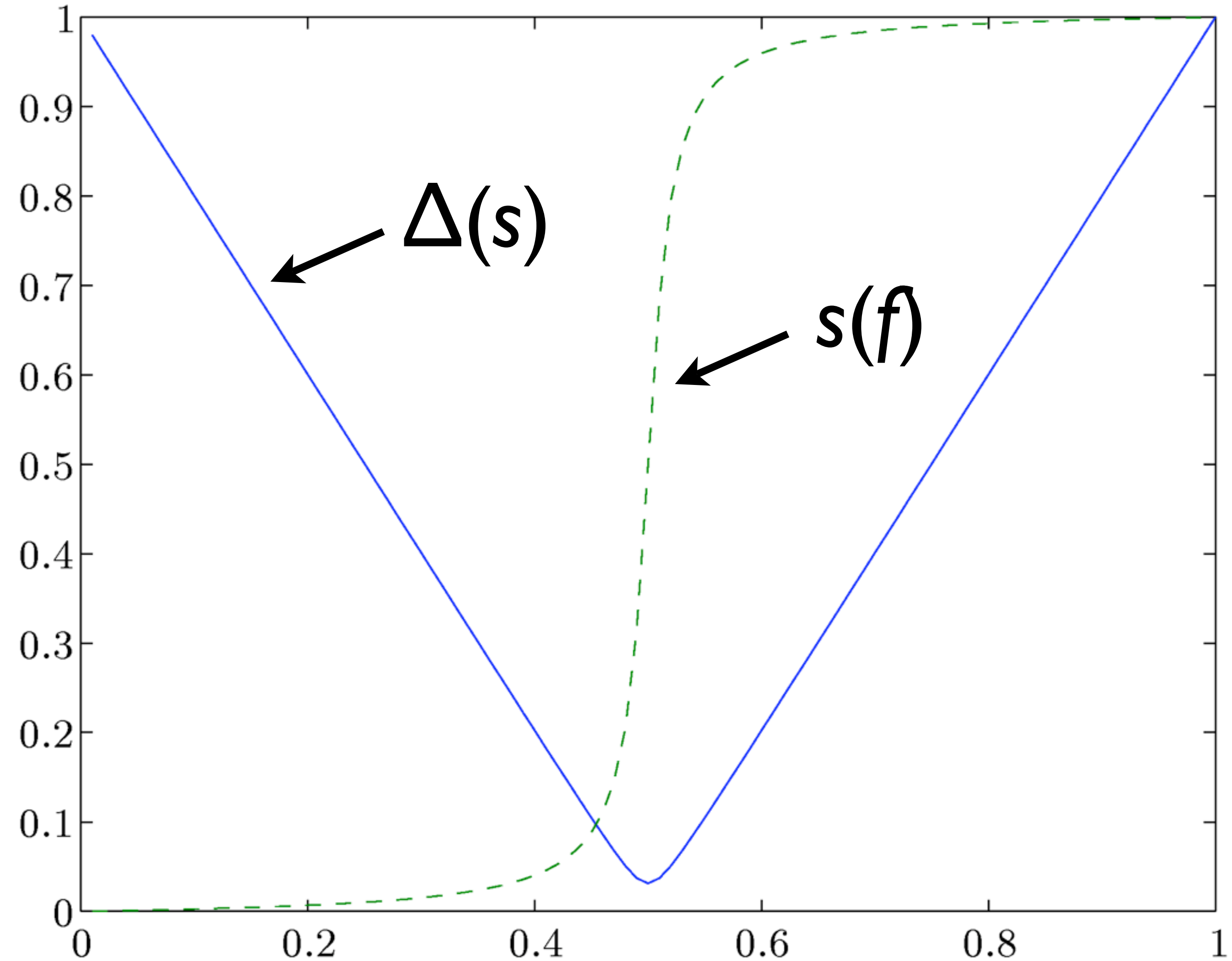
$$\tilde{H}(s) = [1 - f(s)]H_B + f(s)\,H_P$$



[Roland, Cerf 2002; van Dam et al. 2001]

Example: Transverse Ising model

$$H_P = \sum_{j \in \mathbb{Z}_n} \frac{1}{2}\left(1 - \sigma_z^{(j)} \sigma_z^{(j+1)}\right)$$ "agree"

$$H_B = -\sum_{j=1}^{n} \sigma_x^{(j)}$$ with ground state $|s\rangle = |+ \cdots +\rangle = \sum_{z \in \{0,1\}^n} |z\rangle$

$$\tilde{H}(s) = (1-s) H_B + s\, H_P$$

Diagonalize by fermionization (Jordan-Wigner transformation)

Result: $\Delta \propto \frac{1}{n}$ (at critical point of quantum phase transition)

$$|E_0(s \approx 0)\rangle \approx |+ \cdots +\rangle$$

$$|E_0(s \approx 1)\rangle \approx \tfrac{1}{\sqrt{2}}(|0 \cdots 0\rangle + |1 \cdots 1\rangle)$$

[Farhi et al. 2000]

Example: The Fisher problem

$$H_P = \sum_{j \in \mathbb{Z}_n} \frac{J_j}{2} \left(1 - \sigma_z^{(j)} \sigma_z^{(j+1)}\right)$$

J_j=1 or 2, chosen randomly

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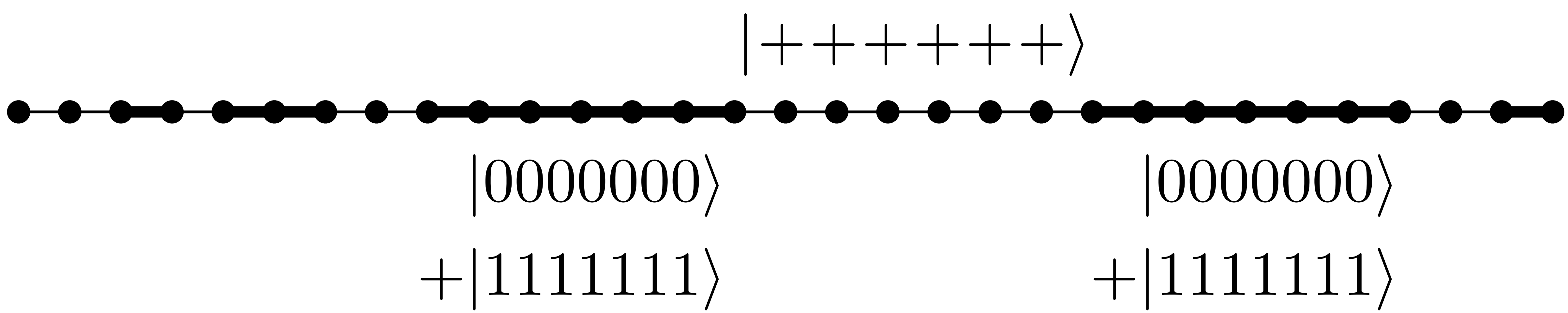

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Random satisfiability problems

Consider random instances of some satisfiability problem (e.g. 3SAT, Exact cover, ...) with a fixed ratio of clauses/bits.

Few clauses: underconstrained. Many solutions, easy to find.
Many clauses: overconstrained. No solutions, easy to find a contradiction.

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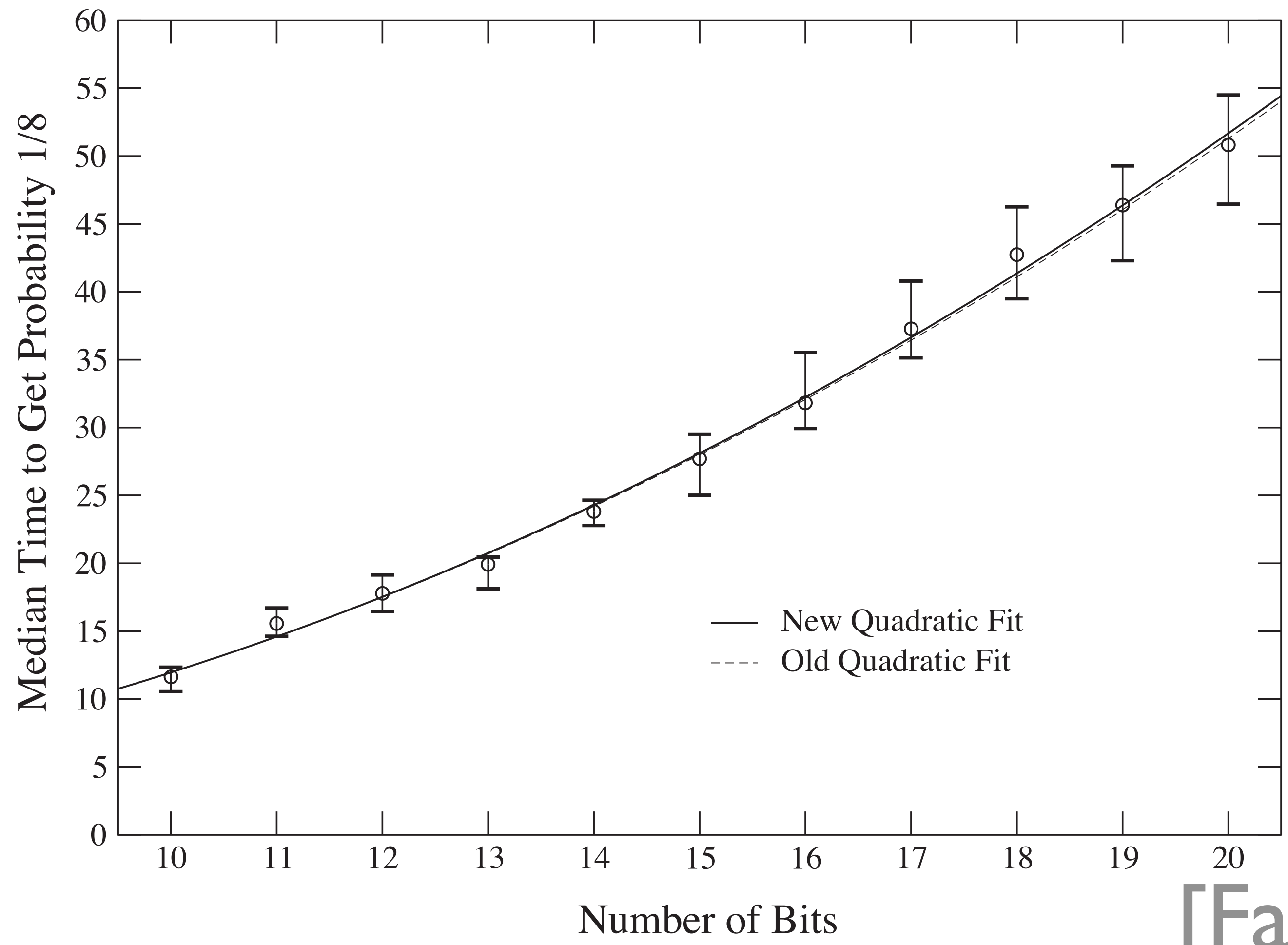
Simulation results for random exact cover instances with unique satisfying assignments:



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Basic idea [Aharonov et al. 2004]: Use this as -H_P.

Final ground state: $$\frac{1}{\sqrt{k}} \sum_{j=1}^{k} U_j U_{j-1} \cdots U_1 |0\rangle \otimes |j\rangle$$

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Note: This is adiabatic, but not adiabatic *optimization*.