Constructing elliptic curve isogenies in quantum subexponential time

Andrew Childs
IQC, C&O

David Jao
C&O

Vladimir Soukharev
C&O

University of Waterloo

Public-key cryptography in the quantum world

Shor 94: Quantum computers can efficiently

- factor integers
- calculate discrete logarithms (in any group)

This breaks two common public-key cryptosystems:

- RSA
- elliptic curve cryptography

Public-key cryptography in the quantum world

Shor 94: Quantum computers can efficiently

- factor integers
- calculate discrete logarithms (in any group)

This breaks two common public-key cryptosystems:

- RSA
- elliptic curve cryptography

How do quantum computers affect the security of PKC in general?

Practical question: we'd like to be able to send confidential information even after quantum computers are built

Theoretical question: crypto is a good setting for exploring the potential strengths/limitations of quantum computers

Isogeny-based elliptic curve cryptography

Not all elliptic curve cryptography is known to be quantumly broken!

Couveignes 97, Rostovstev-Stolbunov 06, Stolbunov 10: Public-key cryptosystems based on the assumption that it is hard to construct an *isogeny* between given elliptic curves over $\mathbb{F}_q$

Best known classical algorithm: $O(q^{1/4})$ [Galbraith, Hess, Smart 02]

Isogeny-based elliptic curve cryptography

Not all elliptic curve cryptography is known to be quantumly broken!

Couveignes 97, Rostovstev-Stolbunov 06, Stolbunov 10: Public-key cryptosystems based on the assumption that it is hard to construct an *isogeny* between given elliptic curves over $\mathbb{F}_q$

Best known classical algorithm: $O(q^{1/4})$ [Galbraith, Hess, Smart 02]

Main result of this talk:

Quantum algorithm that constructs an isogeny in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (assuming GRH), where

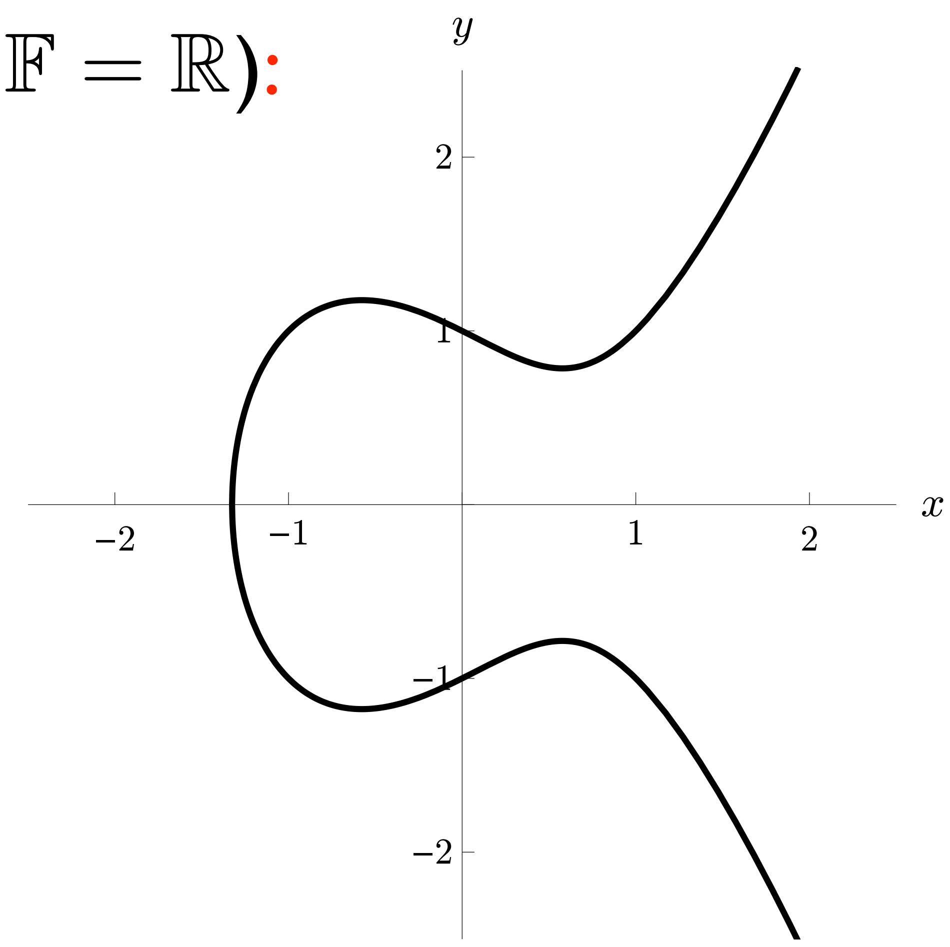
$$L_q(\alpha, c) := \exp\big[(c + o(1))(\ln q)^{\alpha}(\ln \ln q)^{1-\alpha}\big]$$

Elliptic curves

Let $\mathbb{F}$ be a field of characteristic different from 2 or 3

An elliptic curve E is the set of points in $\mathbb{PF}^2$ satisfying an equation of the form $y^2 = x^3 + ax + b$

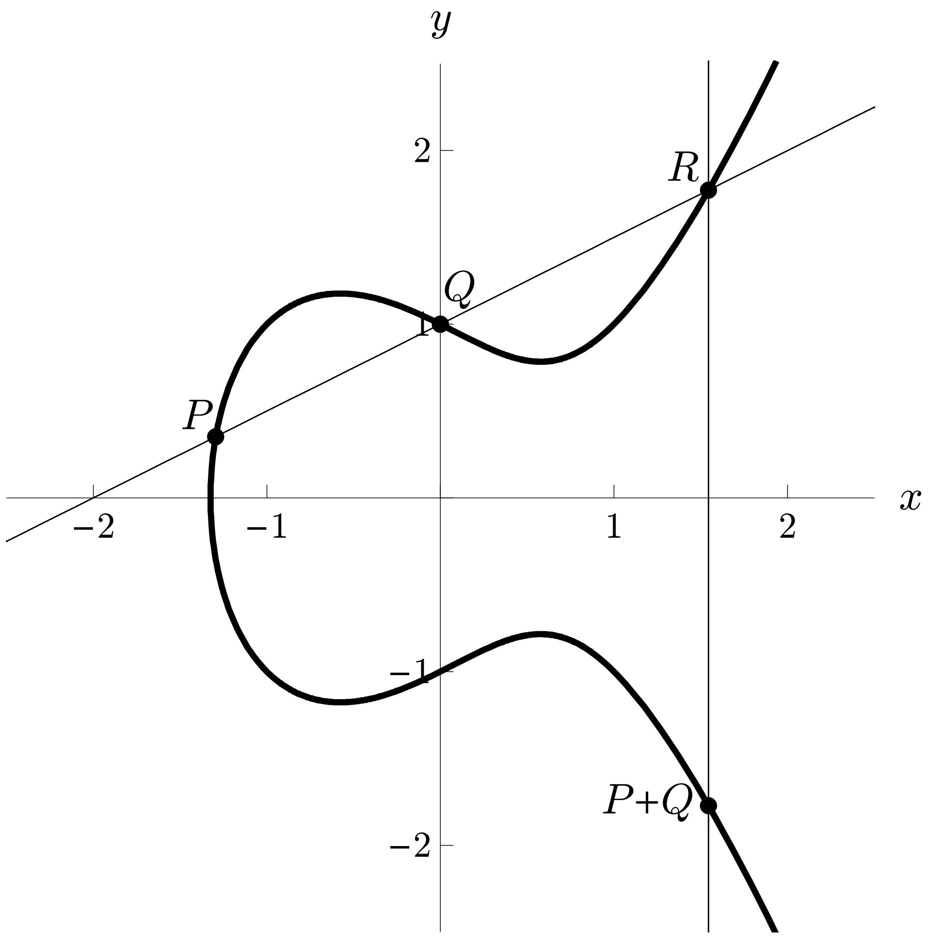
Example ($\mathbb{F} = \mathbb{R}$):



$$y^2 = x^3 - x + 1$$

Elliptic curve group

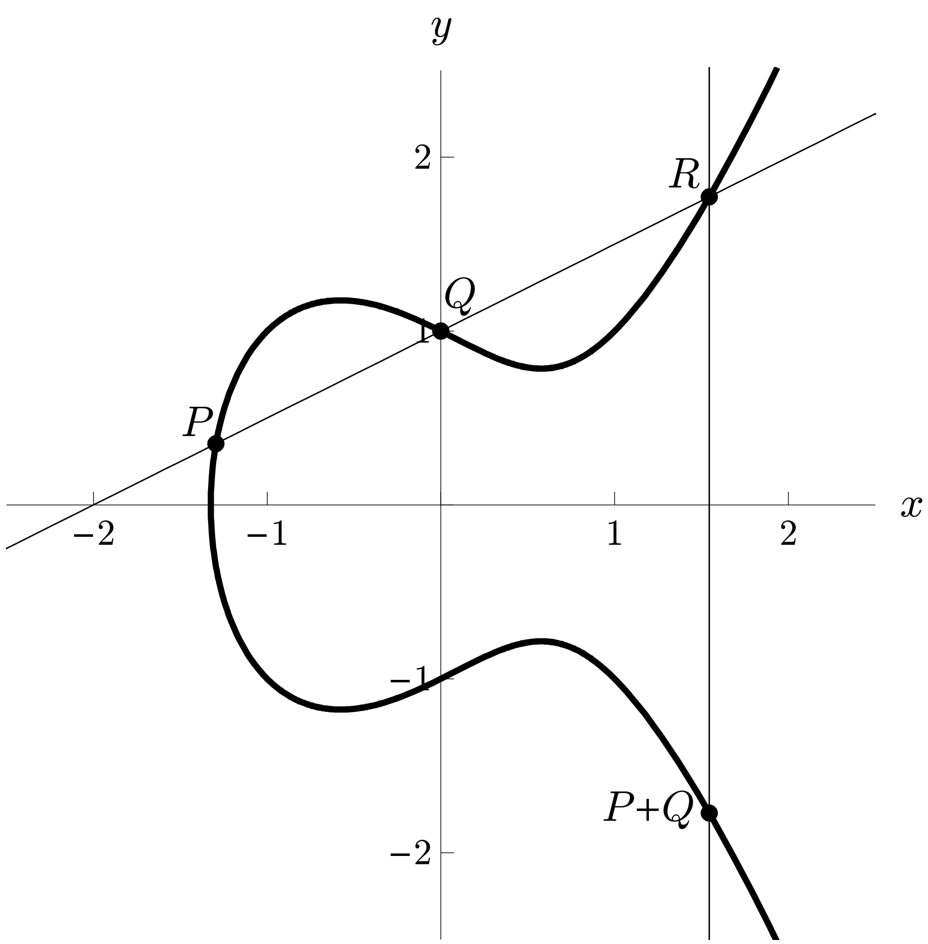
Geometric definition of a binary operation on points of E:



This defines an abelian group with additive identity ∞

Elliptic curve group

Geometric definition of a binary operation on points of E:



Algebraic definition:

for $x_P \neq x_Q$,

$$\lambda := \frac{y_Q - y_P}{x_Q - x_P}$$

$$x_{P+Q} = \lambda^2 - x_P - x_Q$$

$$y_{P+Q} = \lambda(x_P - x_{P+Q}) - y_P$$

(similar expressions for other cases)

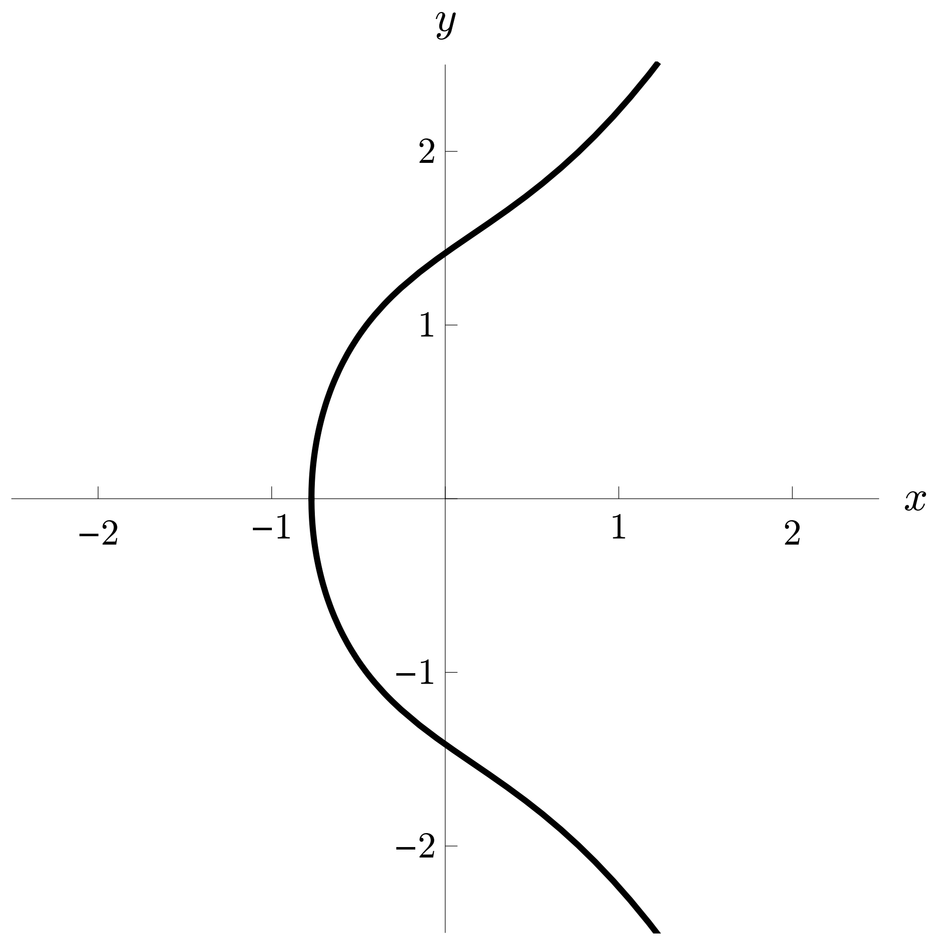
This defines an abelian group with additive identity ∞

Elliptic curves over finite fields

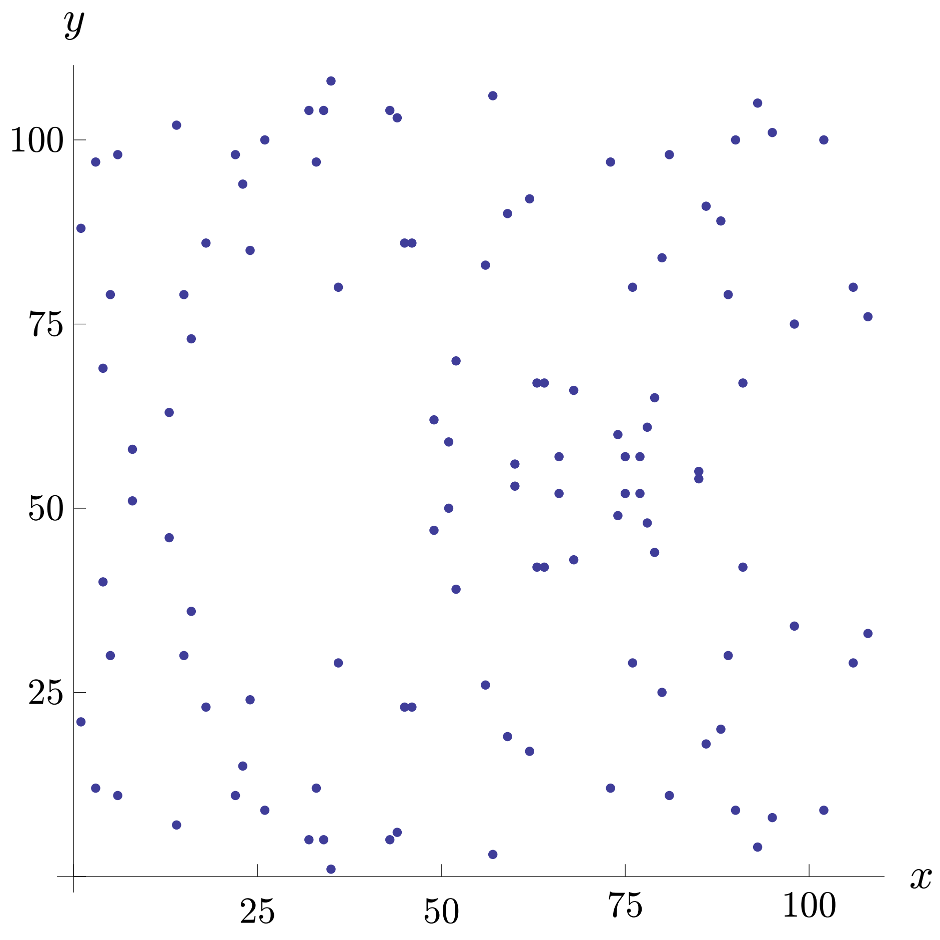
Cryptographic applications use a finite field $\mathbb{F}_q$

Example: $y^2 = x^3 + 2x + 2$

$\mathbb{F} = \mathbb{R}$



$\mathbb{F} = \mathbb{F}_{109}$



Elliptic curve isogenies

Let E_0, E_1 be elliptic curves

An isogeny $\phi : E_0 \to E_1$ is a rational map

$$\phi(x, y) = \left(\frac{f_x(x, y)}{g_x(x, y)}, \frac{f_y(x, y)}{g_y(x, y)} \right)$$

(f_x, f_y, g_x, g_y are polynomials) that is also a group homomorphism:

$$\phi((x, y) + (x', y')) = \phi(x, y) + \phi(x', y')$$

Elliptic curve isogenies

Let E_0, E_1 be elliptic curves

An isogeny $\phi : E_0 \to E_1$ is a rational map

$$\phi(x,y) = \left(\frac{f_x(x,y)}{g_x(x,y)}, \frac{f_y(x,y)}{g_y(x,y)} \right)$$

(f_x, f_y, g_x, g_y are polynomials) that is also a group homomorphism:

$$\phi((x,y) + (x',y')) = \phi(x,y) + \phi(x',y')$$

Example ($\mathbb{F} = \mathbb{F}_{109}$):

$$E_0 : \; y^2 = x^3 + 2x + 2 \qquad \xrightarrow{\phi} \qquad E_1 : \; y^2 = x^3 + 34x + 45$$

$$\phi(x,y) = \left(\frac{x^3 + 20x^2 + 50x + 6}{x^2 + 20x + 100}, \frac{(x^3 + 30x^2 + 23x + 52)y}{x^3 + 30x^2 + 82x + 19} \right)$$

Deciding isogeny

Theorem [Tate 66]: Two elliptic curves over a finite field are isogenous if and only if they have the same number of points.

There is a polynomial-time classical algorithm that counts the points on an elliptic curve [Schoof 85].

Thus a classical computer can decide isogeny in polynomial time.

The endomorphism ring

The set of isogenies from E to itself (over $\bar{\mathbb{F}}$) is denoted $\mathrm{End}(E)$

The endomorphism ring

The set of isogenies from E to itself (over $\bar{\mathbb{F}}$) is denoted $\mathrm{End}(E)$

We assume E is *ordinary* (i.e., not *supersingular*), which is the typical case; then $\mathrm{End}(E) \cong \mathcal{O}_\Delta = \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$ is an imaginary quadratic order of discriminant $\Delta < 0$

The endomorphism ring

The set of isogenies from E to itself (over $\bar{\mathbb{F}}$) is denoted $\mathrm{End}(E)$

We assume E is *ordinary* (i.e., not *supersingular*), which is the typical case; then $\mathrm{End}(E) \cong \mathcal{O}_\Delta = \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$ is an imaginary quadratic order of discriminant $\Delta < 0$

If $\mathrm{End}(E_0) = \mathrm{End}(E_1)$ then we say E_0 and E_1 are *endomorphic*

The endomorphism ring

The set of isogenies from E to itself (over $\bar{\mathbb{F}}$) is denoted $\mathrm{End}(E)$

We assume E is *ordinary* (i.e., not *supersingular*), which is the typical case; then $\mathrm{End}(E) \cong \mathcal{O}_\Delta = \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$ is an imaginary quadratic order of discriminant $\Delta < 0$

If $\mathrm{End}(E_0) = \mathrm{End}(E_1)$ then we say E_0 and E_1 are *endomorphic*

Let $\mathrm{Ell}_{q,n}(\mathcal{O}_\Delta)$ denote the set of elliptic curves over $\mathbb{F}_q$ with n points and endomorphism ring $\mathcal{O}_\Delta$ (up to isomorphism of curves)

Representing isogenies

The degree of an isogeny can be exponential (in $\log q$)

Example: The multiplication by m map,

$$(x, y) \mapsto \underbrace{(x, y) + \cdots + (x, y)}_{m}$$

is an isogeny of degree m^2

Thus we cannot even write down the rational map explicitly in polynomial time

Representing isogenies

The degree of an isogeny can be exponential (in $\log q$)

Example: The multiplication by m map,

$$(x, y) \mapsto \underbrace{(x, y) + \cdots + (x, y)}_{m}$$

is an isogeny of degree m^2

Thus we cannot even write down the rational map explicitly in polynomial time

Fact: Isogenies between endomorphic elliptic curves can be represented by elements of a finite abelian group, the *ideal class group* of the endomorphism ring, denoted $\mathrm{Cl}(\mathcal{O}_\Delta)$

A group action

Thus we can view isogenies in terms of a group action

$$\begin{aligned} *\colon\ \mathrm{Cl}(\mathcal{O}_\Delta) \times \mathrm{Ell}_{q,n}(\mathcal{O}_\Delta) &\to \mathrm{Ell}_{q,n}(\mathcal{O}_\Delta) \\ [\mathfrak{b}] * E &= E_{\mathfrak{b}} \end{aligned}$$

where $E_{\mathfrak{b}}$ is the elliptic curve reached from E by an isogeny corresponding to the ideal class $[\mathfrak{b}]$

A group action

Thus we can view isogenies in terms of a group action

$$\begin{aligned} *\colon\ \mathrm{Cl}(\mathcal{O}_\Delta) \times \mathrm{Ell}_{q,n}(\mathcal{O}_\Delta) &\to \mathrm{Ell}_{q,n}(\mathcal{O}_\Delta) \\ [\mathfrak{b}] * E &= E_{\mathfrak{b}} \end{aligned}$$

where $E_{\mathfrak{b}}$ is the elliptic curve reached from E by an isogeny corresponding to the ideal class $[\mathfrak{b}]$

This action is regular [Waterhouse 69]:
for any E_0, E_1 there is a unique $[\mathfrak{b}]$ such that $[\mathfrak{b}] * E_0 = E_1$

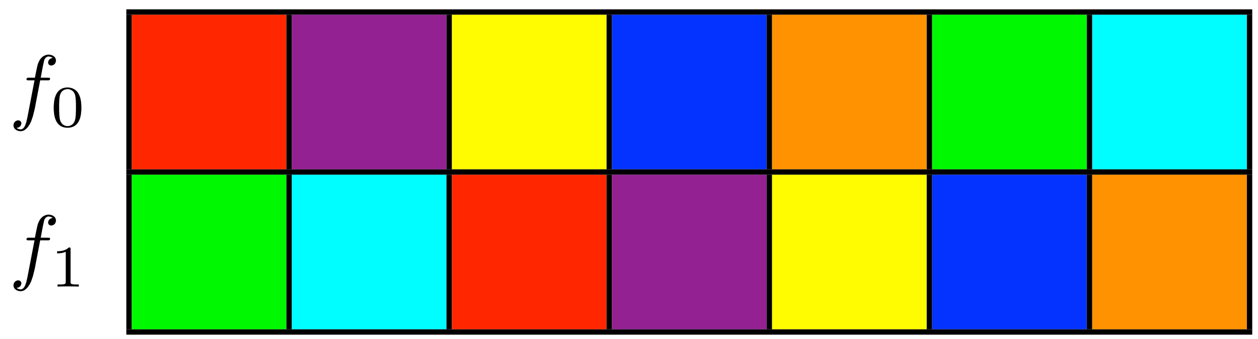
The abelian hidden shift problem

Let A be a known finite abelian group

Let $f_0 : A \to R$ be an injective function (for some finite set R)

Let $f_1 : A \to R$ be defined by $f_1(x) = f_0(xs)$ for some unknown $s \in A$

Problem: find s



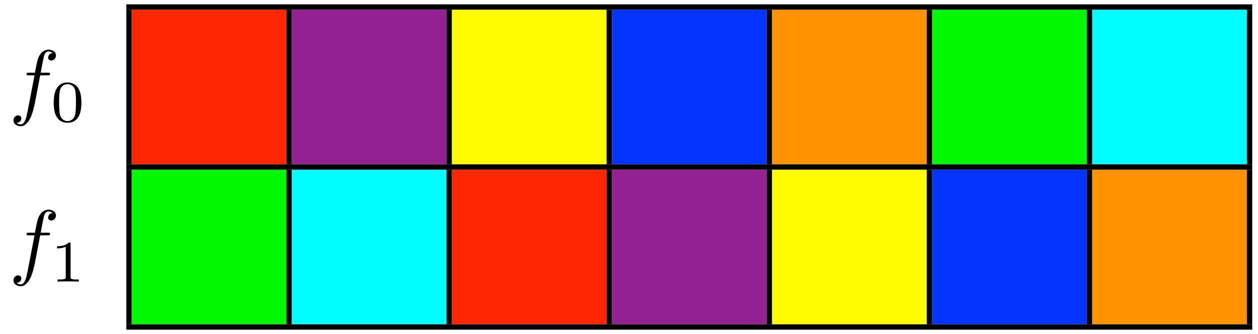
The abelian hidden shift problem

Let A be a known finite abelian group

Let $f_0 : A \to R$ be an injective function (for some finite set R)

Let $f_1 : A \to R$ be defined by $f_1(x) = f_0(xs)$ for some unknown $s \in A$

Problem: find s



For A cyclic, this is equivalent to the dihedral hidden subgroup problem

More generally, this is equivalent to the HSP in the generalized dihedral group $A \rtimes \mathbb{Z}_2$

Isogeny construction as a hidden shift problem

Define $f_0, f_1 : \mathrm{Cl}(\mathcal{O}_\Delta) \to \mathrm{Ell}_{q,n}(\mathcal{O}_\Delta)$ by

$$f_0([\mathfrak{b}]) = [\mathfrak{b}] * E_0$$
$$f_1([\mathfrak{b}]) = [\mathfrak{b}] * E_1$$

E_0, E_1 are isogenous, so there is some $[\mathfrak{s}]$ such that $[\mathfrak{s}] * E_0 = E_1$

Isogeny construction as a hidden shift problem

Define $f_0, f_1 : \mathrm{Cl}(\mathcal{O}_\Delta) \to \mathrm{Ell}_{q,n}(\mathcal{O}_\Delta)$ by

$$f_0([\mathfrak{b}]) = [\mathfrak{b}] * E_0$$
$$f_1([\mathfrak{b}]) = [\mathfrak{b}] * E_1$$

E_0, E_1 are isogenous, so there is some $[\mathfrak{s}]$ such that $[\mathfrak{s}] * E_0 = E_1$

Therefore this is an instance of the hidden shift problem in $\mathrm{Cl}(\mathcal{O}_\Delta)$ with hidden shift $[\mathfrak{s}]$:

- Since $*$ is regular, f_0 is injective
- Since $*$ is a group action, $f_1([\mathfrak{b}]) = f_0([\mathfrak{b}][\mathfrak{s}])$

Kuperberg's algorithm

Theorem [Kuperberg 03]: There is a quantum algorithm that solves the abelian hidden shift problem in a group of order N with running time $\exp[O(\sqrt{\ln N})] = L_N(\frac{1}{2}, 0)$.

Kuperberg's algorithm

Theorem [Kuperberg 03]: There is a quantum algorithm that solves the abelian hidden shift problem in a group of order N with running time $\exp[O(\sqrt{\ln N})] = L_N(\frac{1}{2}, 0)$.

Thus there is a quantum algorithm to construct an isogeny with running time

$$L_N(\tfrac{1}{2}, 0) \times c(N)$$

where $c(N)$ is the cost of evaluating the action

Kuperberg's algorithm

Theorem [Kuperberg 03]: There is a quantum algorithm that solves the abelian hidden shift problem in a group of order N with running time $\exp[O(\sqrt{\ln N})] = L_N(\frac{1}{2}, 0)$.

Thus there is a quantum algorithm to construct an isogeny with running time

$$L_N(\tfrac{1}{2}, 0) \times c(N)$$

where $c(N)$ is the cost of evaluating the action

But previously it was not known how to compute the action in subexponential time

Computing the action

Problem: Given E, Δ, $\mathfrak{b} \in \mathcal{O}_\Delta$, compute $[\mathfrak{b}] * E$

Computing the action

Problem: Given E, Δ, $\mathfrak{b} \in \mathcal{O}_\Delta$, compute $[\mathfrak{b}] * E$

Direct computation (using modular polynomials) takes time $O(\ell^3)$ for an ideal of norm ℓ

Computing the action

Problem: Given E, Δ, $\mathfrak{b} \in \mathcal{O}_\Delta$, compute $[\mathfrak{b}] * E$

Direct computation (using modular polynomials) takes time $O(\ell^3)$ for an ideal of norm ℓ

Instead we use an indirect approach:

- Choose a factor base of small prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_f$
- Find a factorization $[\mathfrak{b}] = [\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_f^{e_f}]$ where $e_1, \ldots, e_f$ are small
- Compute $[\mathfrak{b}] * E$ one small prime at a time

Computing the action

Problem: Given E, Δ, $\mathfrak{b} \in \mathcal{O}_\Delta$, compute $[\mathfrak{b}] * E$

Direct computation (using modular polynomials) takes time $O(\ell^3)$ for an ideal of norm ℓ

Instead we use an indirect approach:

- Choose a factor base of small prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_f$
- Find a factorization $[\mathfrak{b}] = [\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_f^{e_f}]$ where $e_1, \ldots, e_f$ are small
- Compute $[\mathfrak{b}] * E$ one small prime at a time

By optimizing the size of the factor base, this approach can be made to work in time $L(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (assuming GRH)

Computing the action

Problem: Given E, Δ, $\mathfrak{b} \in \mathcal{O}_\Delta$, compute $[\mathfrak{b}] * E$

Direct computation (using modular polynomials) takes time $O(\ell^3)$ for an ideal of norm ℓ

Instead we use an indirect approach:

- Choose a factor base of small prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_f$
- Find a factorization $[\mathfrak{b}] = [\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_f^{e_f}]$ where $e_1, \ldots, e_f$ are small
- Compute $[\mathfrak{b}] * E$ one small prime at a time

By optimizing the size of the factor base, this approach can be made to work in time $L(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (assuming GRH)

Note: This assumes *only* GRH (previous related algorithms required stronger heuristic assumptions)

Polynomial space

Kuperberg's algorithm uses space $\exp[\Theta(\sqrt{\ln N})]$

Regev 04 presented a modified algorithm using only polynomial space for the case $A = \mathbb{Z}_{2^n}$, with running time

$$\exp[O(\sqrt{n \ln n})] = L_{2^n}(\tfrac{1}{2}, O(1))$$

Combining Regev's ideas with techniques used by Kuperberg for the case of a general abelian group (of order N), and performing a careful analysis, we find an algorithm with running time $L_N(\frac{1}{2}, \sqrt{2})$

Thus there is a quantum algorithm to construct elliptic curve isogenies using only polynomial space in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2} + \sqrt{2})$

Conclusions

Given two isogenous, endomorphic, ordinary elliptic curves over $\mathbb{F}_q$, there is a quantum algorithm that constructs an isogeny between them in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (or in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2} + \sqrt{2})$ using $\mathrm{poly}(\log q)$ space)

Conclusions

Given two isogenous, endomorphic, ordinary elliptic curves over $\mathbb{F}_q$, there is a quantum algorithm that constructs an isogeny between them in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (or in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2} + \sqrt{2})$ using $\mathrm{poly}(\log q)$ space)

Consequences:

- Isogeny-based cryptography may be less secure than more mainstream cryptosystems (e.g., lattices)

Conclusions

Given two isogenous, endomorphic, ordinary elliptic curves over $\mathbb{F}_q$, there is a quantum algorithm that constructs an isogeny between them in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (or in time $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2} + \sqrt{2})$ using $\mathrm{poly}(\log q)$ space)

Consequences:

- Isogeny-based cryptography may be less secure than more mainstream cryptosystems (e.g., lattices)
- Computing properties of algebraic curves may be a fruitful direction for new quantum algorithms
  - Can we break isogeny-based cryptography in polynomial time?
  - Computing properties of a single curve (e.g., endomorphism ring)
  - Generalizations: non-endomorphic curves, supersingular curves