The quantum query complexity of read-many formulas

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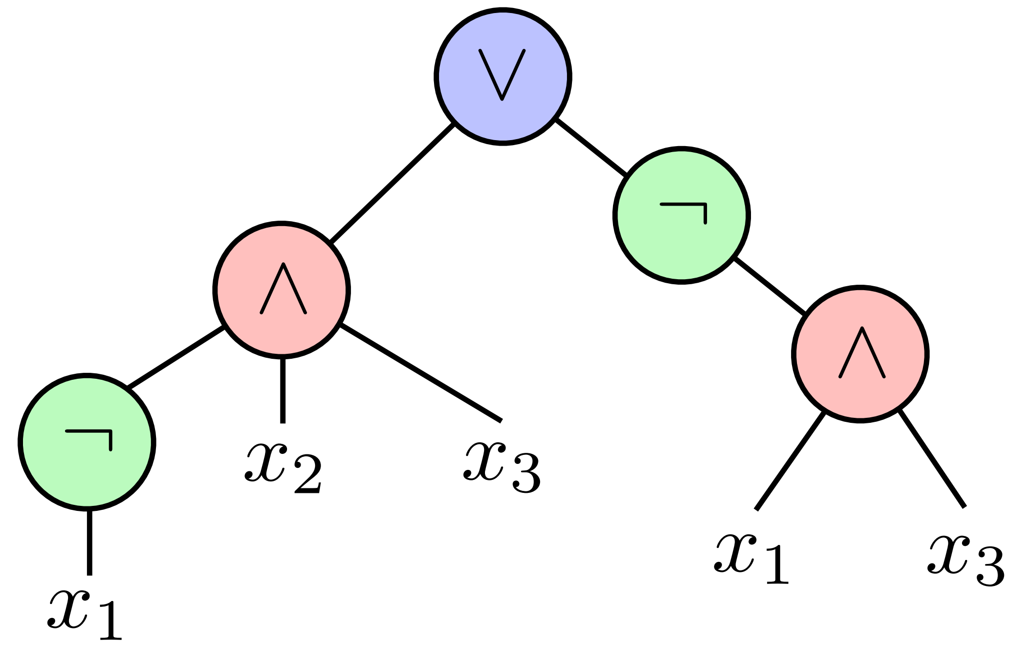
Boolean formulas



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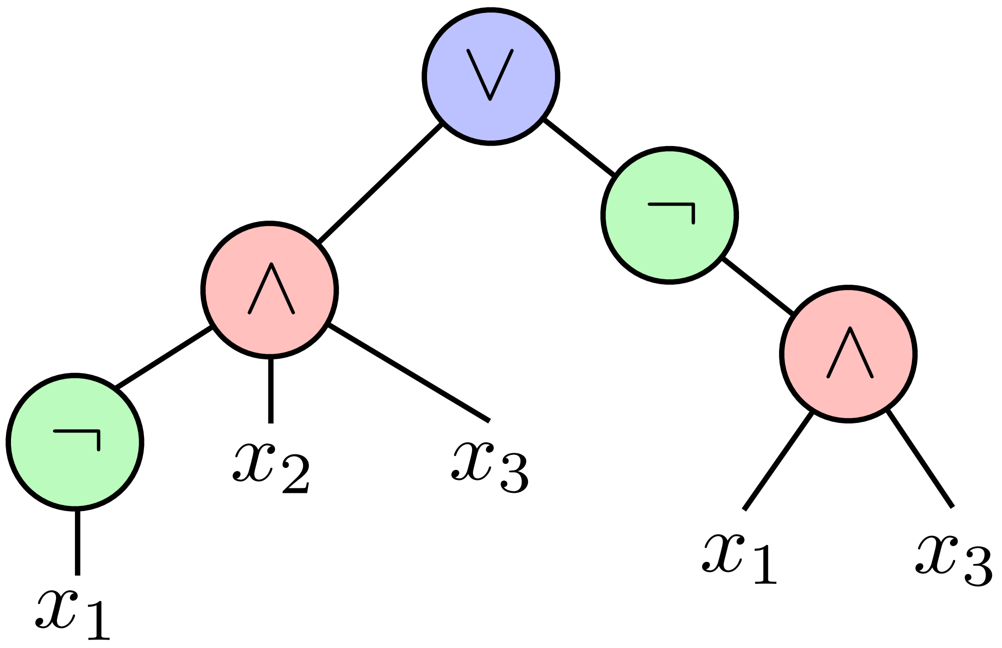
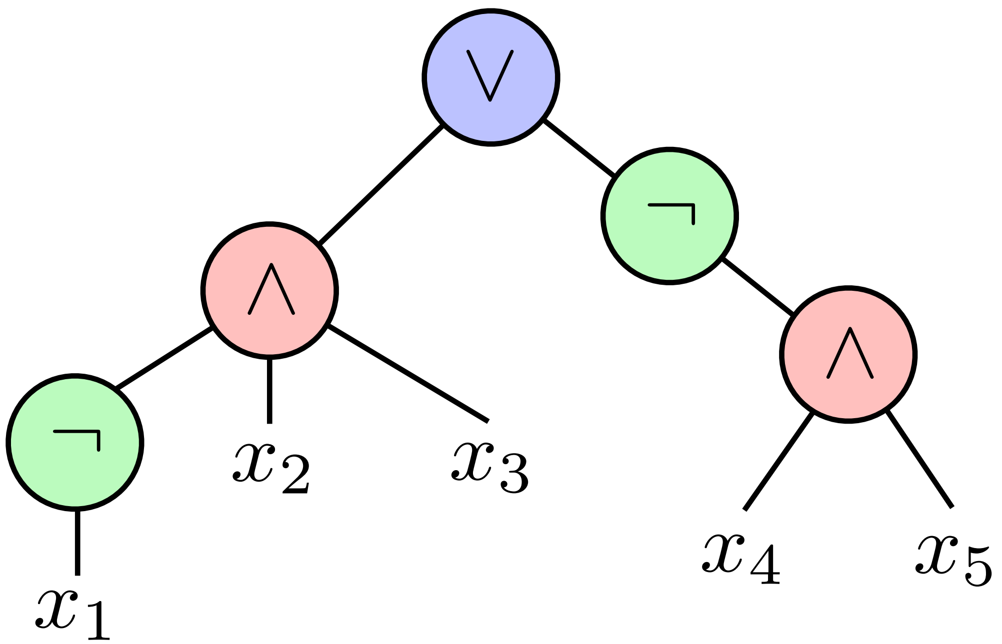


A formula is *read-once* if every input appears at most once.



Evaluating read-once formulas

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Upper bounds:

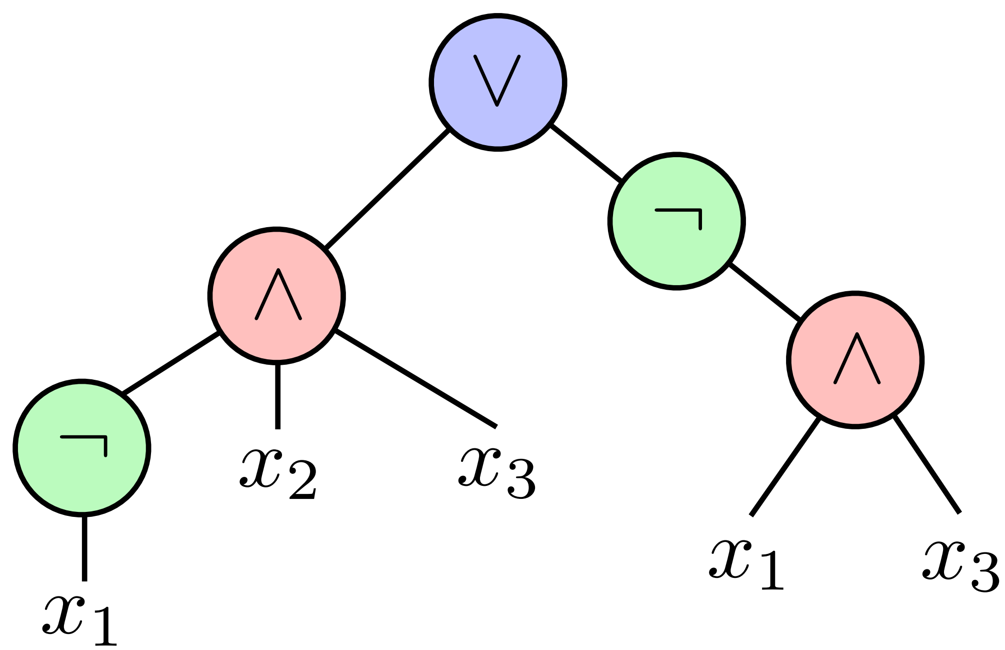
- Grover 96: $O(\sqrt{n})$ for OR
- Buhrman, Cleve, Wigderson 98: $\tilde{O}(\sqrt{n})$ for balanced, constant-depth
- Høyer, Mosca, de Wolf 03: $O(\sqrt{n})$ for balanced, constant-depth
- Farhi, Goldstone, Gutmann 07: $n^{\frac{1}{2}+o(1)}$ for balanced, binary
- Ambainis, Childs, Reichardt, Špalek, Zhang 07: $O(\sqrt{n})$ for approximately balanced formulas, $n^{\frac{1}{2}+o(1)}$ in general
- Reichardt 11: $O(\sqrt{n})$ for any formula

Lower bound:

- Barnum, Saks 04: $\Omega(\sqrt{n})$

Formula size

The size S of a formula is its total number of inputs, counted with multiplicity.



$$n = 3$$
$$S = 5$$

Every Boolean function can be computed by some formula. The formula size is a natural complexity measure.

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Can be expressed by a simple formula:

$$\bigvee_{\text{edges } (v,w)} x_v \wedge x_w$$

n inputs

size $S = 2m = O(n^2)$

More parameters

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Depth: Length of a longest path from the output to an input (not counting NOT gates)

Results

The quantum query complexity of evaluating a formula with n inputs, size S, and G gates is $O(\min\{n, \sqrt{S}, n^{1/2}G^{1/4}\})$.

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There is a depth-2 circuit of linear gate count that requires $\Omega(n^{0.555})$ queries to evaluate (compare $O(n^{3/4})$, trivial lower bound of $\Omega(\sqrt{n})$).

Quantum applications

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$\Omega(n^{1.055})$ lower bound for checking Boolean matrix multiplication

Given $n \times n$ Boolean matrices A, B, C,

decide whether $C_{ij} = \bigvee_{k=1}^{n} A_{ik} \wedge B_{kj}$ for all i, j.

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Constant-depth, bounded-fanout *circuits* with n inputs and G gates (i.e., circuit size G) have query complexity $\tilde{\Theta}(\min\{n, n^{1/2}G^{1/4}\})$.

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Formula *gate count* lower bound of $\Omega(n^2)$ for PARITY (improving over [Khrapchenko 71]).

Constant-depth circuit of size $O(n)$ that requires $\Omega(n^{2-\epsilon})$ gates to express as a formula.

(Best previous result we know of this kind gave a similar lower bound for formula size [Nechiporuk 66, Jukna 12], which is weaker.)

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Lemma: Using $O(n^{1/2}G^{1/4})$ queries, we can produce a formula of size $O(n\sqrt{G})$ with the same value on the given input.

Then apply the read-once formula evaluation algorithm.

Pruning a formula

Call an input *high-degree* if it feeds into more than $\sqrt{G}$ OR gates.

Repeatedly search for a marked high-degree input.

- We delete at least $\sqrt{G}$ OR gates each time, so we repeat $k = O(\sqrt{G})$ times.
- jth iteration takes time $O(\sqrt{n/m_j})$, where m_j is the number of marked high-degree inputs
- m_j decreases each step $\Rightarrow m_{k-j} \geq j$
- Total query complexity: $\sum_{j=1}^{O(\sqrt{G})} O\left(\sqrt{\frac{n}{j}}\right) = O(n^{1/2}G^{1/4})$

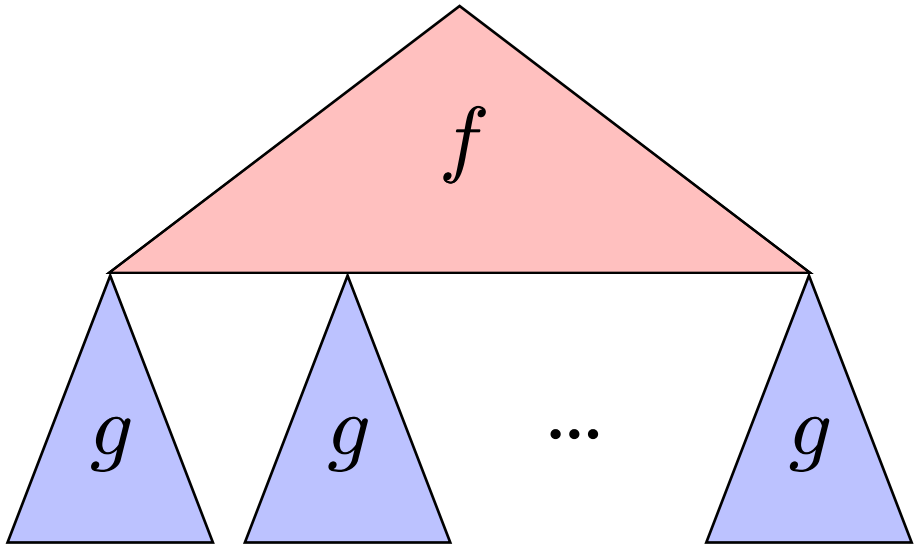
When there are no marked high-degree inputs, we can delete all wires from high-degree inputs to OR gates.

Same thing for AND gates.

Every input has degree at most $\sqrt{G} \Rightarrow$ formula size is $O(n\sqrt{G})$.

Note: No log factors in the analysis.

Lower bounds for composed formulas



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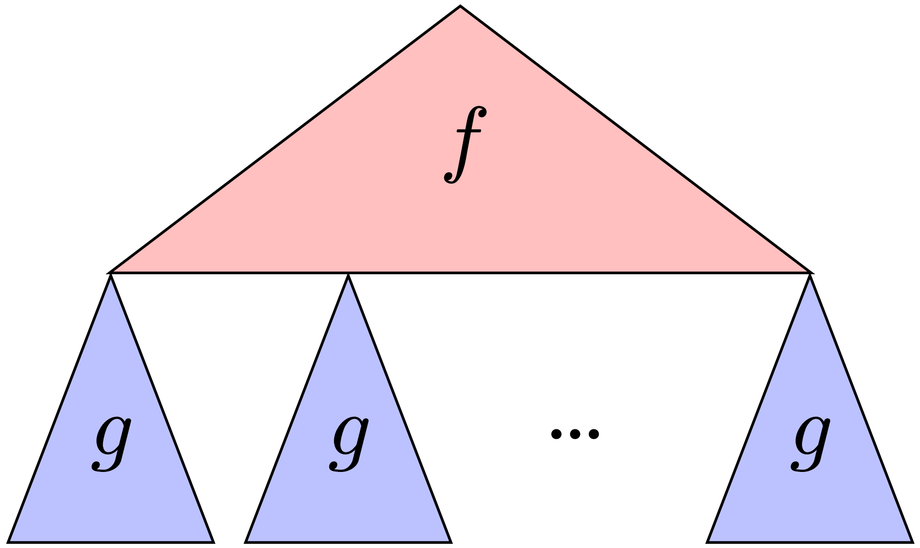

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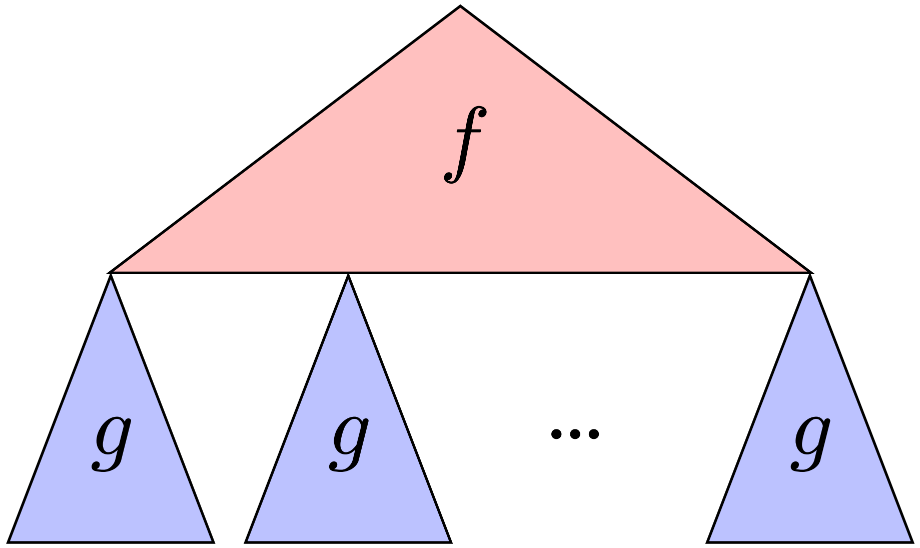

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Lemma: Let f, g be circuits with n_f, n_g inputs, depth k_f, k_g, size G_f, G_g. Then there exists a circuit h with $n_h = 4n_f n_g$ inputs, depth $k_h = k_f + k_g - 1$, size $G_h \leq 2G_f + 4n_f G_g$, such that $Q(h) = \Omega(Q(f)Q(g))$. Furthermore, if f is a formula and $k_g = 1$, then h is a formula of size $S_h = S_f S_g$.

Optimality of the formula evaluation algorithm

Claim: For any n, S, G, there is a formula with n inputs, size at most S, and at most G gates that requires $\Omega(\min\{n, \sqrt{S}, n^{1/2}G^{1/4}\})$ queries to evaluate.

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If the min is n, consider PARITY:

Query complexity $\Omega(n)$ [BBCMW 98, FGGS 98]

Formula size $O(n^2)$ (use $x \oplus y = (x \wedge \bar{y}) \vee (\bar{x} \wedge y)$ recursively)

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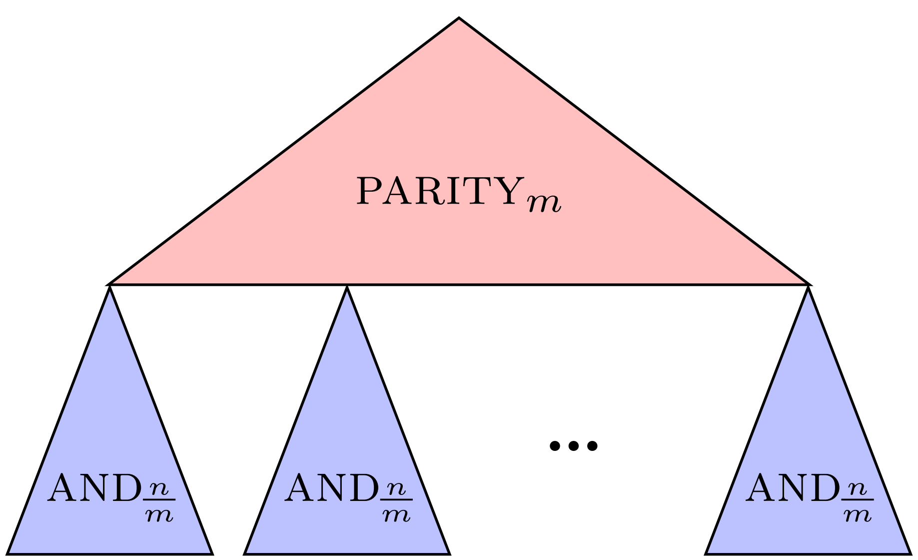

$\Theta(n)$ inputs

size $S = O(m^2(n/m)) = O(nm)$

gate count $G = O(m^2)$

query complexity $\Omega(n\sqrt{n/m}) = \Omega(\sqrt{nm})$

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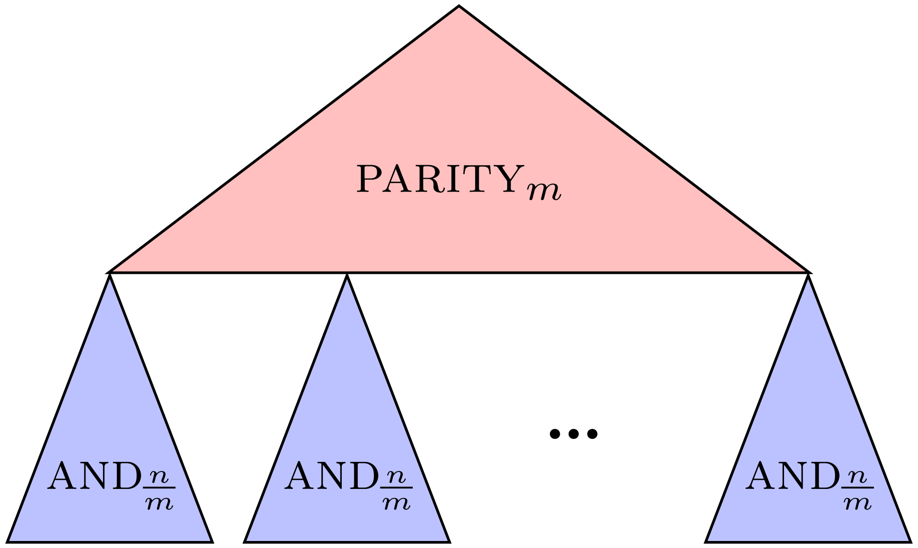

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Choosing m appropriately gives the desired result.

($m = S/n$ if the min is $\sqrt{S}$; $m = \sqrt{G}$ if the min is $n^{1/2}G^{1/4}$)

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Depth-3 formula of size $\tilde{\Theta}(N^2)$: $\textsc{onto}(f) = \bigwedge_{j \in [n]} \bigvee_{i \in [2n-2]} \bigwedge_{\ell=0}^{\log n - 1} f(i)_\ell^{j_\ell}$

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Using this in place of PARITY gives the same lower bounds for depth-3 formulas, up to a log factor.

Depth-2 formulas

Element distinctness

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Using composition to produce a circuit of size n gives a lower bound of $\tilde{\Omega}(n^{5/9}) = \Omega(n^{0.555})$.

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Claim: Checking whether $C = AB$ requires $\Omega(n^{1.055})$ queries to the entries of A, B, C.

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$AB = J$ is the logical AND of n instances of the above problem

$\Rightarrow$ lower bound of $\tilde{\Omega}(\sqrt{n} \cdot n^{5/9}) = \tilde{\Omega}(n^{19/18}) = \Omega(n^{1.055})$

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Since $G < S$, this improves the classic result that the formula size of PARITY is $\Omega(n^2)$ [Khrapchenko 71].

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We show that there is a constant-depth circuit of linear size that requires $\Omega(n^{2-\epsilon})$ *gates* to express as a formula.

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Main idea:

- ONTO has query complexity $\tilde{\Omega}(n)$, circuit size $\tilde{O}(n^2)$
- Recursively composing ONTO with itself gives a circuit with smaller size but nearly the same query complexity

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Graph collision as a depth-2 circuit of quadratic size or a depth-3 circuit of linear size