Universal computation by quantum walk

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A Markov process on a graph $G = (V, E)$.

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Stochastic matrix $W \in \mathbb{R}^{|V| \times |V|}$ $(\sum_k W_{kj} = 1)$

with $W_{kj} \neq 0$ iff $(j, k) \in E$

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probability of taking a step from j to k

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Ex: Simple random walk. $W_{kj} = \begin{cases} \frac{1}{\deg j} & (j, k) \in E \\ 0 & (j, k) \notin E \end{cases}$

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Generator matrix $M \in \mathbb{R}^{|V| \times |V|}$ $(\sum_k M_{kj} = 0)$

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Ex: Laplacian walk. $M_{kj} = L_{kj} = \begin{cases} -\deg j & j = k \\ 1 & (j, k) \in E \\ 0 & (j, k) \notin E \end{cases}$

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Ex: Adjacency matrix. $H_{kj} = A_{kj} = \begin{cases} 1 & (j,k) \in E \\ 0 & (j,k) \notin E \end{cases}$

Aside: Discrete-time quantum walk

We can also define a quantum walk that proceeds by discrete steps.

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In this talk we will focus on the continuous-time model.

Quantum walk algorithms

- Exponential speedup for black box graph traversal [CCDFGS 03]
- Search on graphs [Shenvi, Kempe, Whaley 02], [CG 03, 04], [Ambainis, Kempe, Rivosh 04]
- Element distinctness [Ambainis 03]
- Triangle finding [Magniez, Santha, Szegedy 03]
- Checking matrix multiplication [Buhrman, Špalek 04]
- Testing group commutativity [Magniez, Nayak 05]
- Formula evaluation [Farhi, Goldstone, Gutmann 07], [ACRSZ 07], [Cleve, Gavinsky, Yeung 07], [Reichardt, Špalek 08]
- Unstructured search (many applications) [Grover 96], ...

The question

How powerful is quantum walk?

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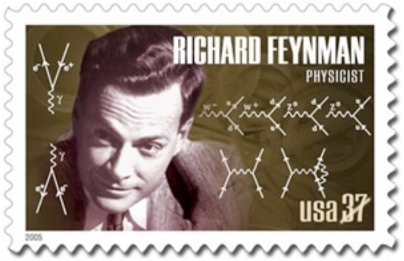
Loosely interpreted (any fixed Hamiltonian): Yes! [Feynman 85]



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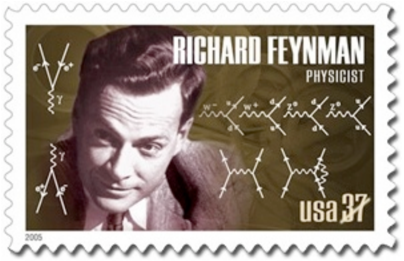


But what if we take the narrowest possible interpretation?

Continuous-time quantum walk on a constant-degree graph,
Hamiltonian given by the adjacency matrix (no edge weights)

The plan

- Scattering theory on graphs
- Gate widgets
- Simplifying the initial state: Momentum filtering and separation

Scattering theory

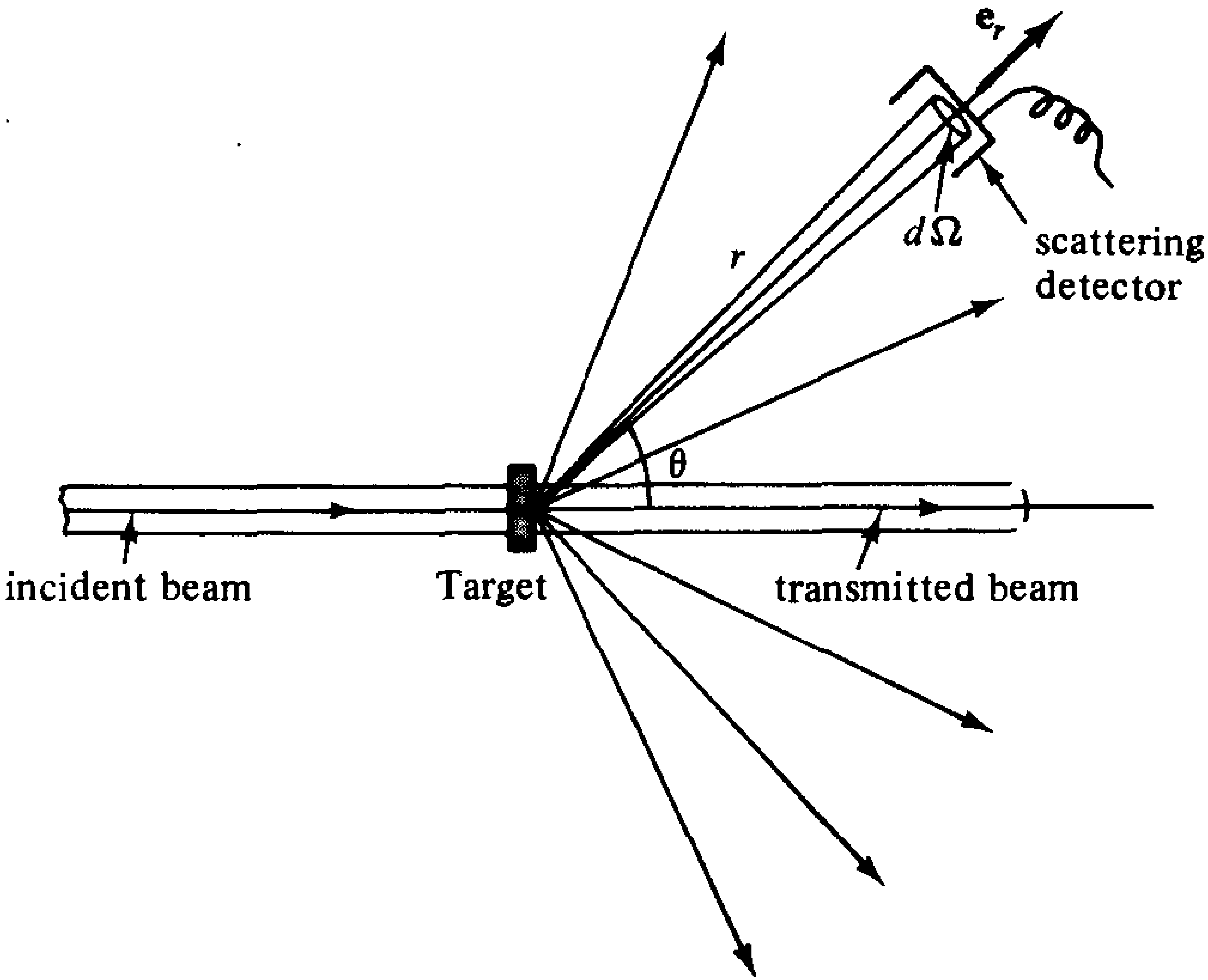
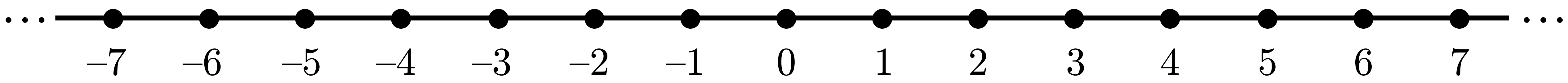


FIGURE 14.1 Scatterhıg configuration.

[Liboff, *Introductory Quantum Mechanics*]

Momentum states

Consider an infinite line:



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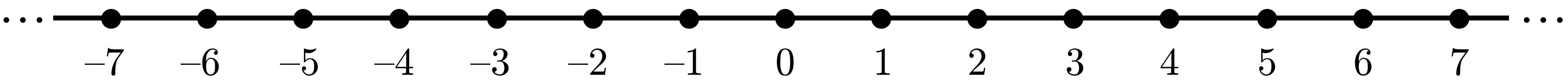

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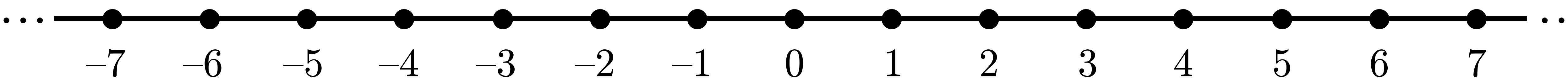

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$$\langle x|\tilde{k}\rangle := e^{\mathrm{i}kx} \qquad k \in [-\pi, \pi)$$

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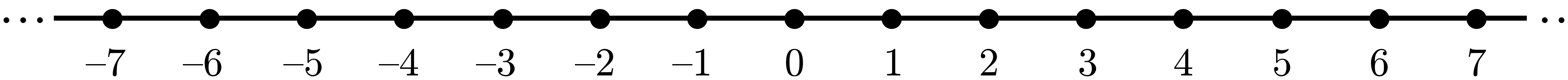

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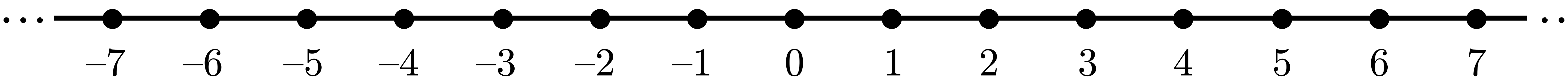

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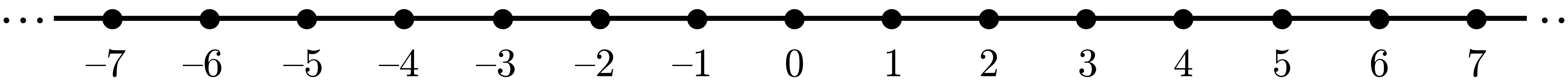

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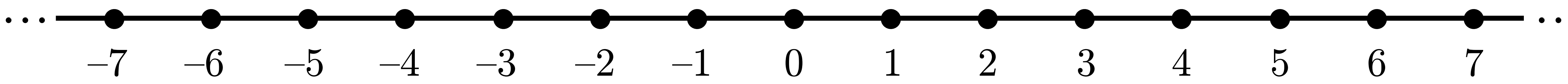

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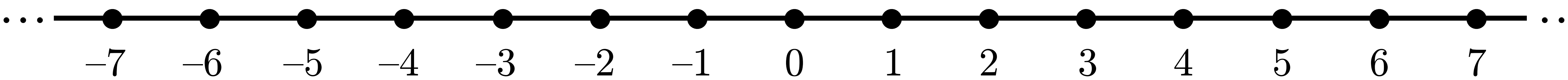

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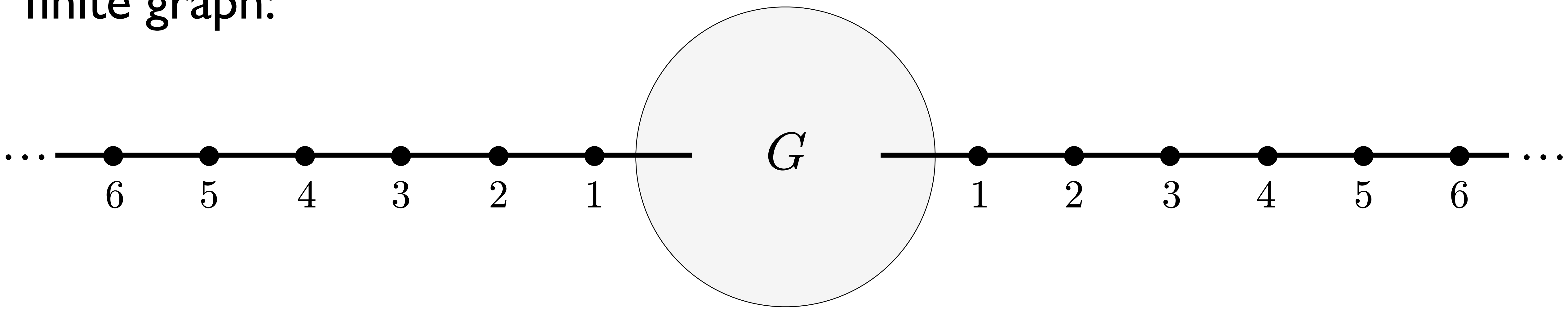
so this is an eigenstate with eigenvalue $2\cos k$.

Scattering on graphs

Now consider adding semi-infinite lines to two vertices of an arbitrary finite graph:



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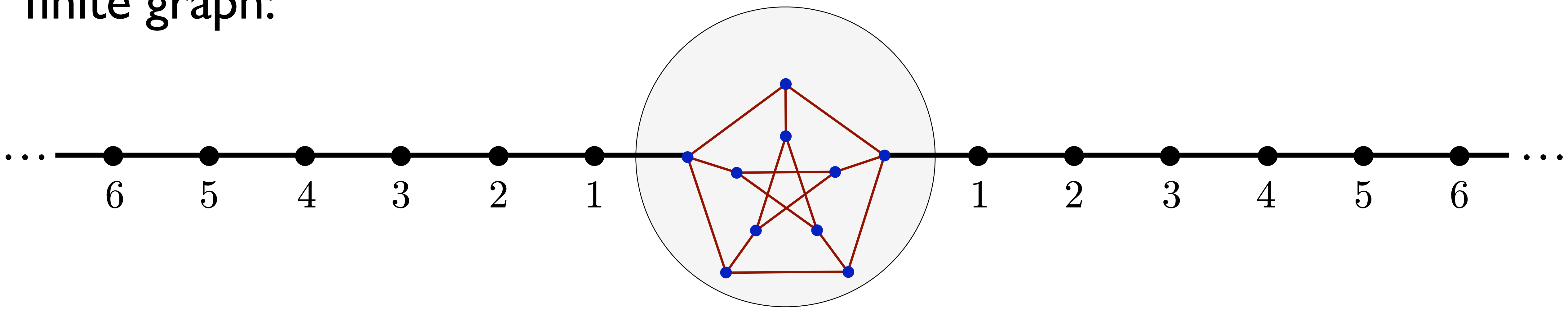

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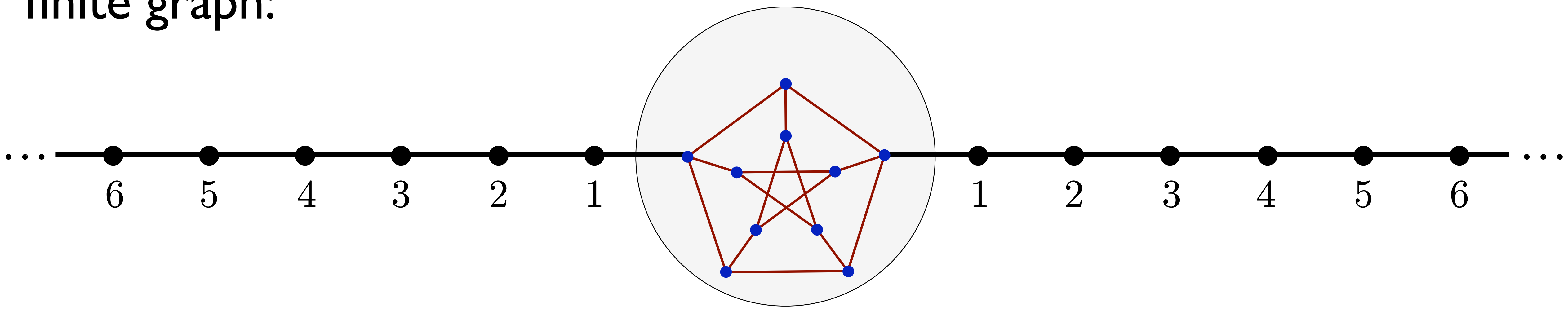

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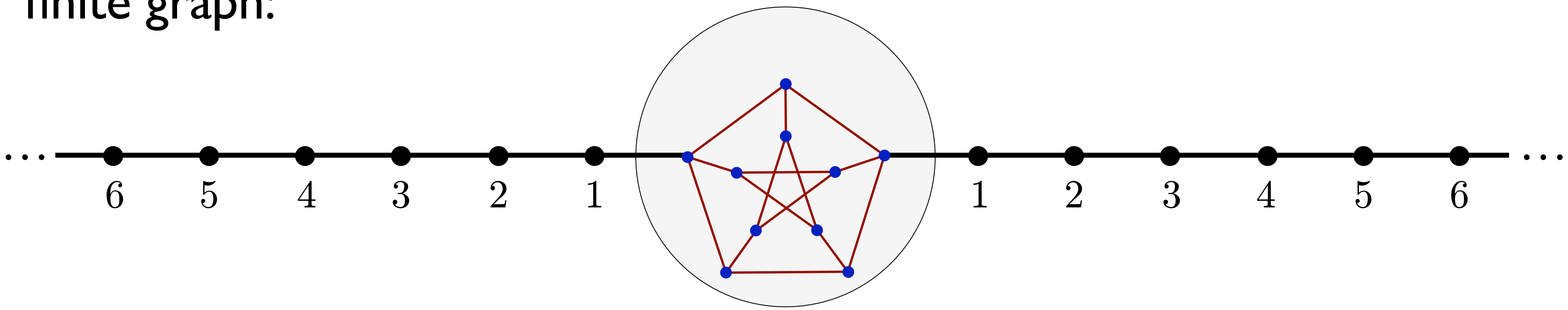

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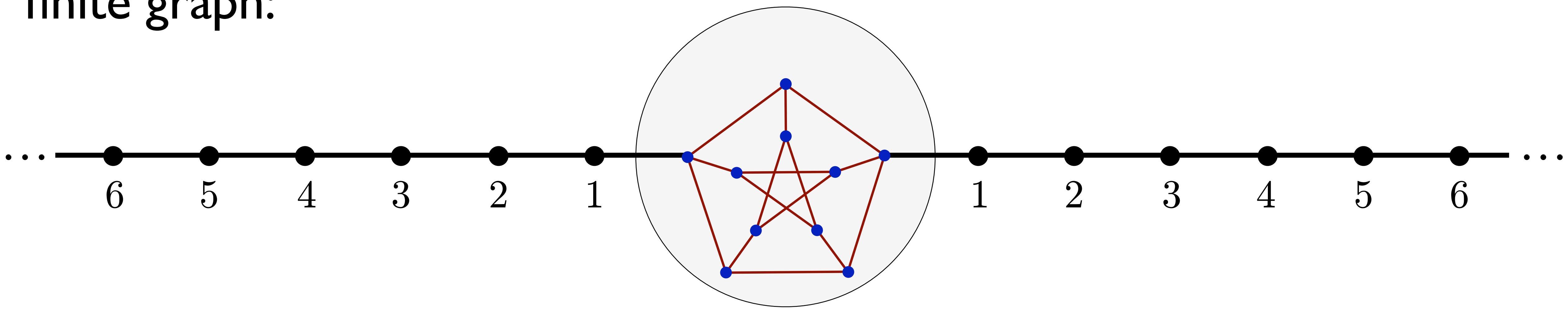

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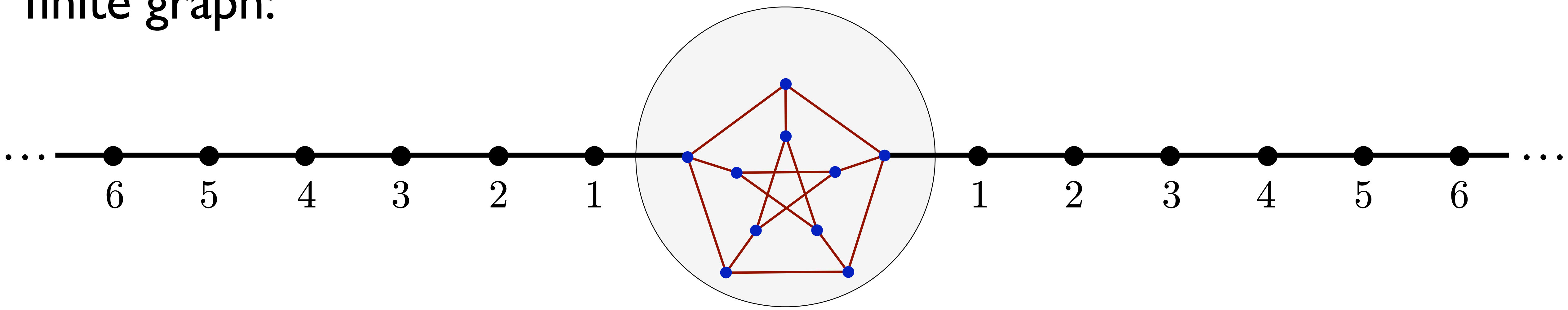

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$$\langle x, \text{left}|\tilde{\kappa}, \text{bd}^{\pm}\rangle = (\pm e^{-\kappa})^{x} \qquad \langle x, \text{right}|\tilde{\kappa}, \text{bd}^{\pm}\rangle = B^{\pm}(\kappa)(\pm e^{-\kappa})^{x}$$

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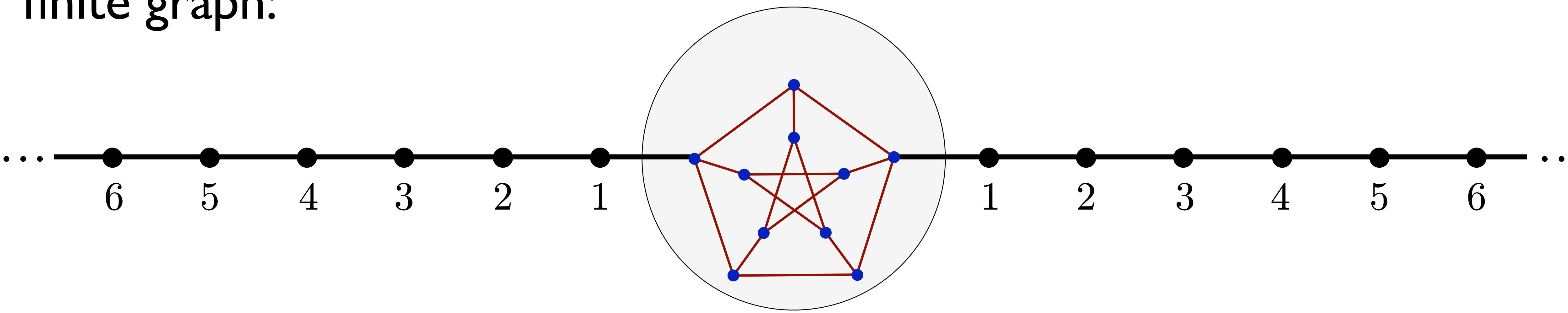

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It can be shown that these states form a complete, orthonormal basis of the Hilbert space, where $k \in [-\pi, 0]$ and $\kappa > 0$ takes certain discrete values.

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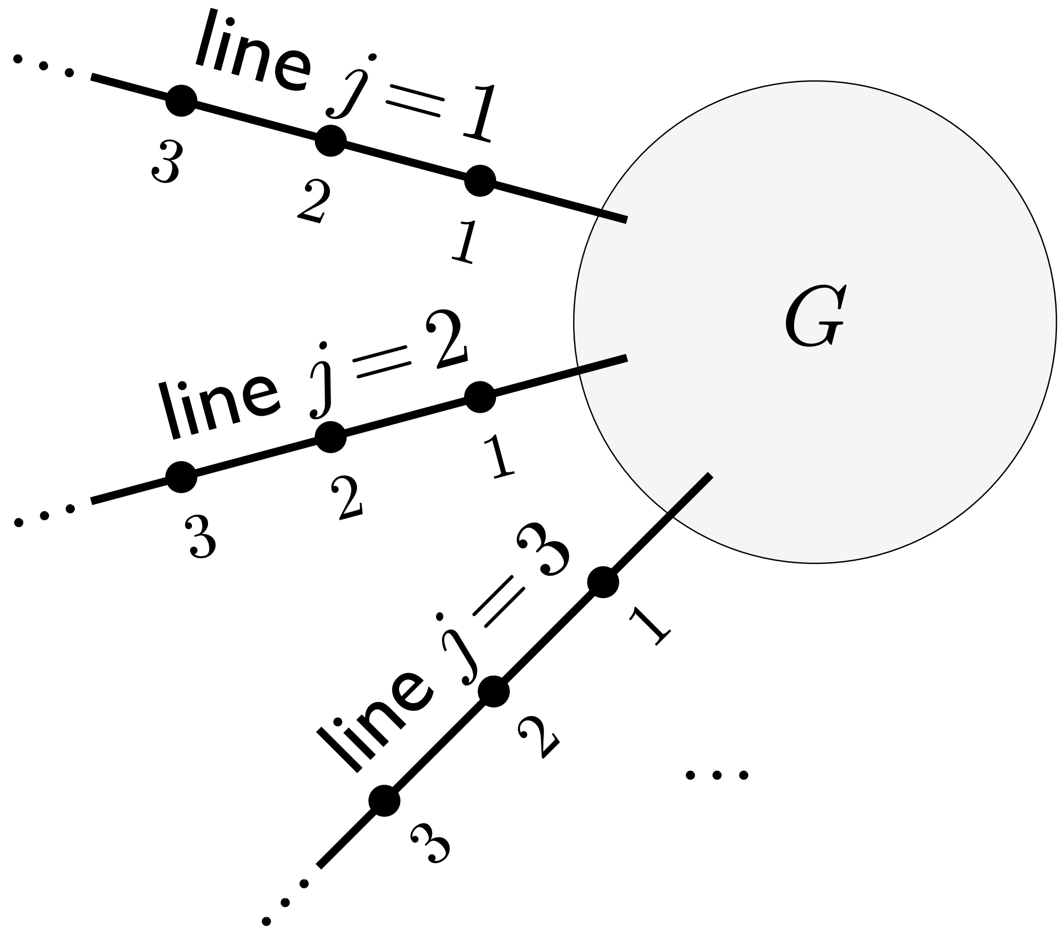
This generalizes to any number of semi-infinite lines attached to any finite graph.



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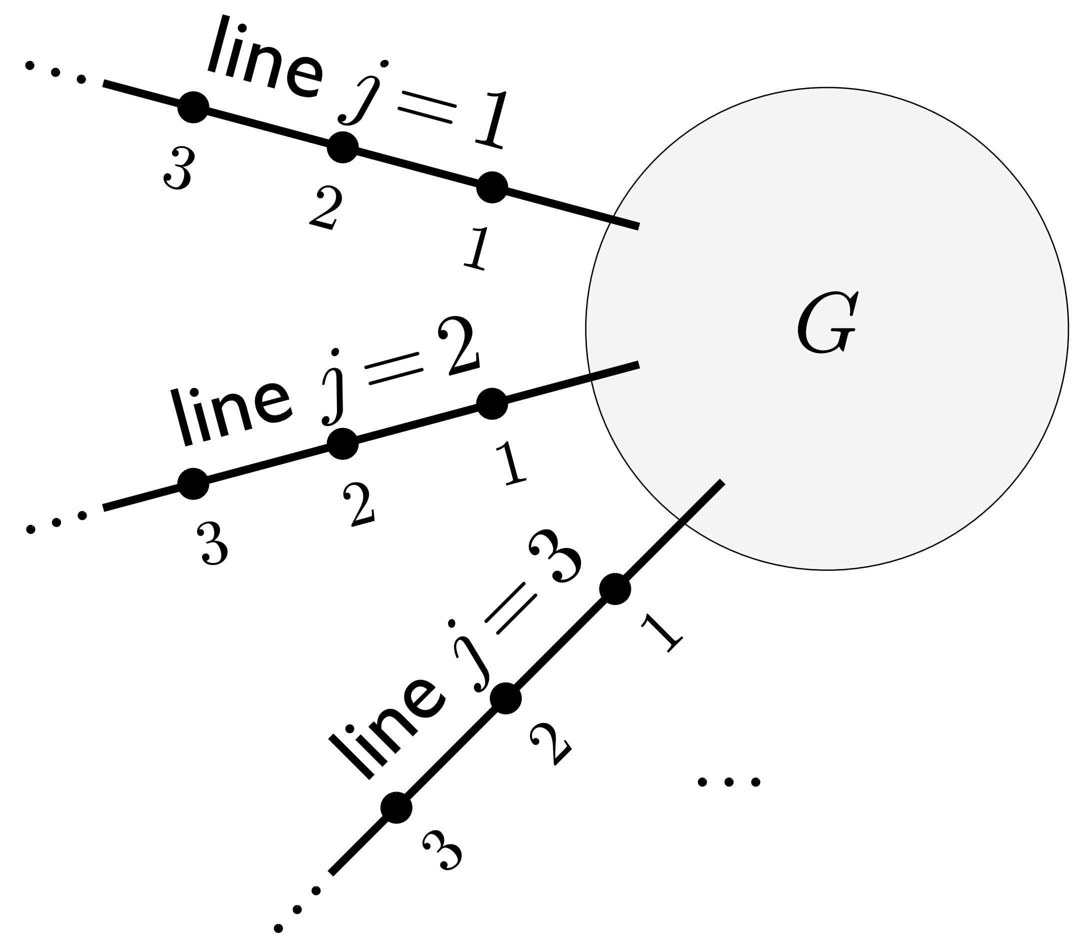

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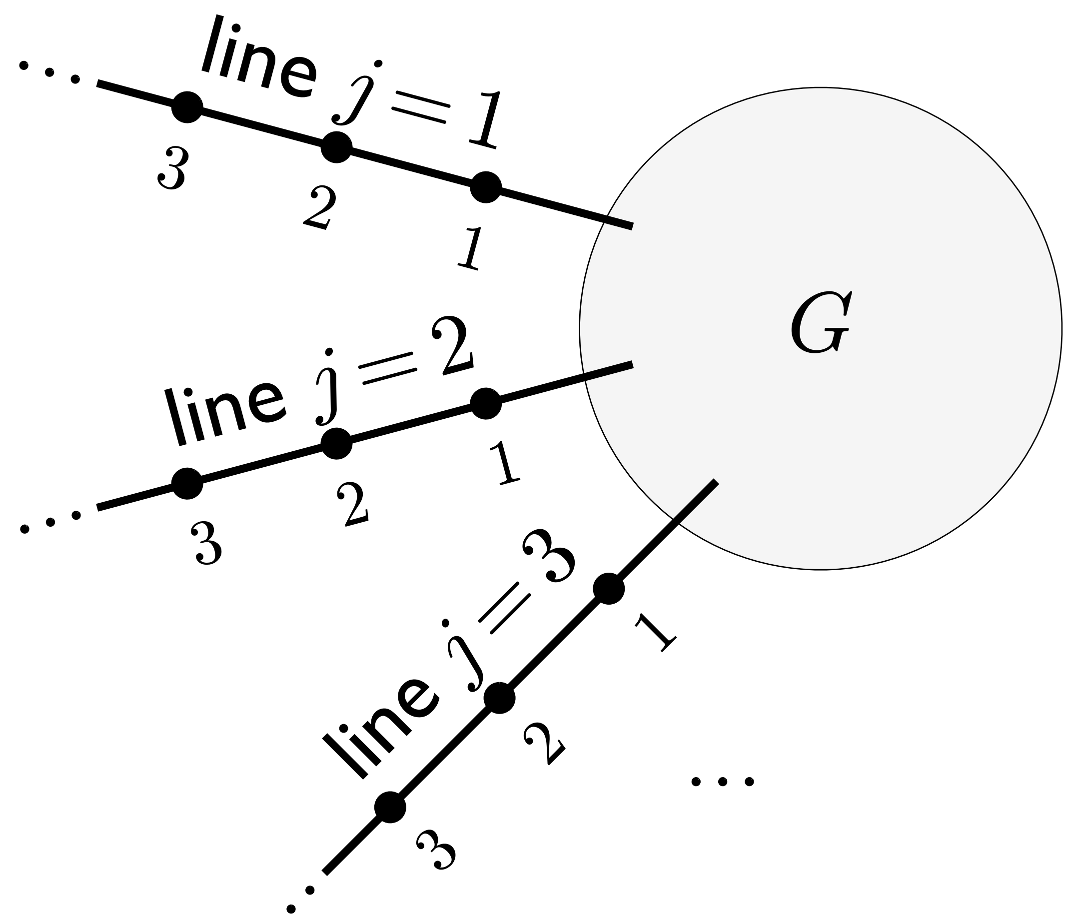

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Bound states:

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Solution of the quantum walk equation:

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$$\langle y, j'|e^{-\mathrm{i}Ht}|x, j\rangle = \sum_{\bar{j}=1}^{N} \int_{-\pi}^{0} e^{-2\mathrm{i}t\cos k} \langle y, j'|\tilde{k}, \mathrm{sc}_{\bar{j}}^{\rightarrow}\rangle\langle\tilde{k}, \mathrm{sc}_{\bar{j}}^{\rightarrow}|x, j\rangle \,\text{đ}k$$

$$+ \sum_{\kappa,\pm} e^{\mp 2\mathrm{i}t\cosh\kappa} \langle y, j'|\tilde{\kappa}, \mathrm{bd}^{\pm}\rangle\langle\tilde{\kappa}, \mathrm{bd}^{\pm}|x, j\rangle$$

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The method of stationary phase

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In scattering on graphs, we have

$$\langle y, j'|e^{-\mathrm{i}Ht}|x, j\rangle \approx \int_{-\pi}^{0} e^{\mathrm{i}k(x+y)-2\mathrm{i}t\cos k} T_{j,j'}(k) đk$$

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In scattering on graphs, we have

$$\langle y, j' | e^{-\mathrm{i}Ht} | x, j\rangle \approx \int_{-\pi}^{0} e^{\mathrm{i}k(x+y) - 2\mathrm{i}t\cos k} T_{j,j'}(k) đk$$

The phase is stationary for k satisfying $x + y + \ell_{j,j'}(k) = v(k)t$

$$v(k) := \frac{\mathrm{d}}{\mathrm{d}k} 2\cos k = -2\sin k \qquad \text{group velocity}$$

$$\ell_{j,j'}(k) := \frac{\mathrm{d}}{\mathrm{d}k} \arg T_{j,j'}(k) \qquad \text{effective length}$$

Finite lines suffice

To obtain a finite graph, truncate the semi-infinite lines at a length $O(t)$, where t is the total evolution time.

This gives nearly the same behavior since the walk on a line has a maximum propagation speed of 2.

E.g., from stationary phase: $|v(k)| = 2|\sin k| \leq 2\,.$

Computation by scattering

Encode quantum circuits into graphs.

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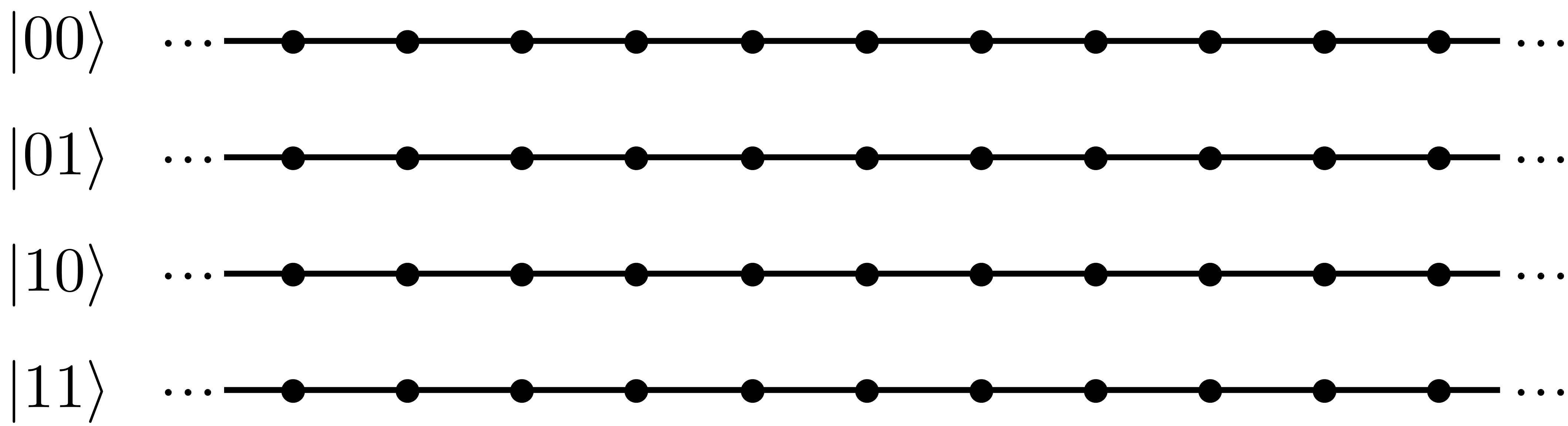

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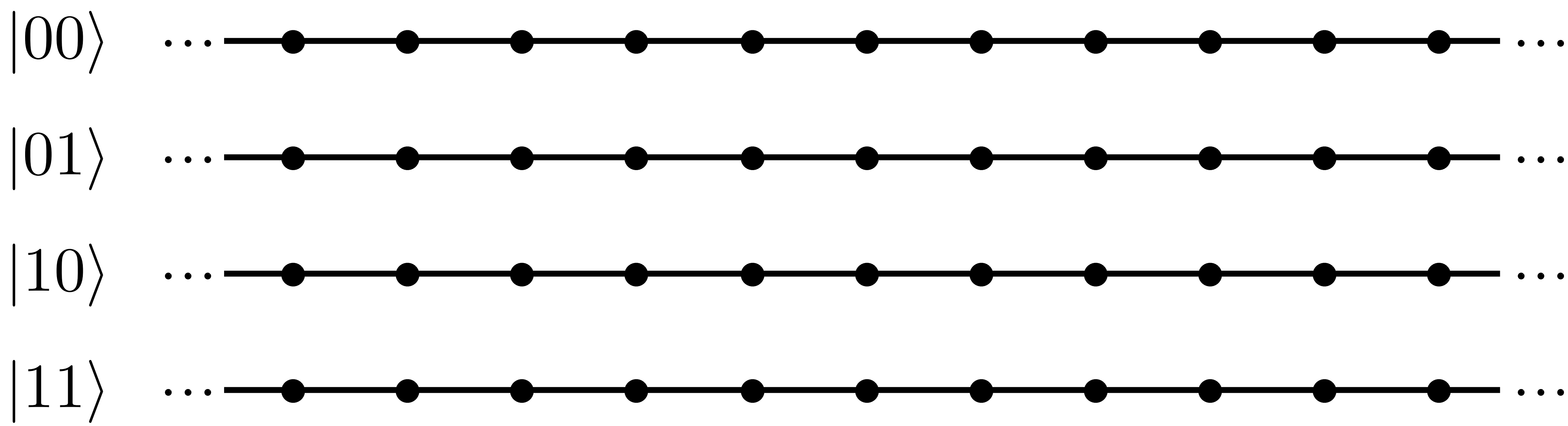

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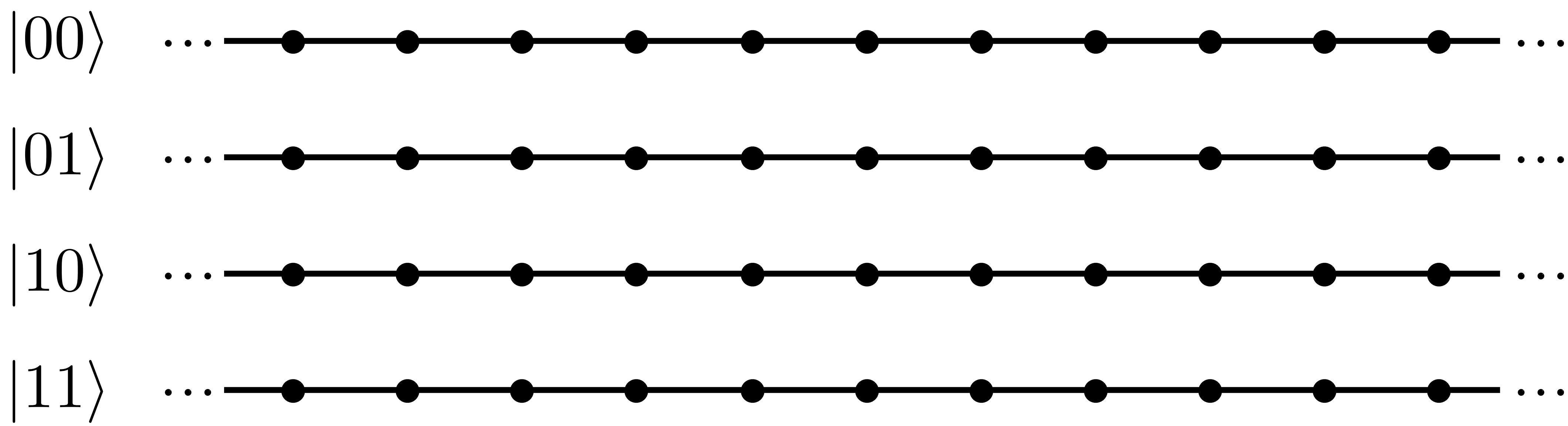

Quantum information propagates from left to right.

To perform gates, attach graphs along/connecting the wires.

A universal gate set

Theorem. Any unitary operation on n qubits can be approximated arbitrarily closely by a product of gates from the set

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\mathrm{i}} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

[Boykin et al. 00]

We can implement these elementary gates (and indeed, any product of these gates) by scattering on graphs.

Controlled-not

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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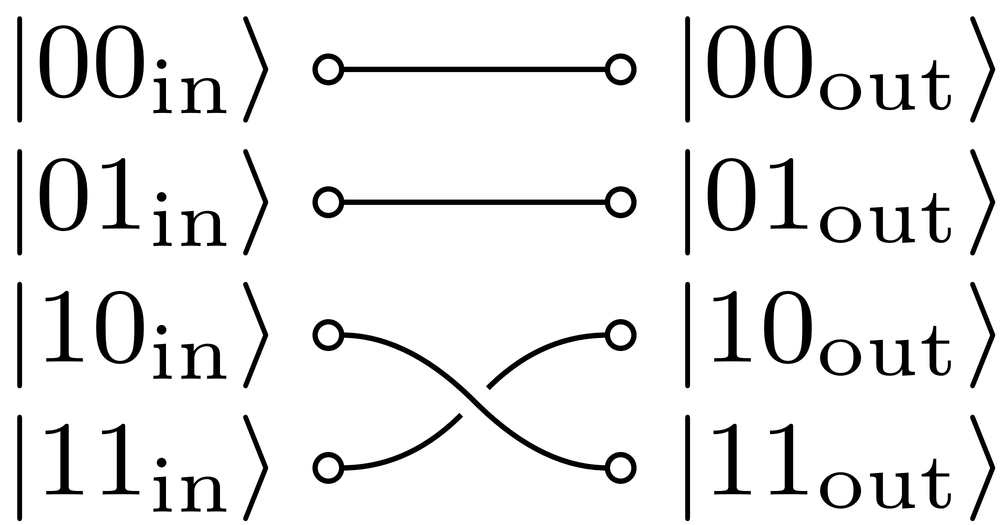

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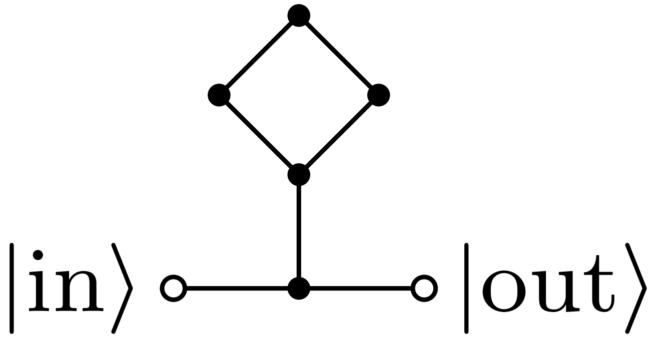

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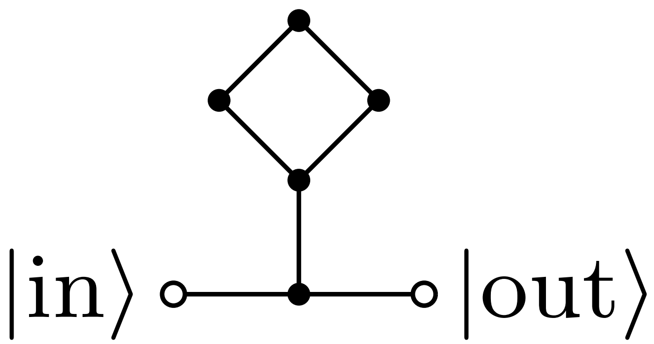

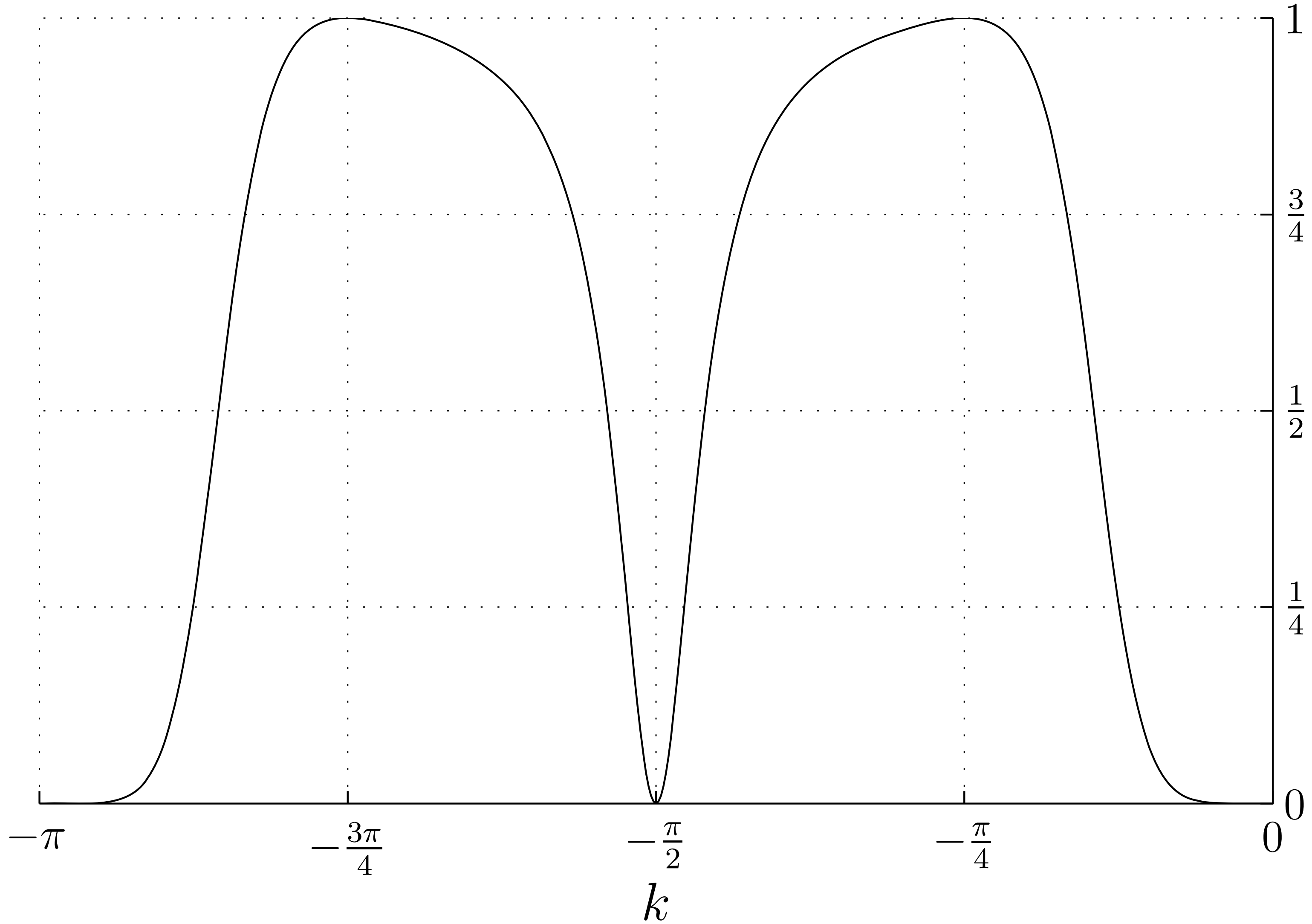

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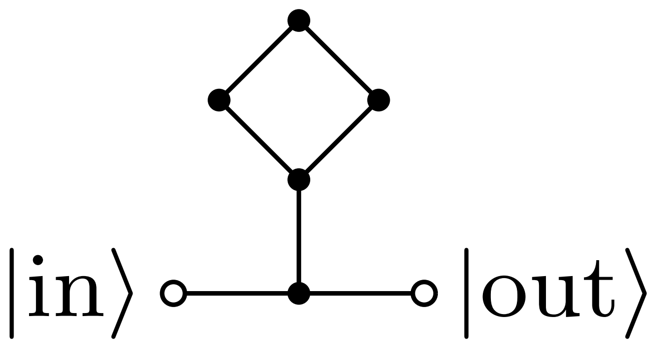

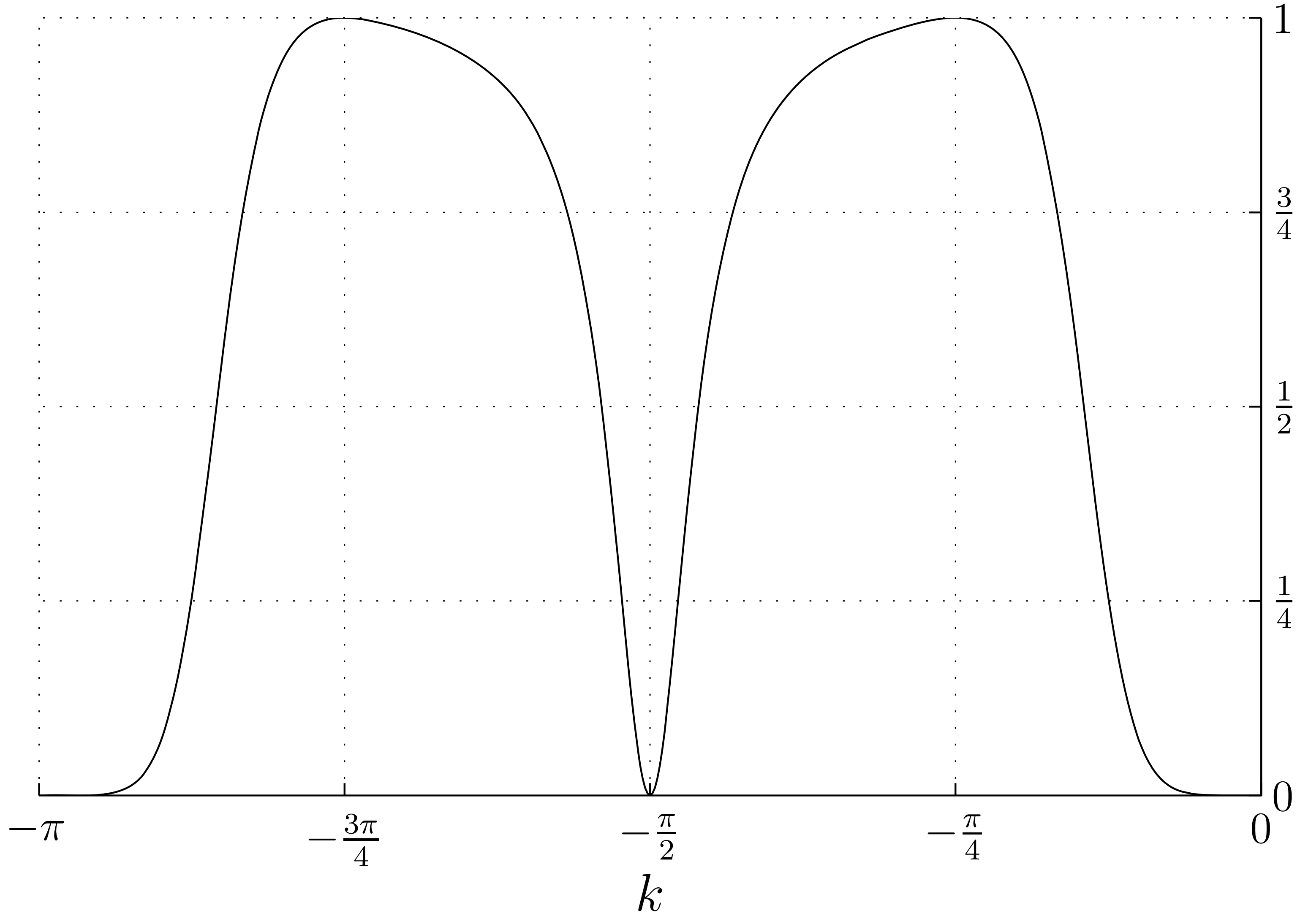

A basis-changing gate

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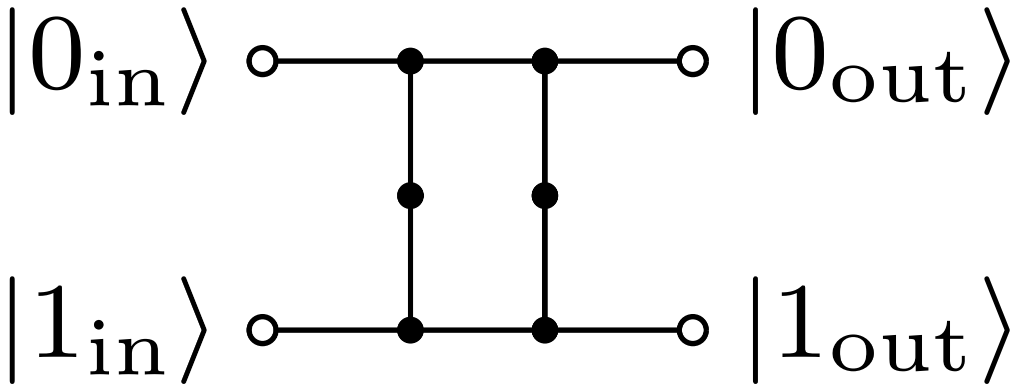

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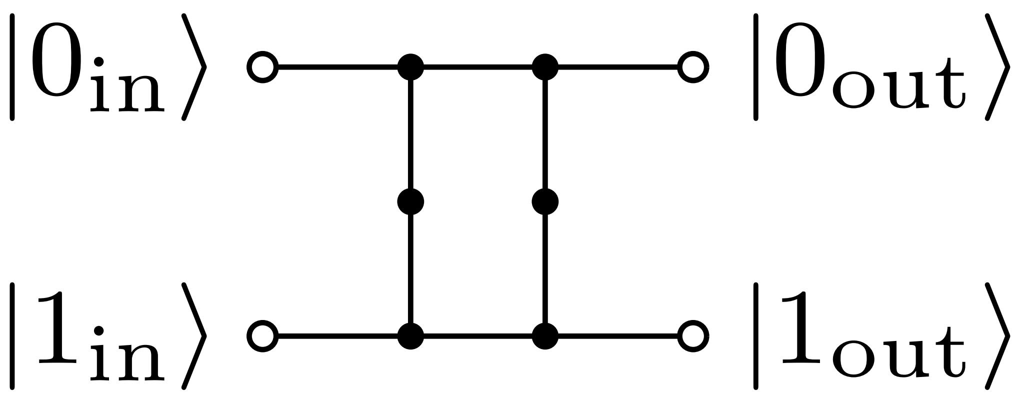

$$T_{0_{\text{in}},0_{\text{out}}}(k) = \frac{e^{ik}(\cos k + i\sin 3k)}{2\cos k + i(\sin 3k - \sin k)}$$

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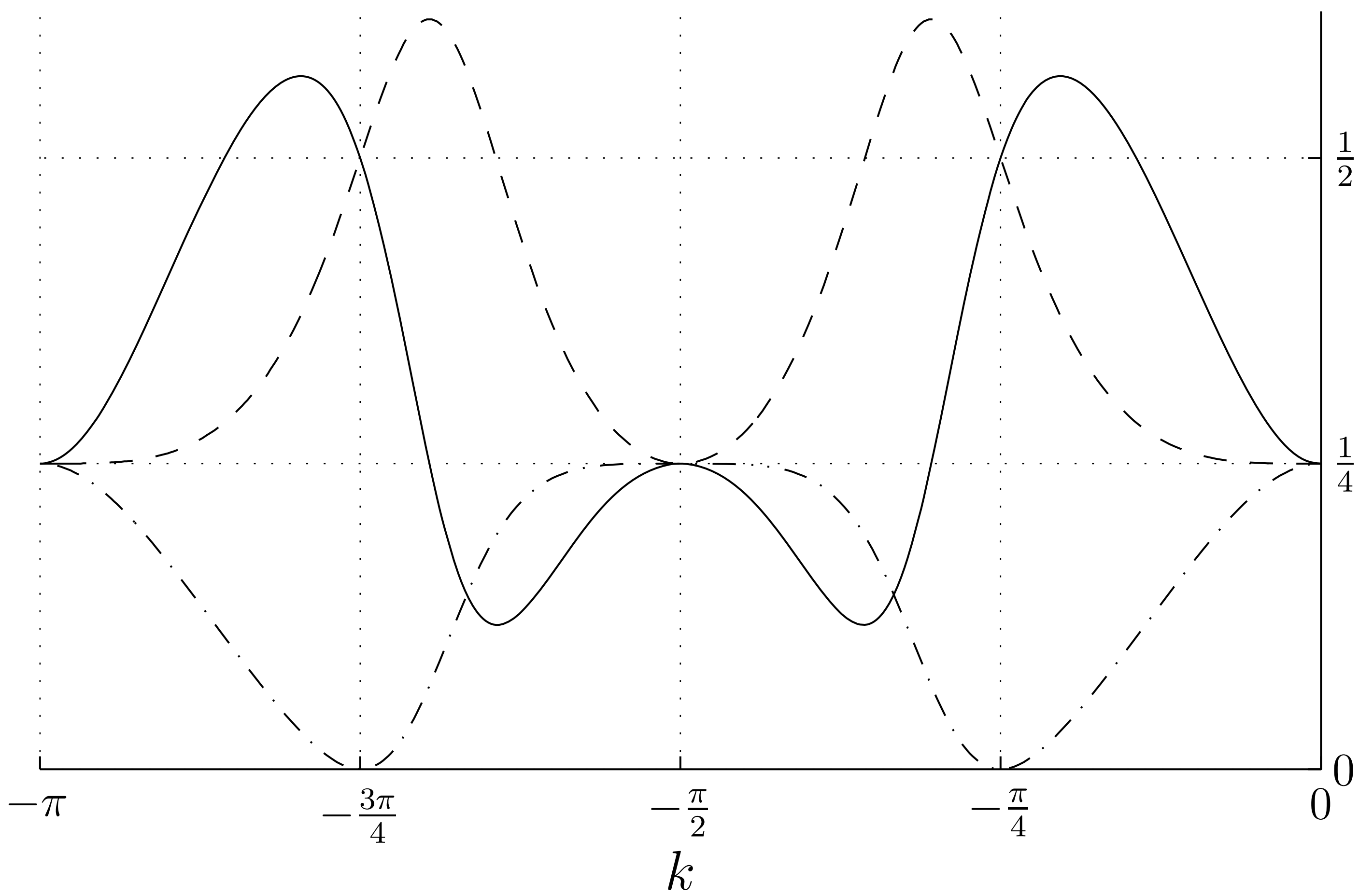

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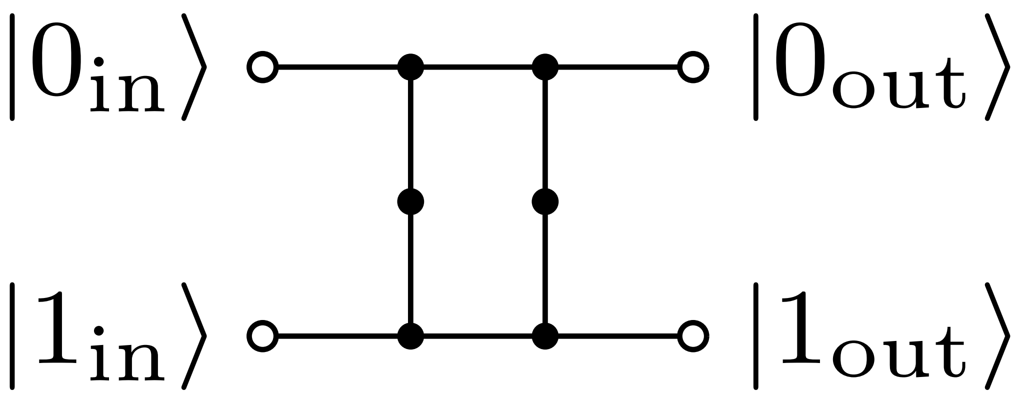

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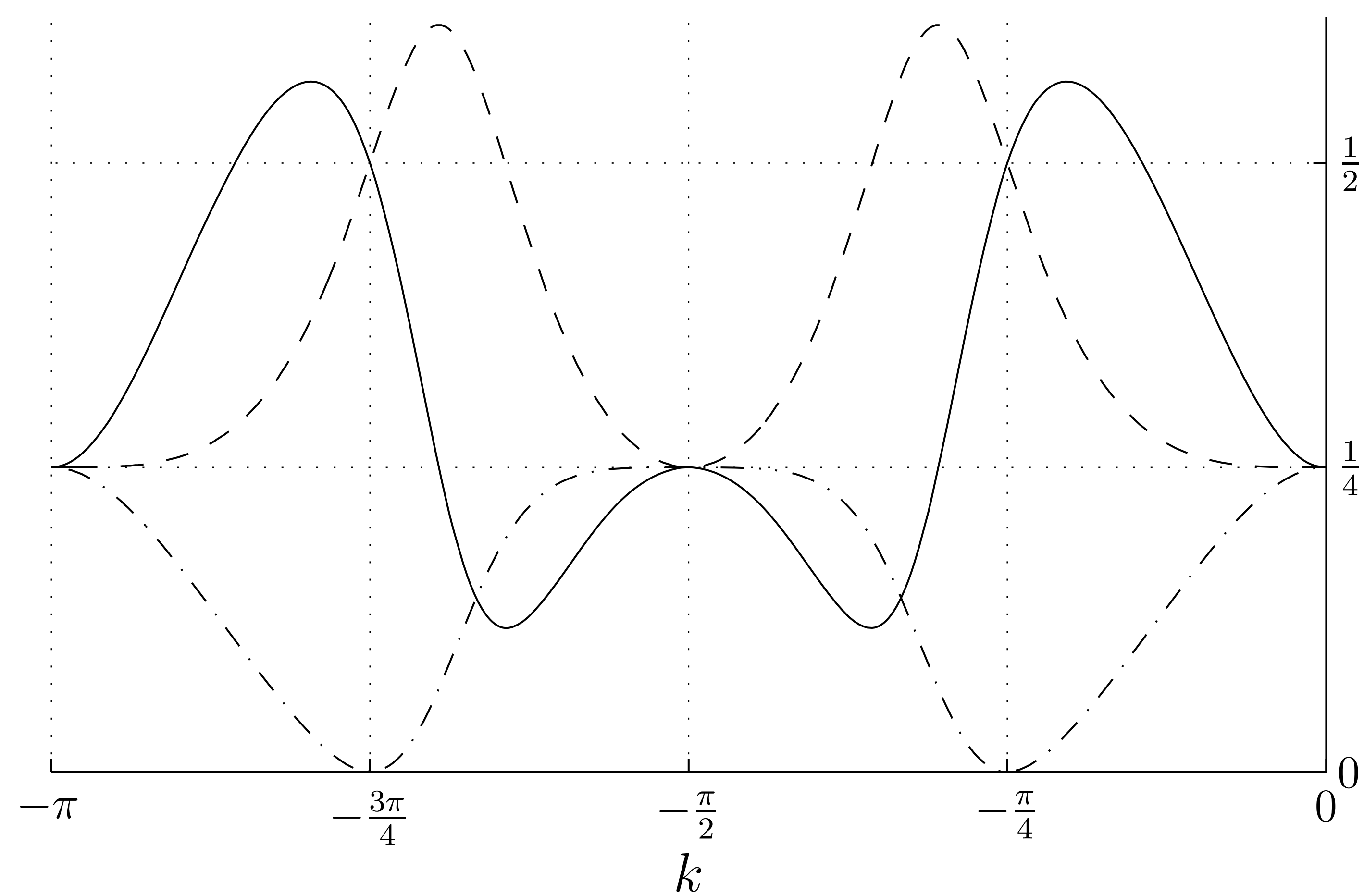
$$R_{0_{\rm in}}(k) = T_{0_{\rm in},1_{\rm in}}(k) = -\frac{e^{ik}\cos 2k}{2\cos k + i(\sin 3k - \sin k)}$$

At $k = -\pi/4$ this implements the unitary transformation

$$U = -\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

from inputs to outputs



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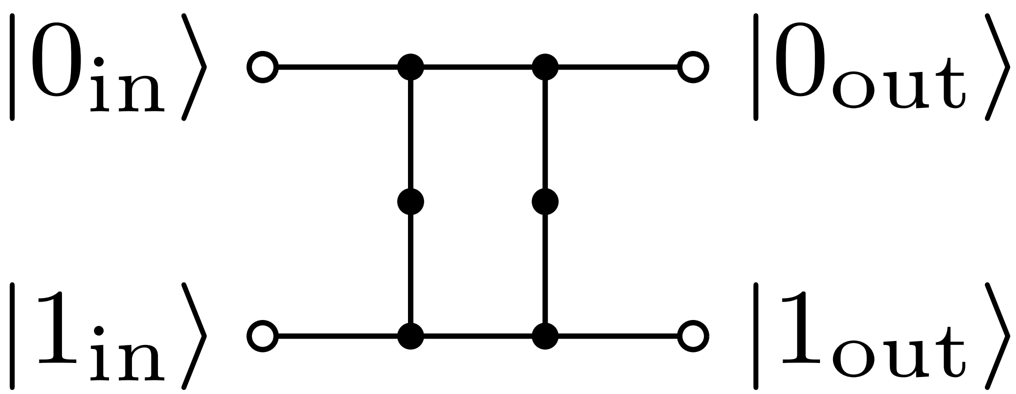

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$$\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{i} & 1\\ 1 & \mathrm{i}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix} = e^{\mathrm{i}\phi}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

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impleme

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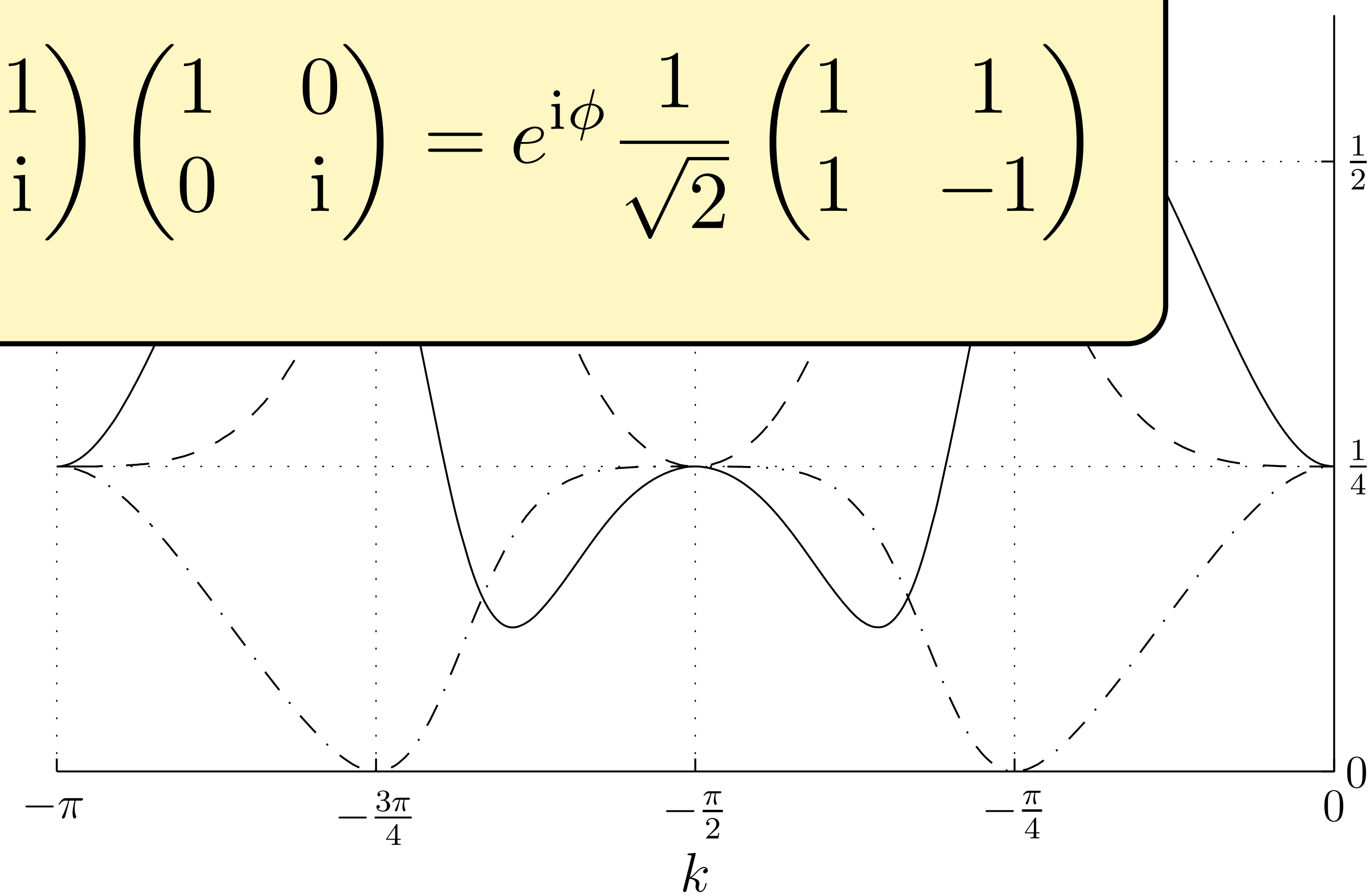

Tensor product structure

To embed an m-qubit gate in an n-qubit system, simply include the gate widget 2^{n-m} times, once for every possible computational basis state of the $n-m$ qubits not acted on by the gate.

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Note: The graph has $2^n \cdot \mathrm{poly}(n)$ vertices (exponentially many), corresponding to the dimension of the Hilbert space used by the simulation. Vertices correspond to basis states, *not* qubits.

Despite its exponential size, the graph has a succinct description in terms of the circuit being simulated.

In particular, the quantum walk can be efficiently simulated by a universal quantum computer.

Composition law

To perform a sequence of gates, simply connect the output wires to the next set of input wires.

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Arrange the transmission/reflection coefficients as transformations from inputs to outputs:

$$\mathcal{T}_{j,j'} = T_{j_{\rm in},j'_{\rm out}} \qquad \mathcal{R}_{j,j'} = \begin{cases} R_{j_{\rm in}} & j = j' \\ T_{j_{\rm in},j'_{\rm in}} & j \neq j' \end{cases}$$

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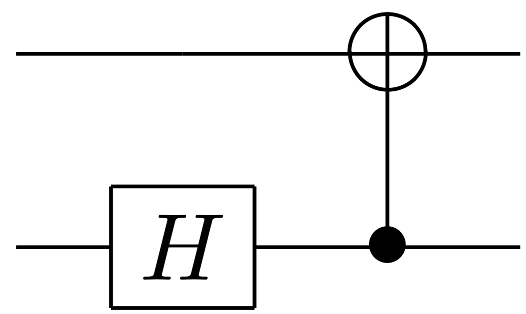
Then we have

$$\begin{aligned} \mathcal{T}_{12} &= \mathcal{T}_1 (1 - \mathcal{R}_2 \bar{\mathcal{R}}_1)^{-1} \mathcal{T}_2 \\ \mathcal{R}_{12} &= \mathcal{R}_1 + \mathcal{T}_1 (1 - \mathcal{R}_2 \bar{\mathcal{R}}_1)^{-1} \mathcal{R}_2 \bar{\mathcal{T}}_1 \\ \bar{\mathcal{T}}_{12} &= \bar{\mathcal{T}}_2 (1 - \bar{\mathcal{R}}_1 \mathcal{R}_2)^{-1} \bar{\mathcal{T}}_1 \\ \bar{\mathcal{R}}_{12} &= \bar{\mathcal{R}}_2 + \bar{\mathcal{T}}_2 (1 - \bar{\mathcal{R}}_1 \mathcal{R}_2)^{-1} \bar{\mathcal{R}}_1 \mathcal{T}_2 \end{aligned}$$

Example



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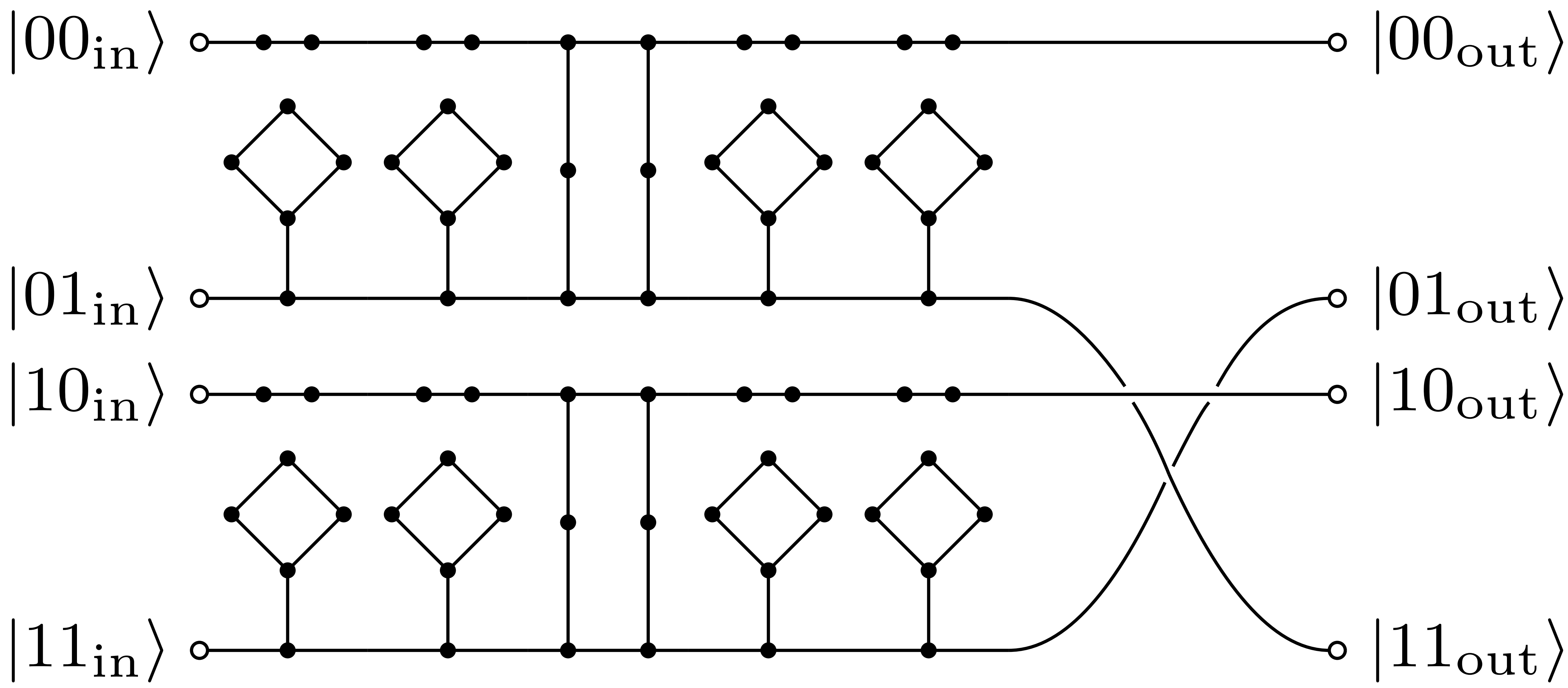

$|00_{\mathrm{in}}\rangle$
$|01_{\mathrm{in}}\rangle$
$|10_{\mathrm{in}}\rangle$
$|11_{\mathrm{in}}\rangle$
$|00_{\mathrm{out}}\rangle$
$|01_{\mathrm{out}}\rangle$
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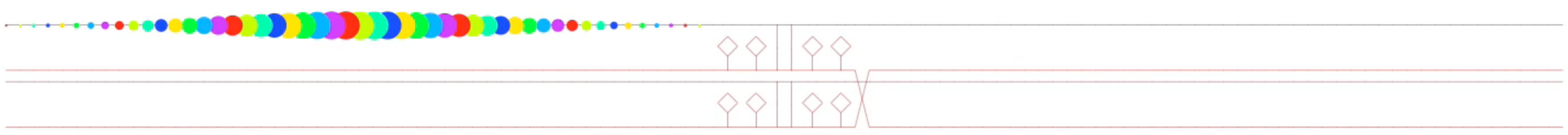
$|11_{\mathrm{out}}\rangle$

Example in action



Simplifying the initial state

So far, we have assumed that the computation takes place using only momenta near $k = -\pi/4$.

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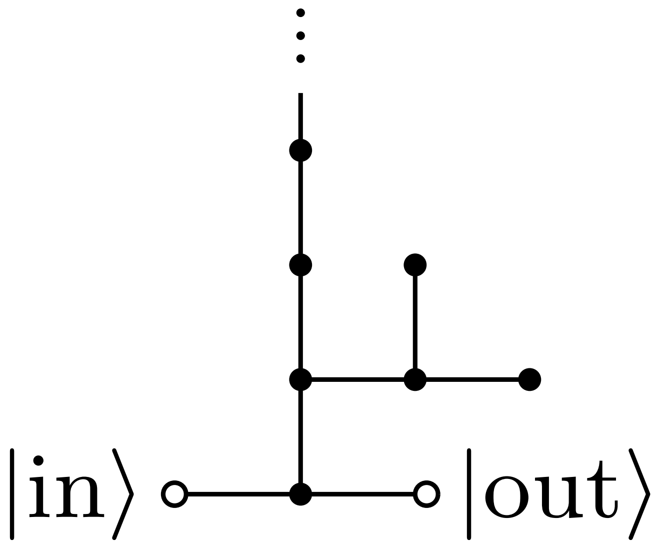
Can we relax this restriction? Start from a single vertex of the graph?

Idea: A single vertex has equal amplitudes for all momenta. Filter out momenta except within $1/\mathrm{poly}(n)$ of $k = -\pi/4$.

Momentum filter



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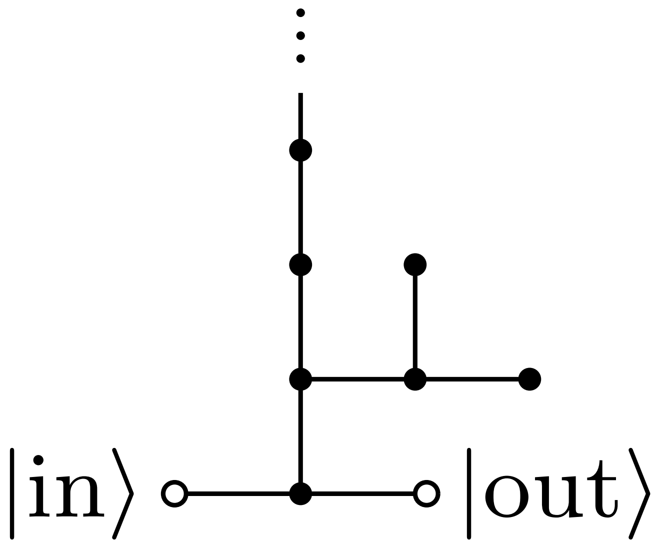

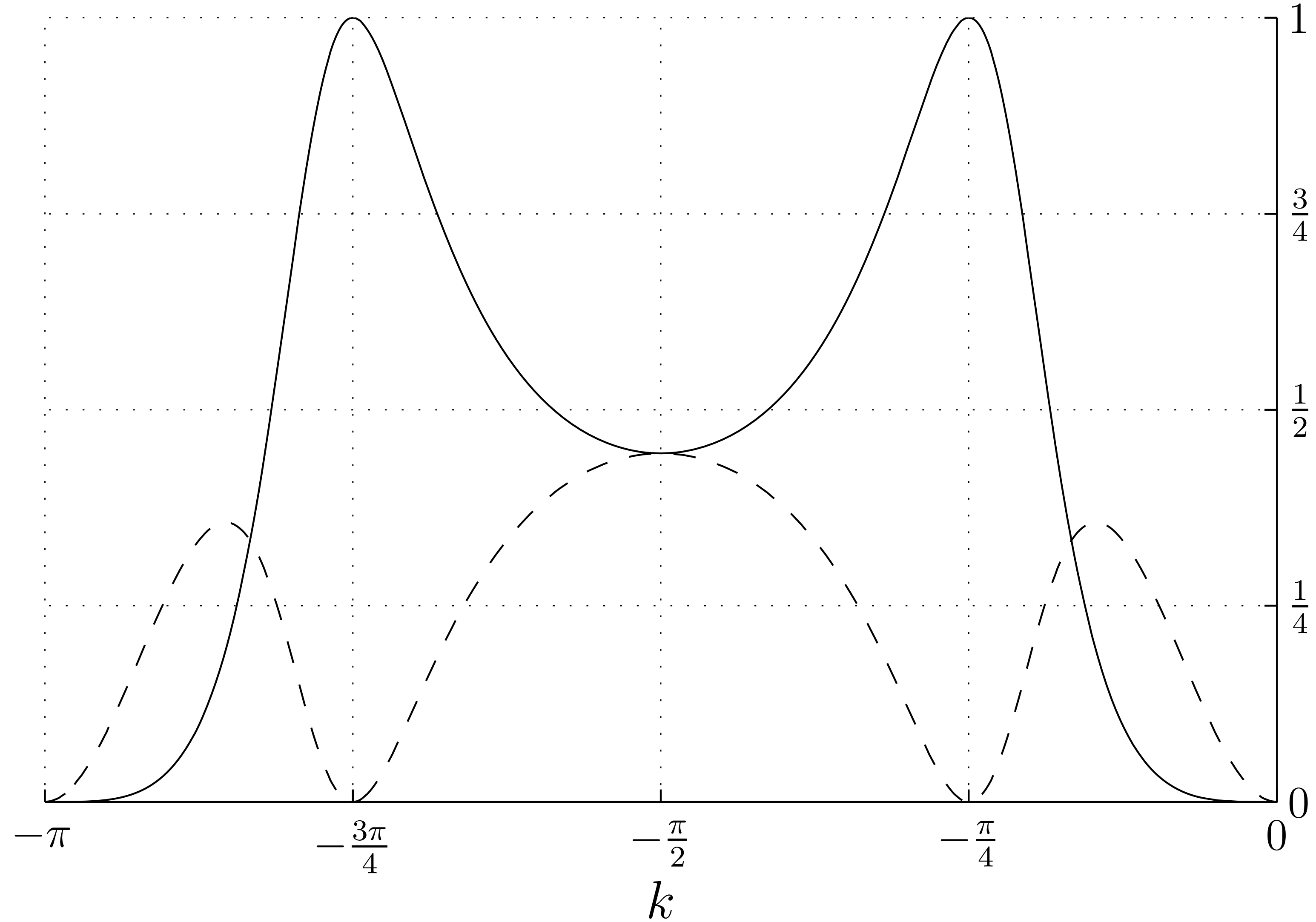

Transfer matrix

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This can be analyzed using a transfer matrix approach.

Write $$\begin{pmatrix} \langle x+1|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \\ \langle x|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \end{pmatrix} = M \begin{pmatrix} \langle x|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \\ \langle x-1|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \end{pmatrix}$$

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For m filters, suppose

$$M^m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$T = \frac{2\mathrm{i}e^{-\mathrm{i}km}\sin k}{-ae^{-\mathrm{i}k} - b + c + de^{\mathrm{i}k}}$$

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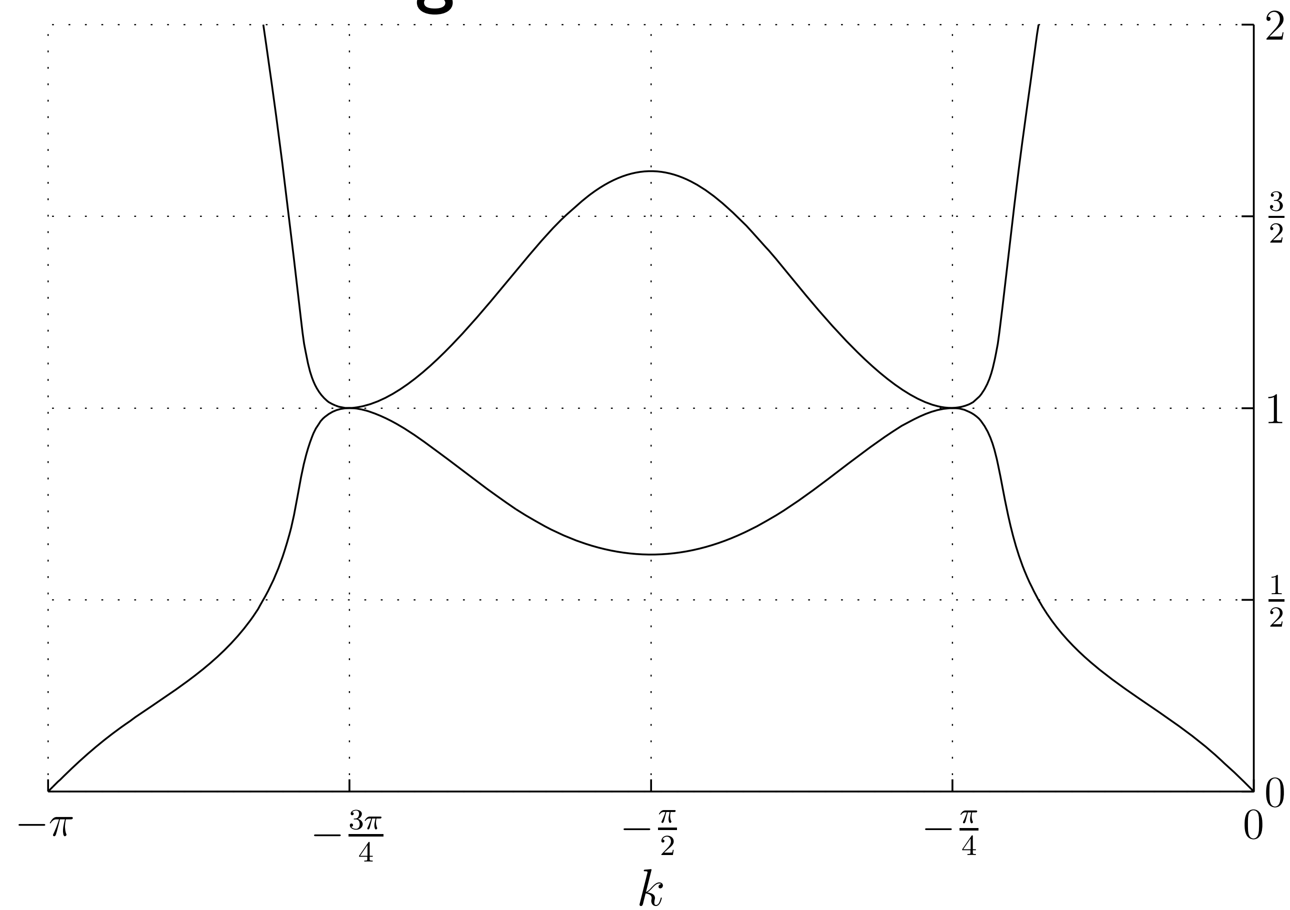
Eigenvalues of M



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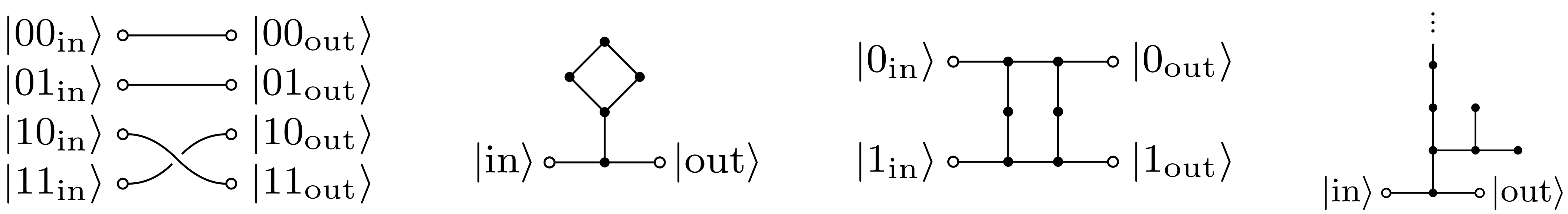

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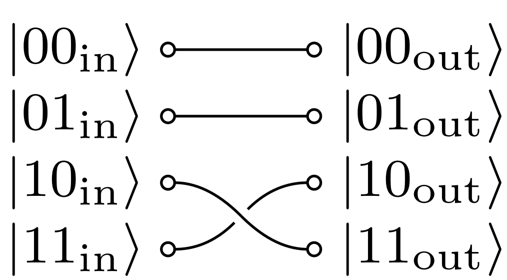
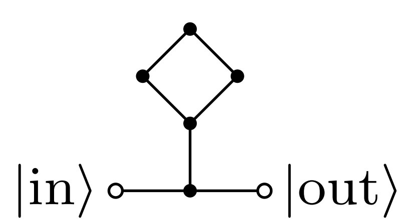
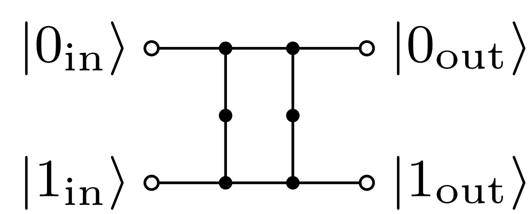
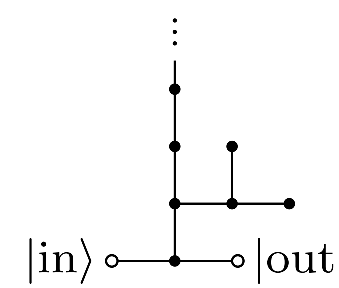


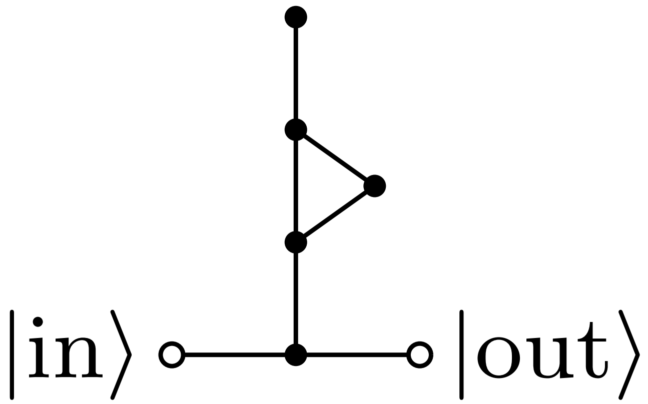






This is because they are all bipartite.

Momentum separator



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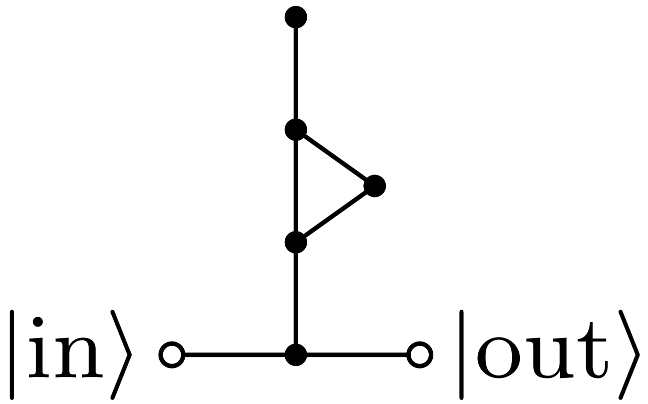
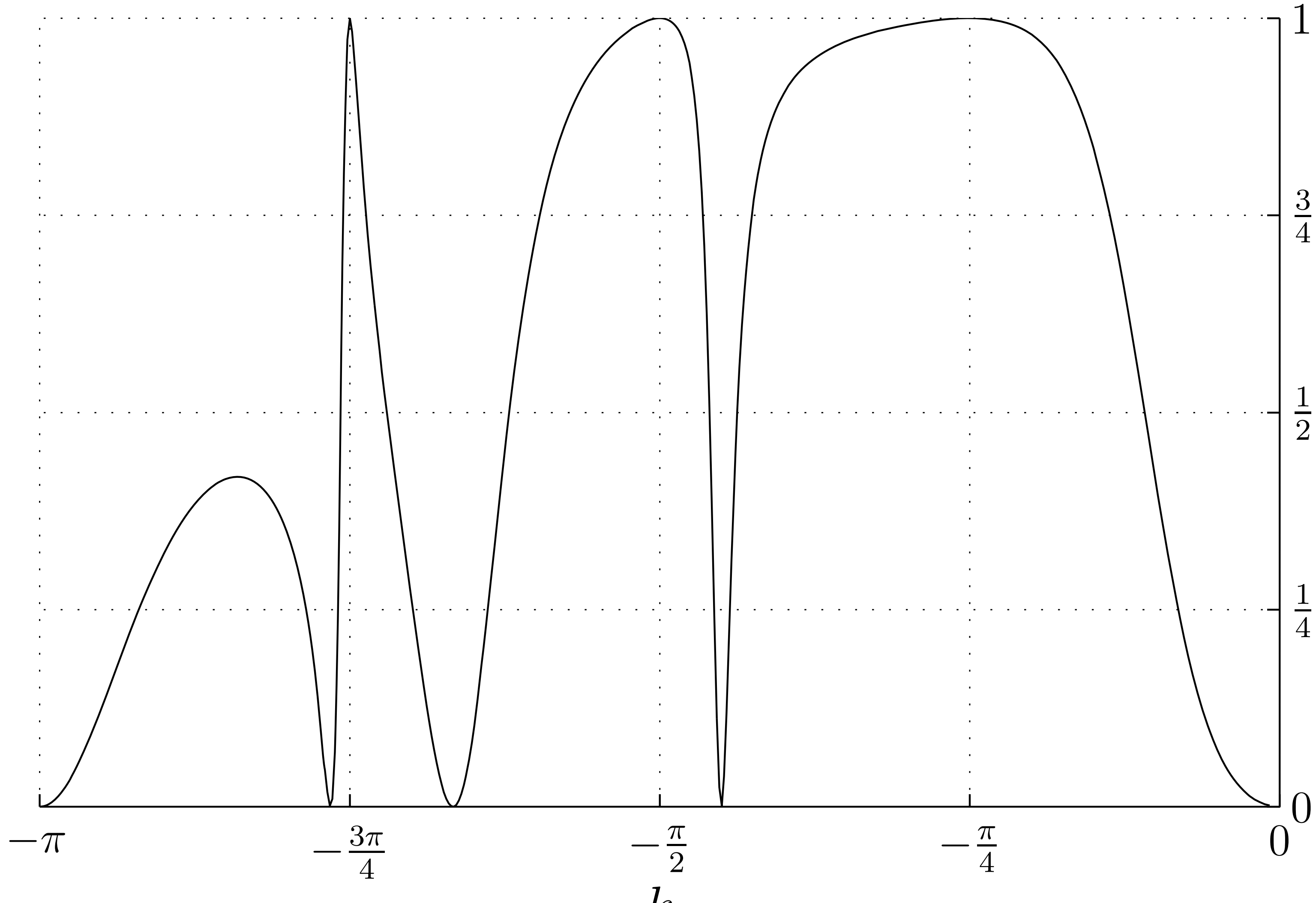
$$T_{\mathrm{in,out}}(k) = \left[1 + \frac{\mathrm{i}(\cos k + \cos 3k)}{\sin k + 2\sin 2k + \sin 3k - \sin 5k}\right]^{-1}$$





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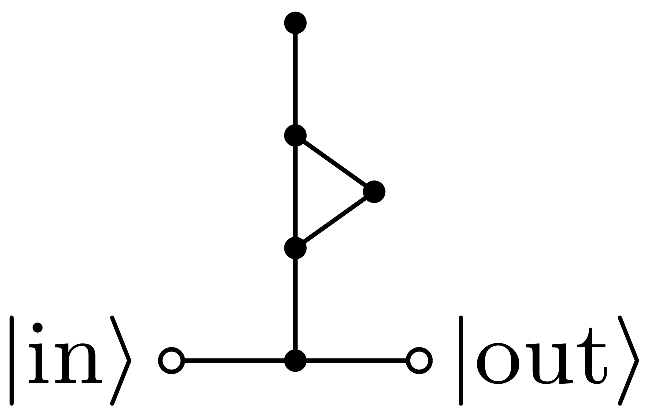

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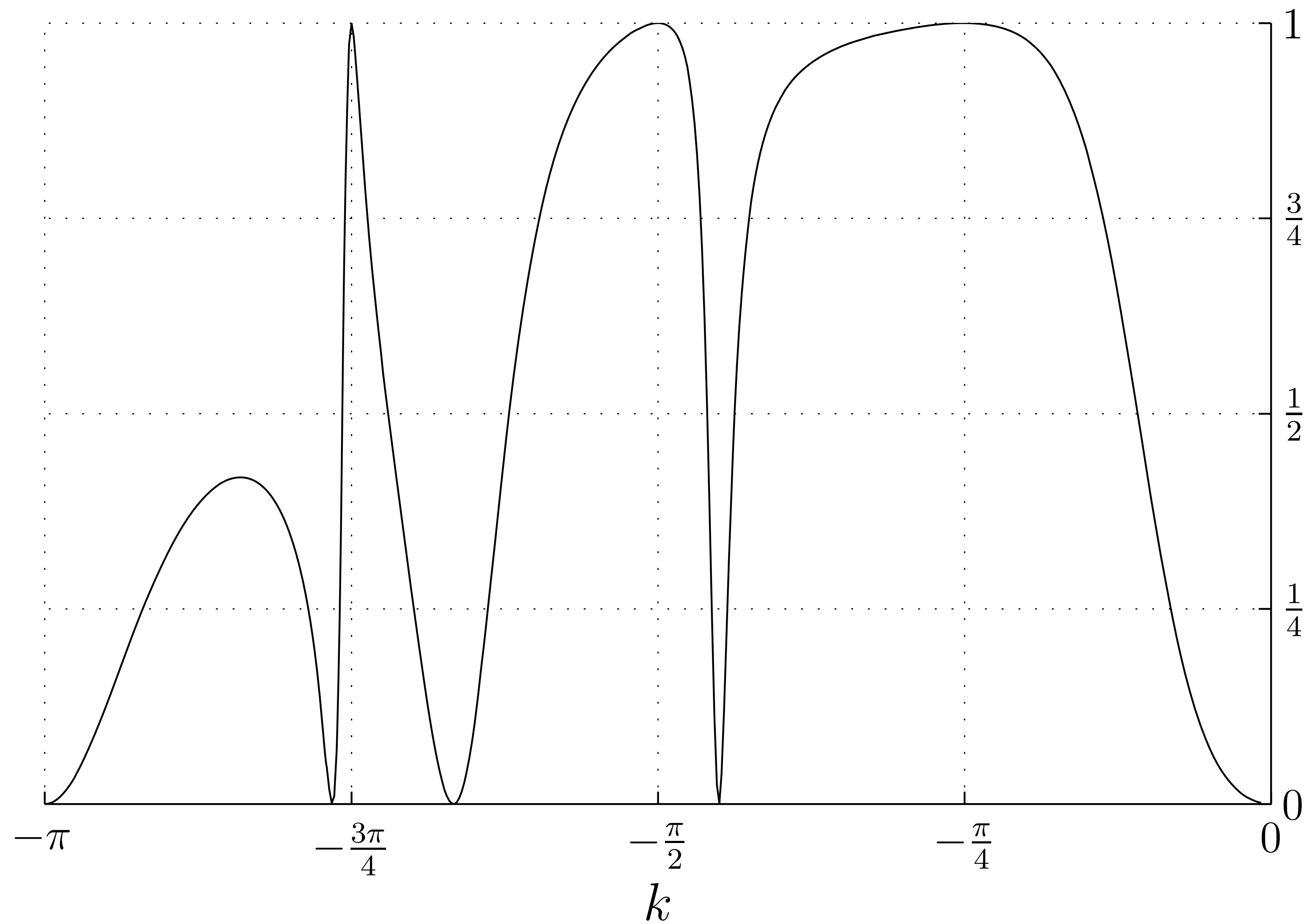
$$\ell_{\mathrm{in,out}}(-\pi/4) = 4(3 - 2\sqrt{2}) \approx 0.686$$

$$\ell_{\mathrm{in,out}}(-3\pi/4) = 4(3 + 2\sqrt{2}) \approx 23.3$$



A universal computer

Consider an m-gate quantum circuit (unitary transformation U).

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Graph:

- $\log \Theta(m^2)$ filter widgets on input line $00\ldots0$
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Simulation:

- Start at vertex $x = \Theta(m^4)$ on input line $00\ldots0$
- Evolve for time $t = \pi\lfloor(x+\ell)/\sqrt{2}\pi\rfloor = O(m^4)$
- Measure in the vertex basis
- Conditioned on reaching vertex 0 on some output line s (which happens with probability $\Omega(1/m^4)$), the distribution over s is approximately $|\langle s|U|00\ldots0\rangle|^2$

Applications?

- Quantum algorithms
- Quantum complexity theory
- Architectures for quantum computers