# Quantum algorithms by optimal state estimation

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## Problems

- Simulating quantum dynamics
- Factoring
- Discrete log
- Pell's equation
- Abelian HSP
- Some nonabelian HSPs
- Estimating gauss sums
- Legendre symbol/polynomial reconstruction
- Graph traversal
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## Techniques

- Fourier sampling
- Quantum walk
- Adiabatic optimization
- Trace estimation
- Optimal measurement

# Outline

- The hidden subgroup problem (HSP)
- Optimal measurements for distinguishing quantum states
- Dihedral HSP
- Heisenberg HSP
- Unlabeled hidden shift problem
- Summary and open problems

# The hidden subgroup problem

Problem: Fix a group $G$ (known) and a subgroup $H$ (unknown). Given a black box that computes $f: G \to S$ that is

- Constant on any particular left coset of $H$ in $G$
- Distinct on different left cosets of $H$ in $G$

(We say that $f$ hides $H$.)

Goal: Find (a generating set for) $H$.
An efficient algorithm runs in time $\mathrm{poly}(\log |G|)$.

# Most interesting cases of the HSP

- Abelian groups
  Applications to factoring, discrete log, Pell's equation, etc.
  Can be solved efficiently

- Dihedral group
  Applications to lattice problems [Regev 2002]
  Subexponential-time algorithm [Kuperberg 2003]

- Symmetric group
  Application to graph isomorphism
  No nontrivial algorithms

# Efficient algorithms for the HSP

- Abelian groups [Shor 1994; Boneh, Lipton 1995; Kitaev 1995]
- Normal subgroups [Hallgren, Russell, Ta-Shma 2000]
- "Almost abelian" groups [Grigni, Schulman, Vazirani² 2001]
- "Near-Hamiltonian" groups [Gavinsky 2004]
- $(\mathbb{Z}_2^n \times \mathbb{Z}_2^n) \rtimes \mathbb{Z}_2$ [Püschel, Rötteler, Beth 1998]
- $\mathbb{Z}_{p^k}^n \rtimes \mathbb{Z}_2$, smoothly solvable groups [Friedl, Ivanyos, Magniez, Santha, Sen 2002]
- $p$-hedral: $\mathbb{Z}_N \rtimes \mathbb{Z}_p$, $p = \phi(N)/\mathrm{poly}(\log N)$ prime, $N$ prime [Moore, Rockmore, Russell, Schulman 2004]
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- ➡ $\mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ [Inui, Le Gall 2004]
- ➡ $\mathbb{Z}_p^r \rtimes \mathbb{Z}_p$, $r$ constant (including Heisenberg, $r=2$) [BCD 2005]

# Standard approach to the HSP

Compute uniform superposition of function values:

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \mapsto \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g, f(g)\rangle$$

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Equivalently, we have the *hidden subgroup state*

$$\rho_H := \frac{1}{|G|}\sum_{g\in G}|gH\rangle\langle gH|$$

# Distinguishing quantum states

Problem: Given a quantum state $\rho$ chosen from an ensemble of states $\rho_i$ with a priori probabilities $p_i$, determine $i$.

This can only be done perfectly if the states are orthogonal. In general, we would just like a high probability of success.

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Good news: This is always sufficient to identify $H$. [Ettinger, Høyer, Knill 1999]

Bad news: For some groups, it is necessary to make joint measurements on $\Omega(\log|G|)$ copies. [Moore, Russell, Schulman 2005-6; Hallgren, Rötteler, Sen 2006]

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Can we use this as a principle to find quantum algorithms?

# Optimal measurement

**Theorem.** [Holevo 1973, Yuen-Kennedy-Lax 1975]
Given an ensemble of quantum states $\rho_i$ with a priori probabilities $p_i$, the measurement with POVM elements $E_i$ maximizes the probability of successfully identifying the state if and only if $R = R^\dagger$ and $R \geq p_i \rho_i$ for all $i$, where

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In general, it is nontrivial to find a POVM that satisfies these conditions (although it is a semidefinite program!).

But for all the cases discussed in this talk, the optimal is a particularly simple POVM, the *pretty good measurement*.

# Pretty good measurement

Given states $\rho_i$ with a priori probabilities $p_i$, define POVM elements

$$E_i := p_i \frac{1}{\sqrt{\Sigma}} \rho_i \frac{1}{\sqrt{\Sigma}} \quad \text{where} \quad \Sigma := \sum_i p_i \rho_i$$

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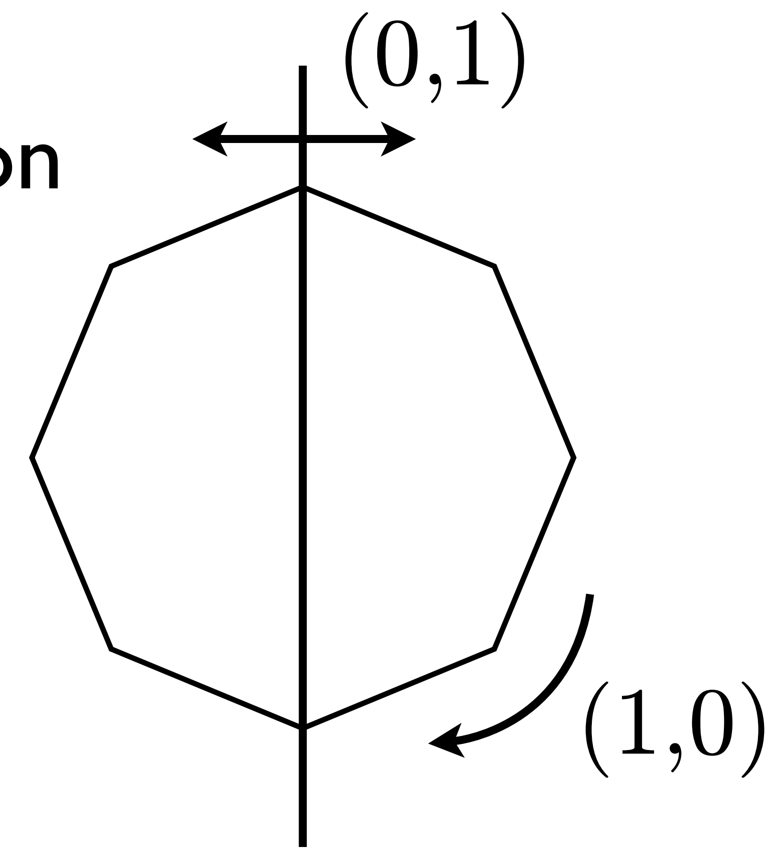
The PGM often does a pretty good job of distinguishing the $\rho_i$.

In fact, sometimes it is optimal! (Check Holevo/YKL conditions)

# Dihedral group $(\mathbb{Z}_N \rtimes \mathbb{Z}_2)$

Symmetry group of an $N$-sided regular polygon



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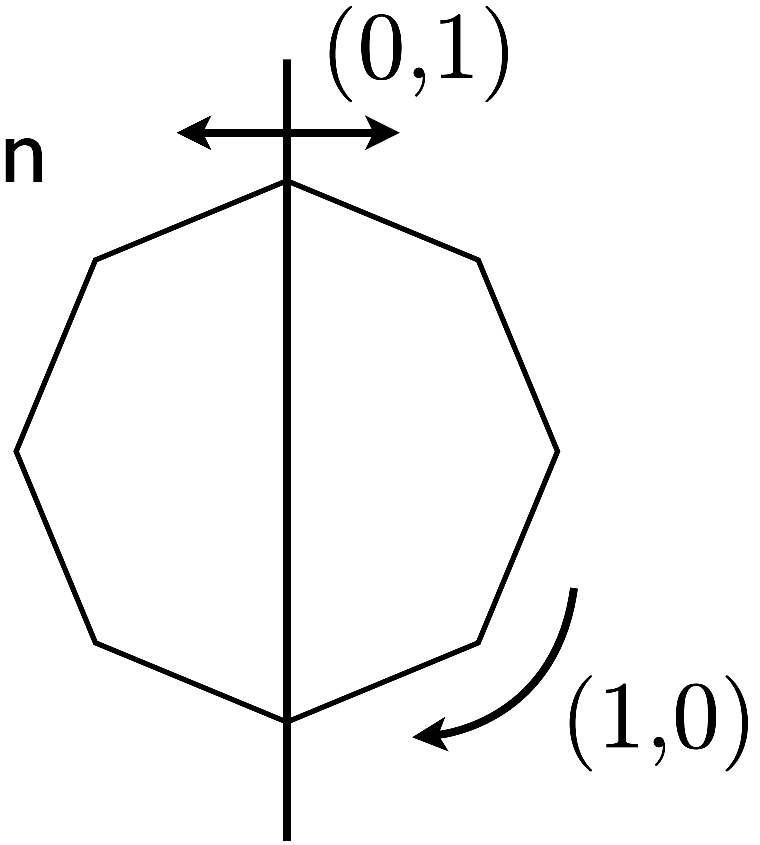

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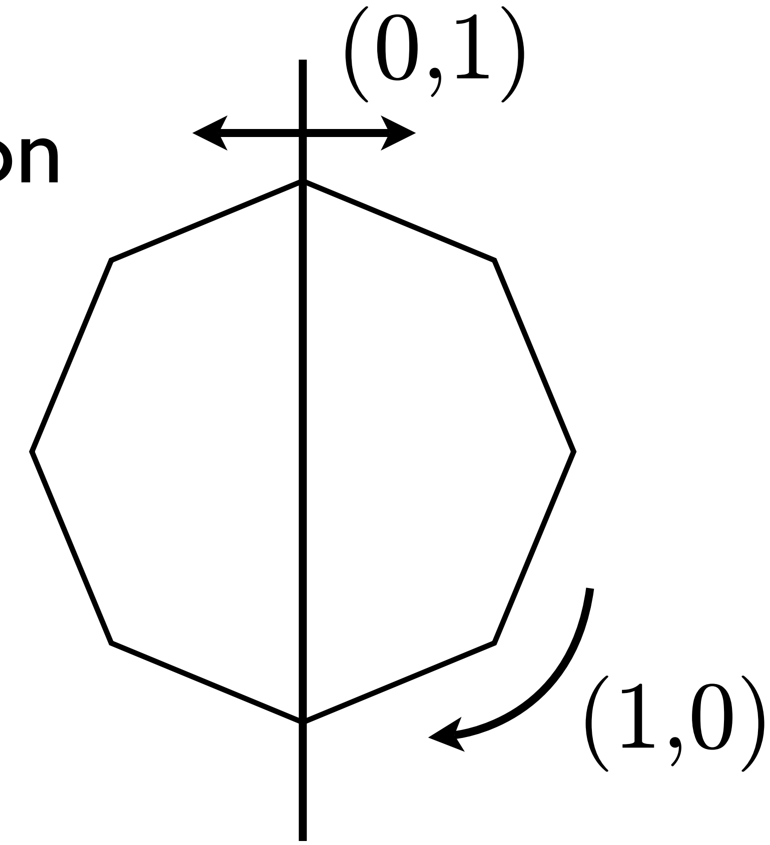


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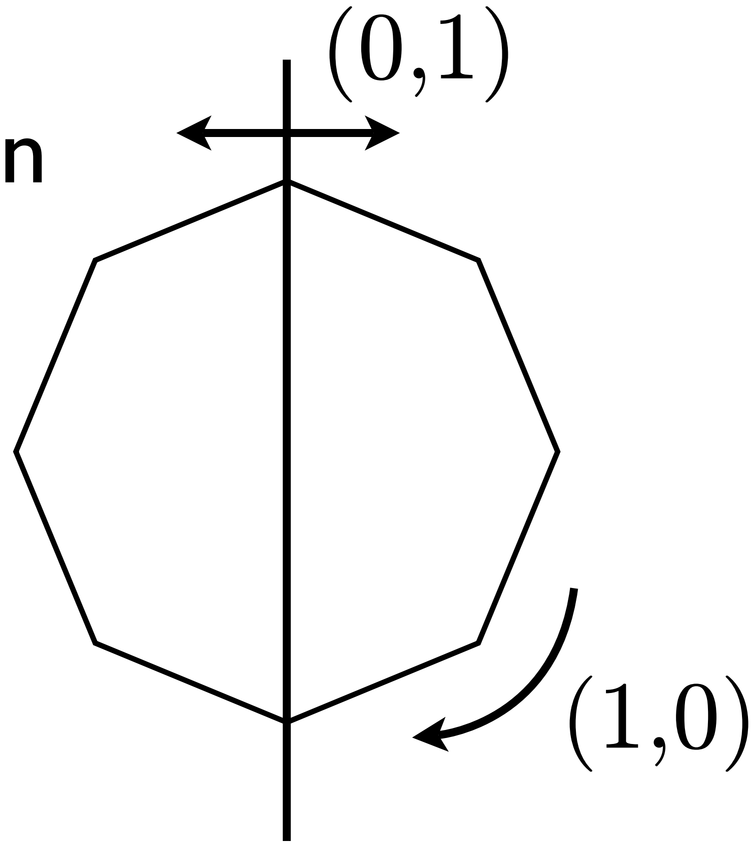


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Coset states: $|(a',0)H\rangle = \frac{1}{\sqrt{2}}(|a',0\rangle + |a+a',1\rangle)$

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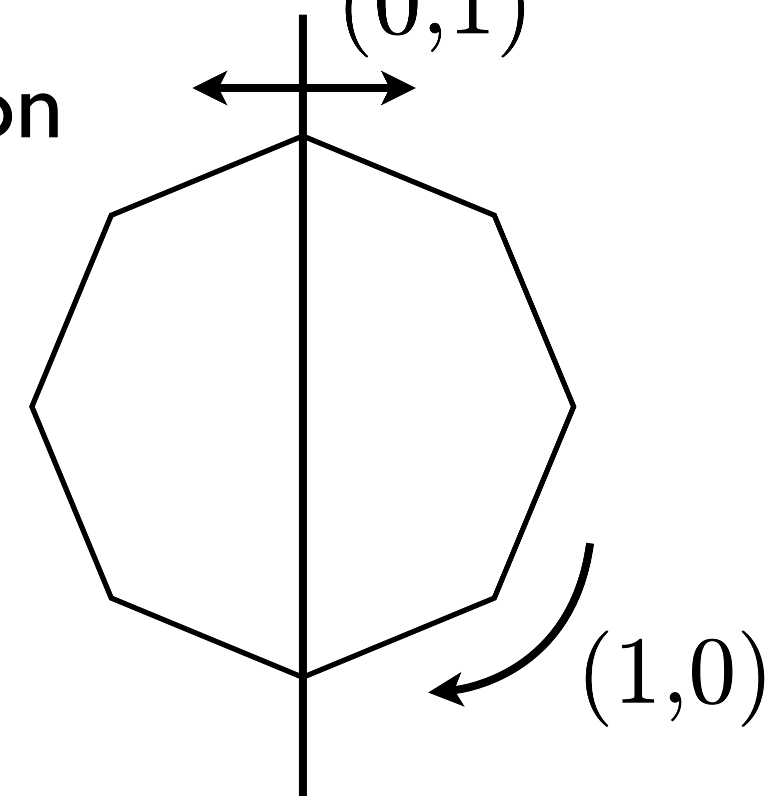


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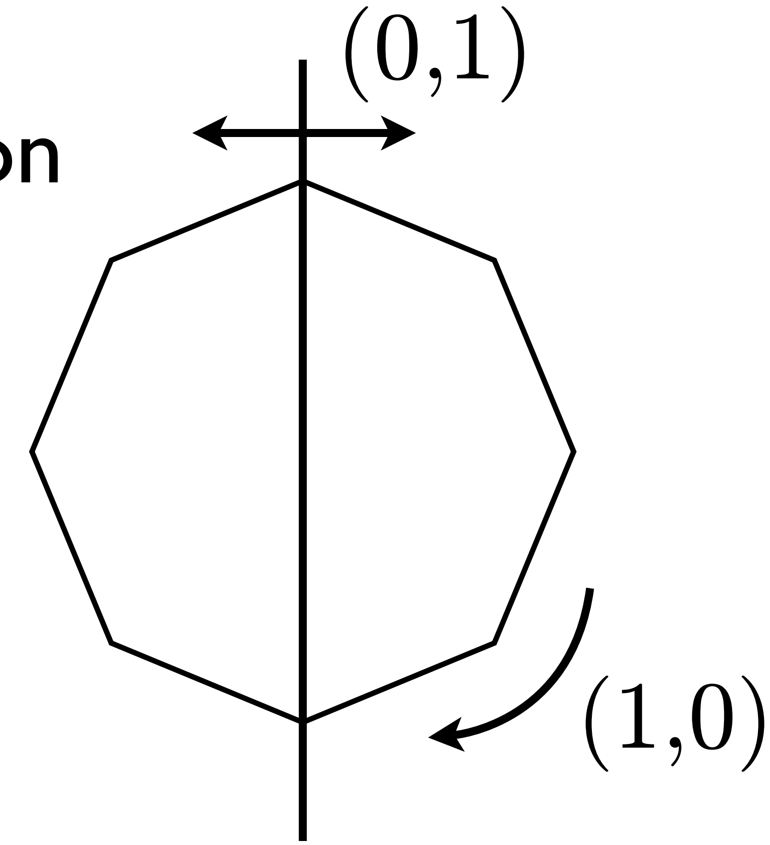


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By symmetry, we can measure $x$ wlog (Fourier sampling: measure which irreducible representation)

# Multiple dihedral coset states

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + \omega^{xa}|1\rangle)\right)^{\otimes k} = \frac{1}{\sqrt{2^k}} \sum_{b \in \mathbb{Z}_2^k} \omega^{(b \cdot x)a} |b\rangle$$

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$$= \frac{1}{\sqrt{2^k}} \sum_{w\in\mathbb{Z}_N} \omega^{wa} \sqrt{\eta_w^x}|S_w^x\rangle$$

solutions of *subset sum problem*:

$$S_w^x := \{b \in \mathbb{Z}_2^k : b \cdot x = w\}$$

$$\eta_w^x := |S_w^x|$$

$$|S_w^x\rangle := \frac{1}{\sqrt{\eta_w^x}} \sum_{b\in S_w^x} |b\rangle$$

# Subset sum and DHSP

The PGM (which is optimal) can be implemented unitarily by doing the inverse of the *quantum sampling* transformation:

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Questions:

- How big must $k$ be so that the solutions of the subset sum problem are nearly uniformly distributed?
- For such values of $k$, can we quantum sample from the subset sum solutions?

# Subset sum problem

Problem: Given $k$ integers $x_1,\ldots,x_k$ from $\mathbb{Z}_N$ and a target $w$ from $\mathbb{Z}_N$, find a subset of the $k$ integers that sum to the target (i.e., find $b_1,\ldots,b_k$ from $\mathbb{Z}_2$ so that $b \cdot x = w$).

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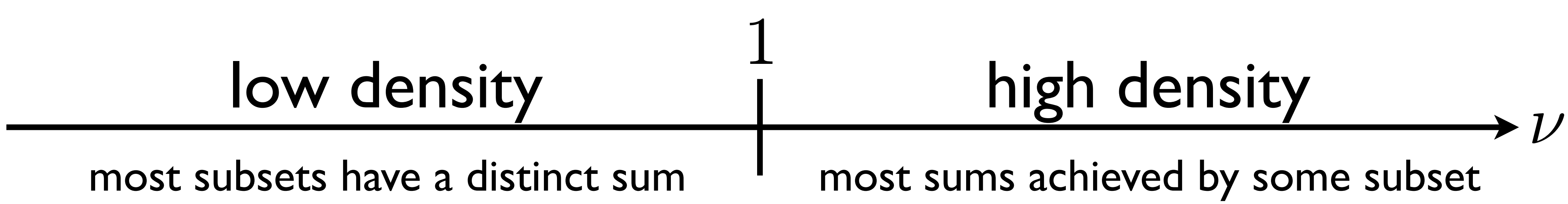

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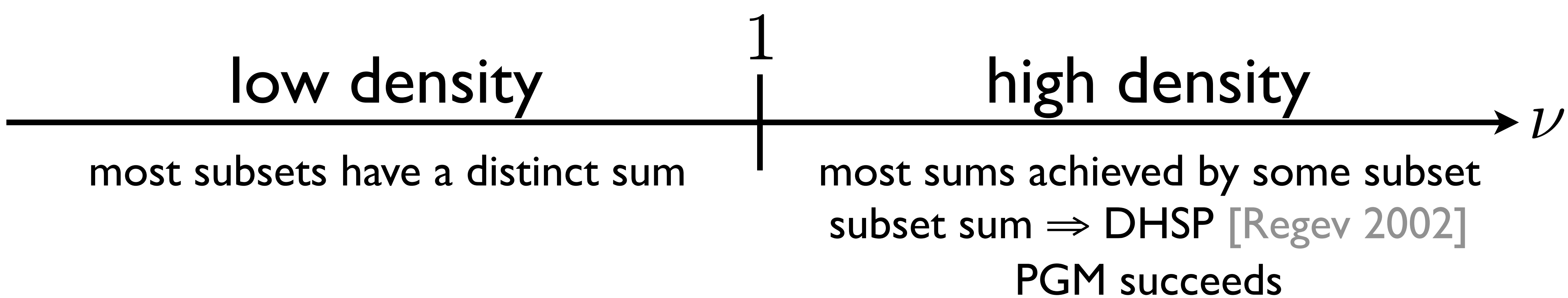

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low density | high density $\nu$

most subsets have a distinct sum

most sums achieved by some subset

subset sum $\Rightarrow$ DHSP [Regev 2002]

PGM succeeds

| | | |
|---|---|---|
| $k < c\sqrt{\log N}$<br>efficient classical algorithm<br>[Lagarias, Odlyzko 1985] | hard? | $k > 2^{c\sqrt{\log N}}$<br>$\mathrm{poly}(k)$ classical algorithm<br>[Flaxman, Przydatek 2004] |

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$\mathrm{poly}(k)$ quantum DHSP algorithm
[Kuperberg 2003]

# General approach

- Cast problem as a state distinguishability problem (e.g., coset states for HSP)
- Express the states in terms of an average-case algebraic problem (e.g., subset sum for dihedral HSP)
- Perform the pretty good measurement on $k$ copies of the states:
  - Choose $k$ large enough that the measurement succeeds with reasonably high probability (this happens if the average-case problem typically has many solutions)
  - Implement the measurement by solving the problem on average (quantum sampling from the set of solutions)

# The Heisenberg group

Subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ a & c & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\}$$

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Semidirect product $\mathbb{Z}_p^2 \rtimes_\varphi \mathbb{Z}_p$

$$\varphi : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_p^2) \quad \textbf{with} \quad \varphi(c)(a,b) = (a + bc, b)$$

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Group of $p \times p$ unitary matrices

$\langle X, Z \rangle = \{\omega^a X^b Z^c : a, b, c \in \mathbb{Z}_p\}$ where

$$X := \sum_{x \in \mathbb{Z}_p} |x + 1\rangle\langle x| \,, \quad Z := \sum_{x \in \mathbb{Z}_p} \omega^x |x\rangle\langle x| \,, \quad \omega := e^{2\pi \mathrm{i}/p}$$

# Heisenberg subgroups

Fact: To solve the HSP in the Heisenberg group, it is sufficient to distinguish the order $p$ subgroups $\langle(a,b,1)\rangle = \{(a,b,1)^j : j \in \mathbb{Z}_p\}$

$$(a,b,1)^2 = (a,b,1)(a,b,1) = (2a+b,2b,2)$$
$$(a,b,1)^3 = (a,b,1)(2a+b,2b,2) = (3a+3b,3b,3)$$
$$(a,b,1)^4 = (a,b,1)(3a+2b,3b,3) = (4a+6b,4b,4)$$
$$\vdots$$
$$(a,b,1)^j = (ja+\tbinom{j}{2}b, jb, j)$$

# Heisenberg coset states

Identity coset:

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Fourier transform and measure the first two registers:

$$\frac{1}{\sqrt{p}} \sum_{j \in \mathbb{Z}_p} \omega^{x\left[ja + \binom{j}{2} b\right] + yjb} |j\rangle$$

$x$,$y$ uniformly random; note $a'$,$b'$ disappear

# Two coset states

$$\left(\frac{1}{\sqrt{p}}\sum_{j\in\mathbb{Z}_p}\omega^{axj+b\left[yj+x\binom{j}{2}\right]}\big|j\big\rangle\right)^{\otimes 2}$$

# Two coset states

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$$= \frac{1}{p} \sum_{j_1, j_2 \in \mathbb{Z}_p} \omega^{a(x_1 j_1 + x_2 j_2) + b\left[y_1 j_1 + y_2 j_2 + x_1 \binom{j_1}{2} + x_2 \binom{j_2}{2}\right]} \big| j_1, j_2 \big\rangle$$

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For typical values of $x_1$,$x_2$,$y_1$,$y_2$,$v$,$w$ there are two solutions $(j_{1,1},j_{2,1})$, $(j_{1,2},j_{2,2})$.

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# Generalized abelian hidden shift problem

Problem: Given a function $f : \{0, 1, \ldots, M-1\} \times \mathbb{Z}_N \to S$ satisfying $f(b, x) = f(b+1, x+s)$ for $b = 0, 1, \ldots, M-2$, find the value of the *hidden shift* $s \in \mathbb{Z}_N$.

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This is an instance of integer programming in $k$ dimensions. Lenstra's algorithm (based on LLL lattice basis reduction) solves this efficiently for $k$ constant. $k{=}\log N / \log M \Rightarrow$ efficient algorithm for any $M{=}N^\epsilon$ for fixed $\epsilon{>}0$.

| Original problem | $k$ | Average-case problem | Solution |
|---|---|---|---|
| Abelian HSP | $1$ | Linear equations | Easy |
| Metacyclic HSP<br>$\mathbb{Z}_N \rtimes \mathbb{Z}_p,\ p = \phi(N)/\operatorname{poly}(\log N)$ | $1$ | Discrete log | Shor's algorithm |
| $\mathbb{Z}_p^r \rtimes \mathbb{Z}_p$<br>($r$=$2$ is Heisenberg) | $r$ | Polynomial equations | Buchburger's algorithm, elimination |
| Generalized abelian hidden shift problem, $M$=$N^\epsilon$ | $1/\epsilon$ | Integer programming | Lenstra's algorithm |
| Dihedral HSP | $\log N$ | Subset sum | ? |
| Symmetric group HSP | $n \log n$ | ? | ? |

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- Is there a problem that is not even information theoretically reconstructible from *single*-register measurements, but for which there is an efficient *multi*-register algorithm?