Quantum walks on graphs

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Information is physical.

Outline

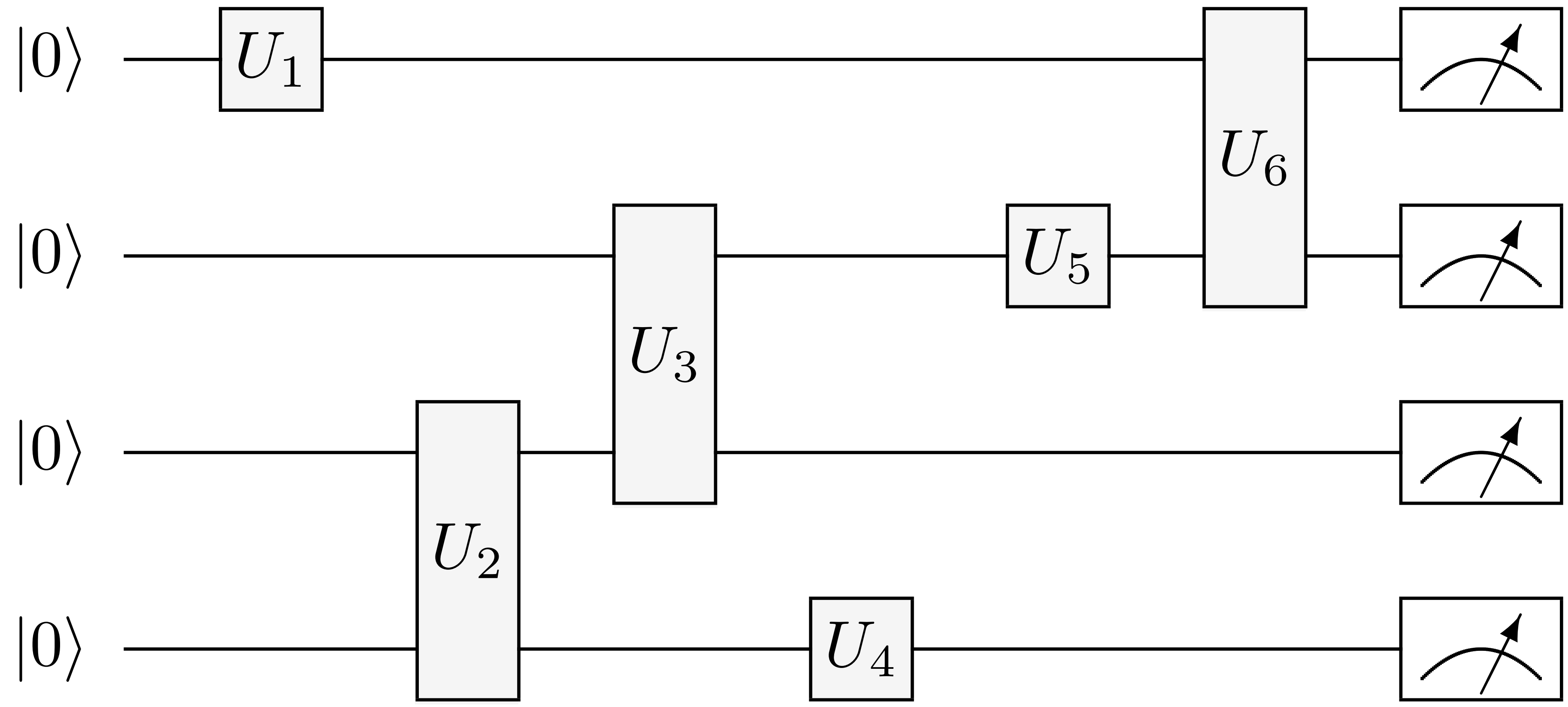
1. Introduction: Quantum systems as information processing machines
2. Exponential speedup by quantum walk
3. Spatial search by quantum walk
4. Evaluating Boolean formulas

1. Introduction

The universal quantum computer

(The ultimate quantum physics lab!)

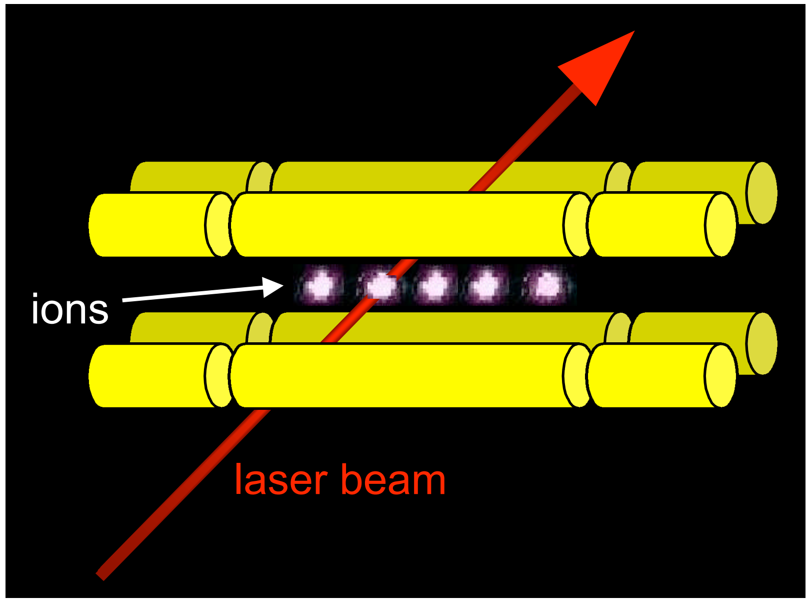
- Prepare n qubits in the state $|00\ldots0\rangle$
- Apply a sequence of unitary operations acting on one or two qubits at a time
- Perform a measurement to get the result



Note: Many equivalent models exist (Hamiltonian dynamics of coupled spins, braiding of nonabelian anyons, quantum cellular automata, ...).

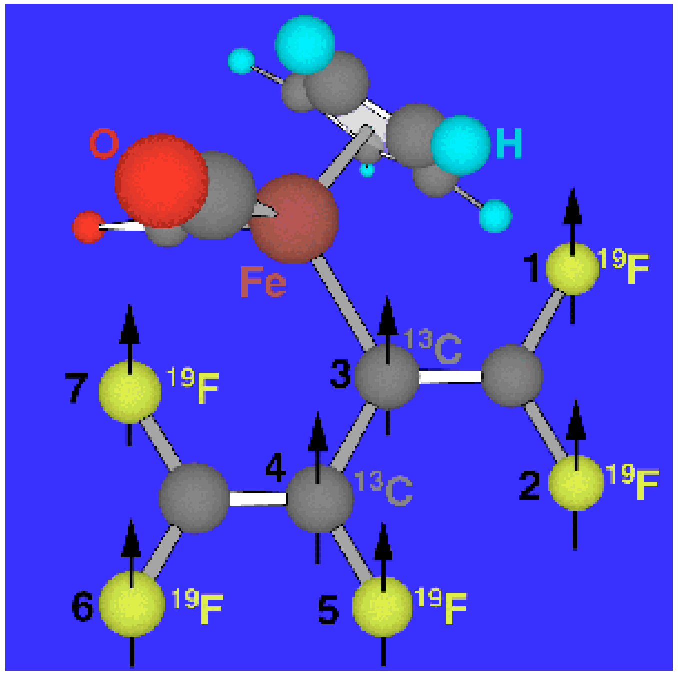
Implementations of quantum computers

Trapped ions



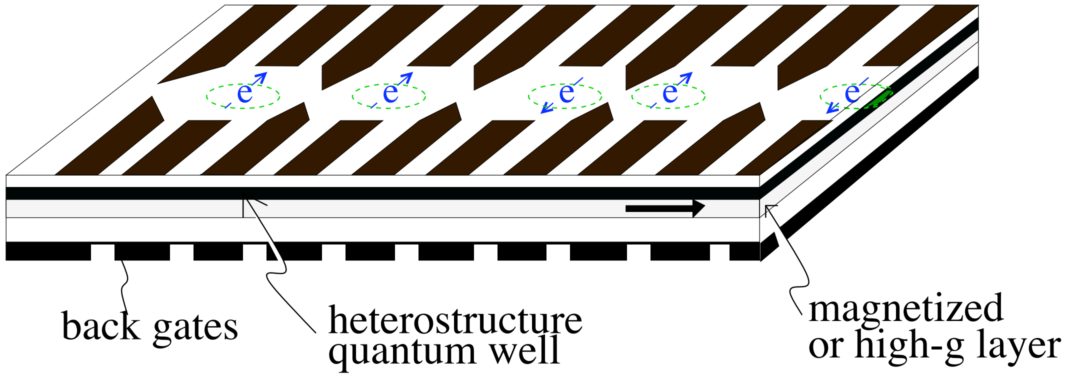
(Monroe & Wineland)

Nuclear spins



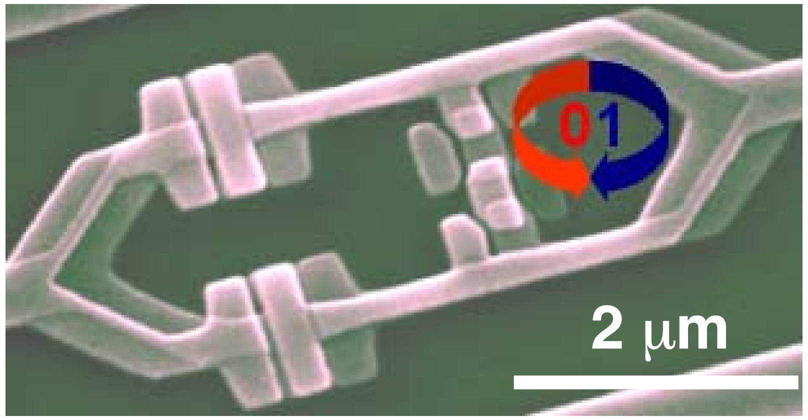
(Chuang et al.)

Quantum dots



(Loss & DiVincenzo)

Superconducting circuits



(Nakamura et al.)

... and many others!

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In the rest of this talk, we assume a perfectly functioning quantum computer.

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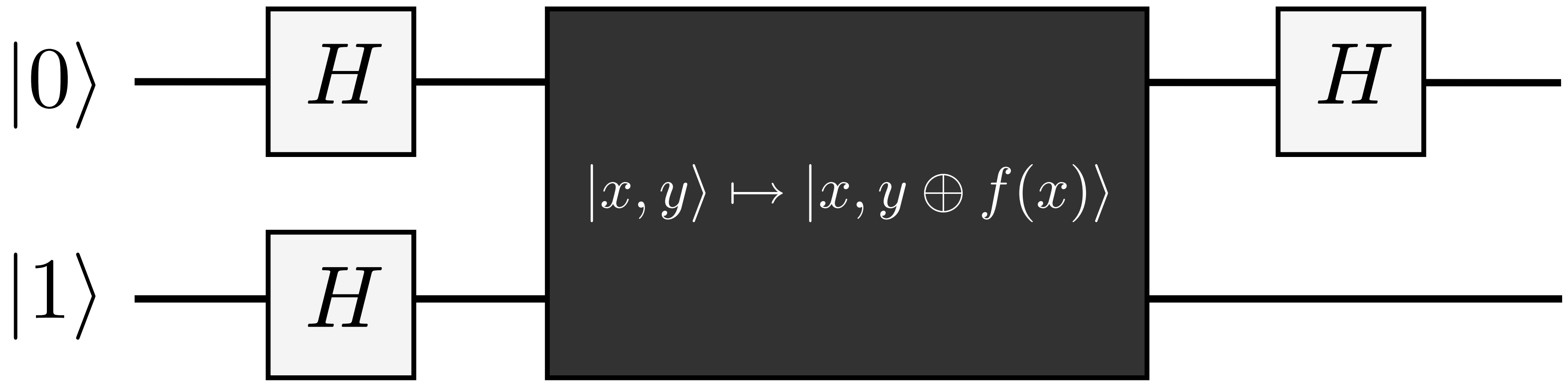

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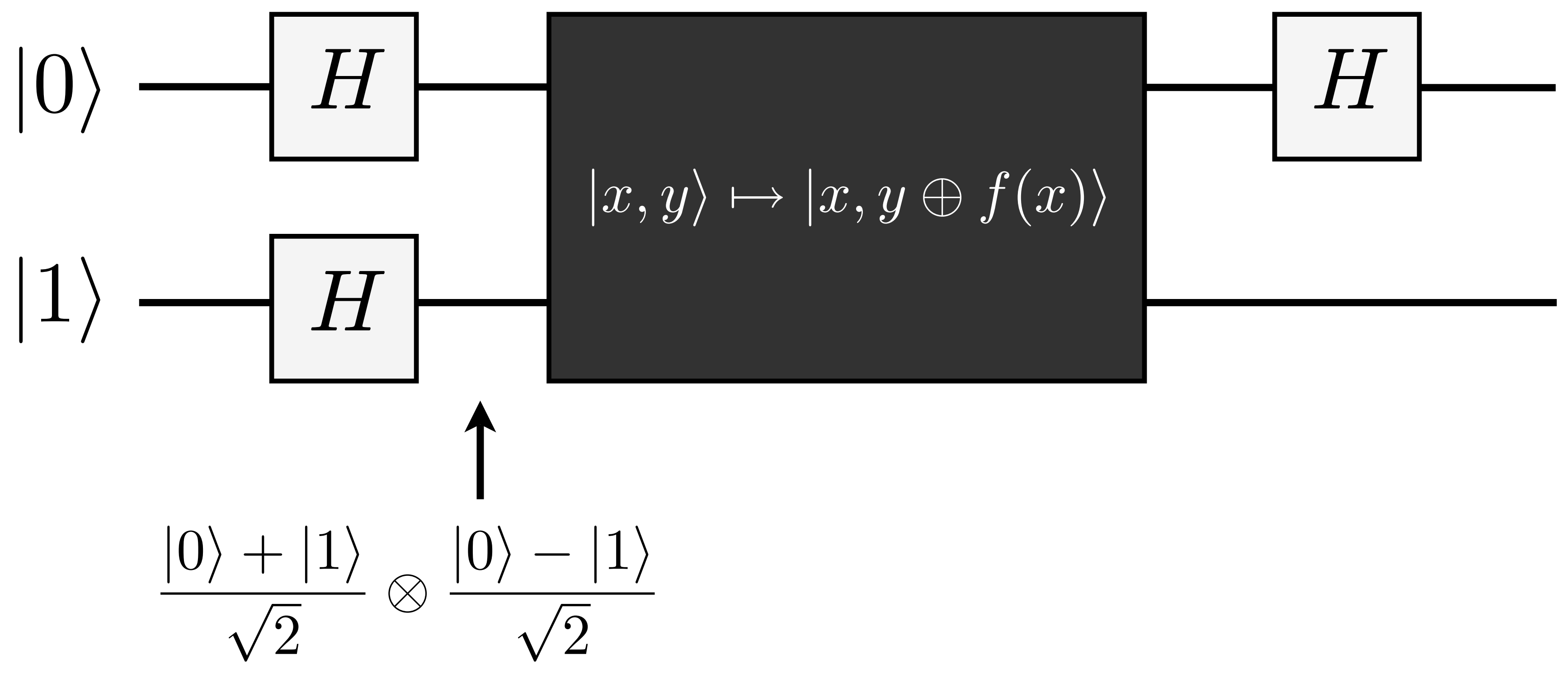

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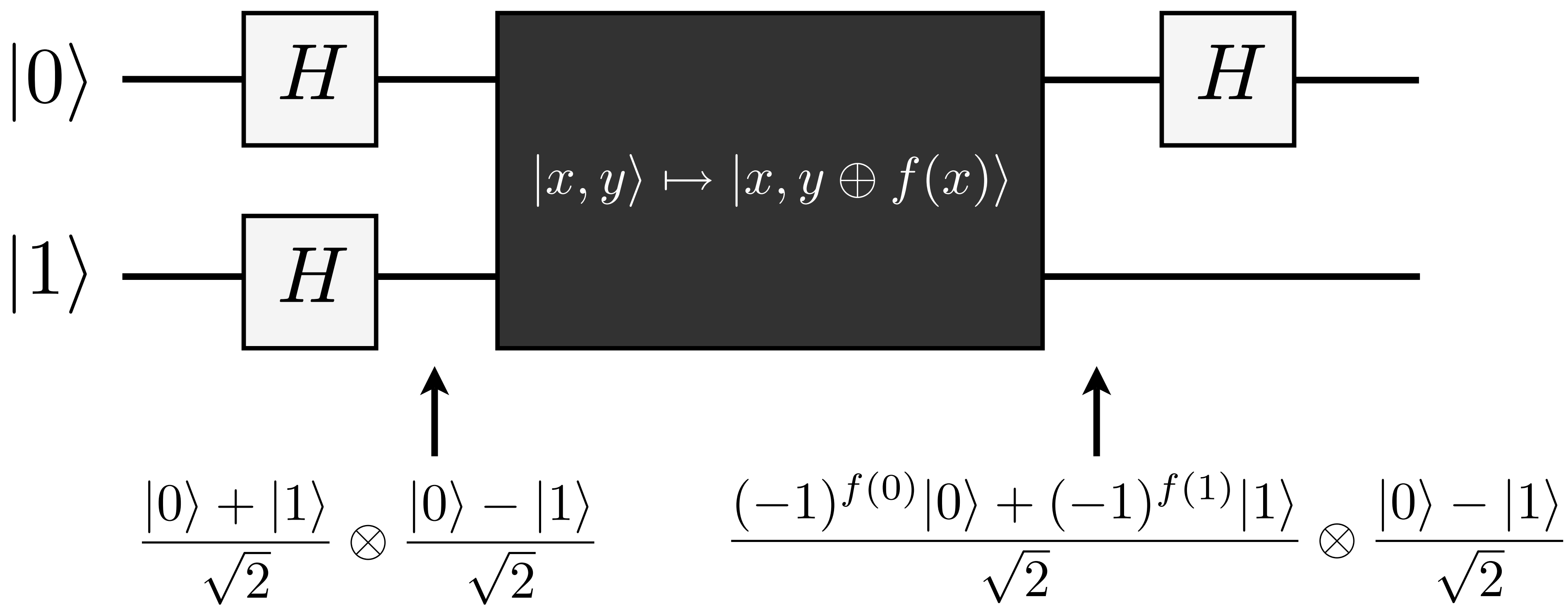

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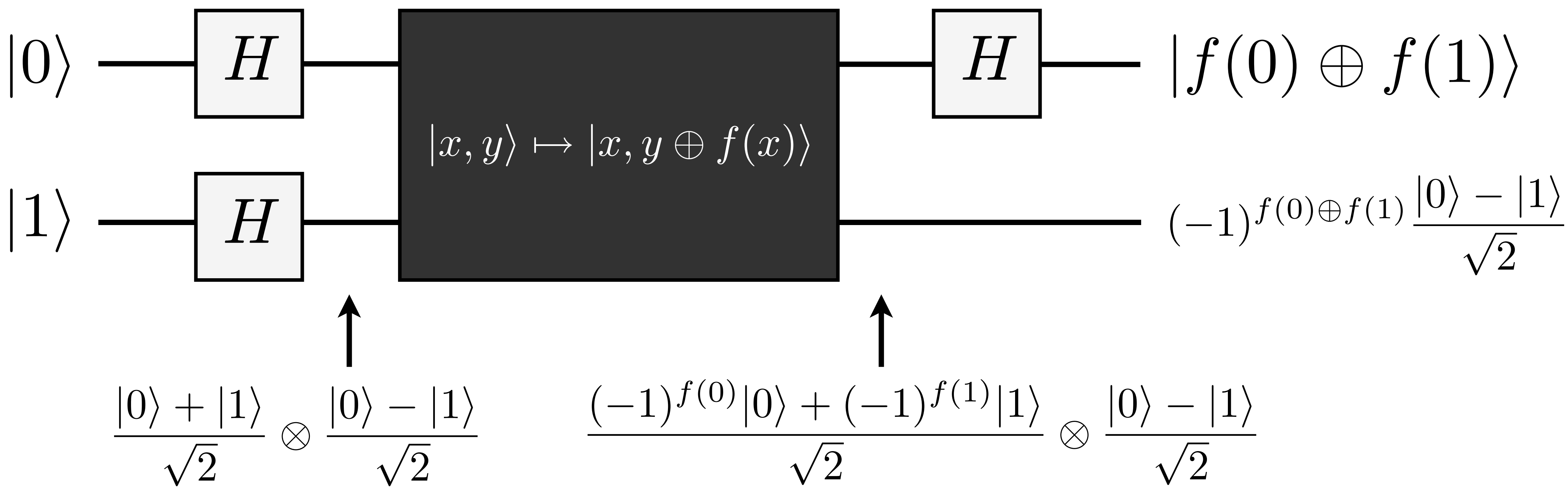

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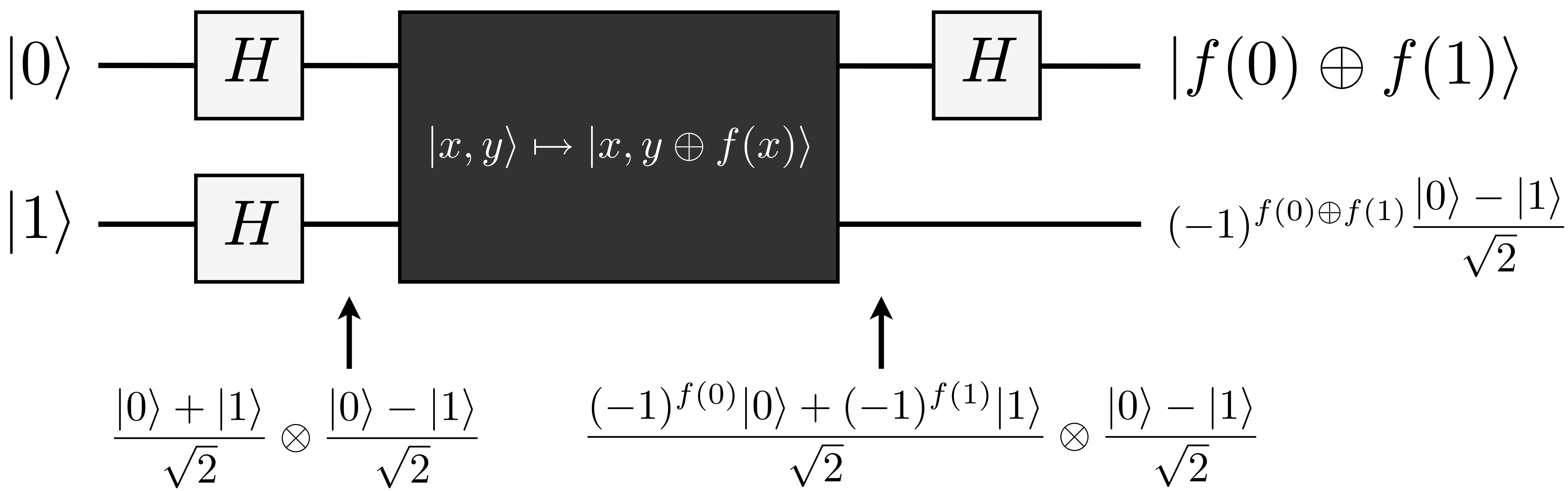

But don't classical systems exhibit interference too?

Hilbert space is a big place

n classical bits: 2^n discrete values

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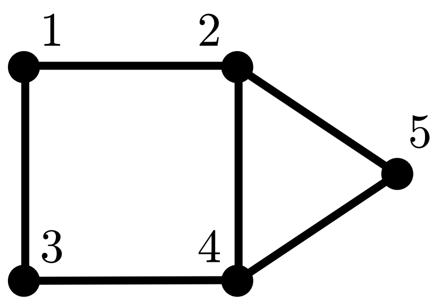
But probabilities do not exhibit interference!

How can we exploit the efficient representation of interference phenomena to perform fast computations?

From random walk to quantum walk

Graph G:



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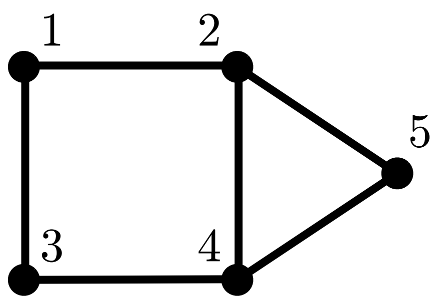

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

adjacency matrix

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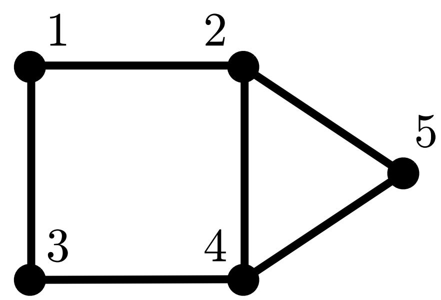

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adjacency matrix *Laplacian*

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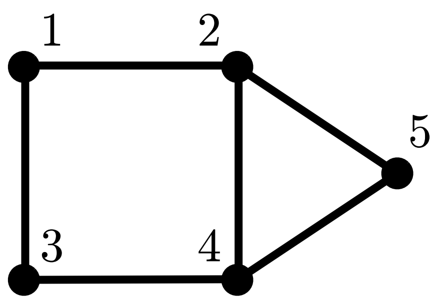

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State: Probability $p_j(t)$ of being at vertex j at time t

Dynamics: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -\gamma L \vec{p}$

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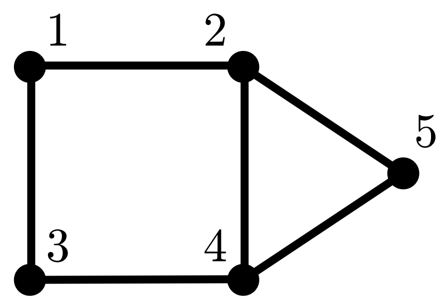

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Laplacian

Random walk on G

State: Probability $p_j(t)$ of being at vertex j at time t

Dynamics: $$\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -\gamma L\vec{p}$$

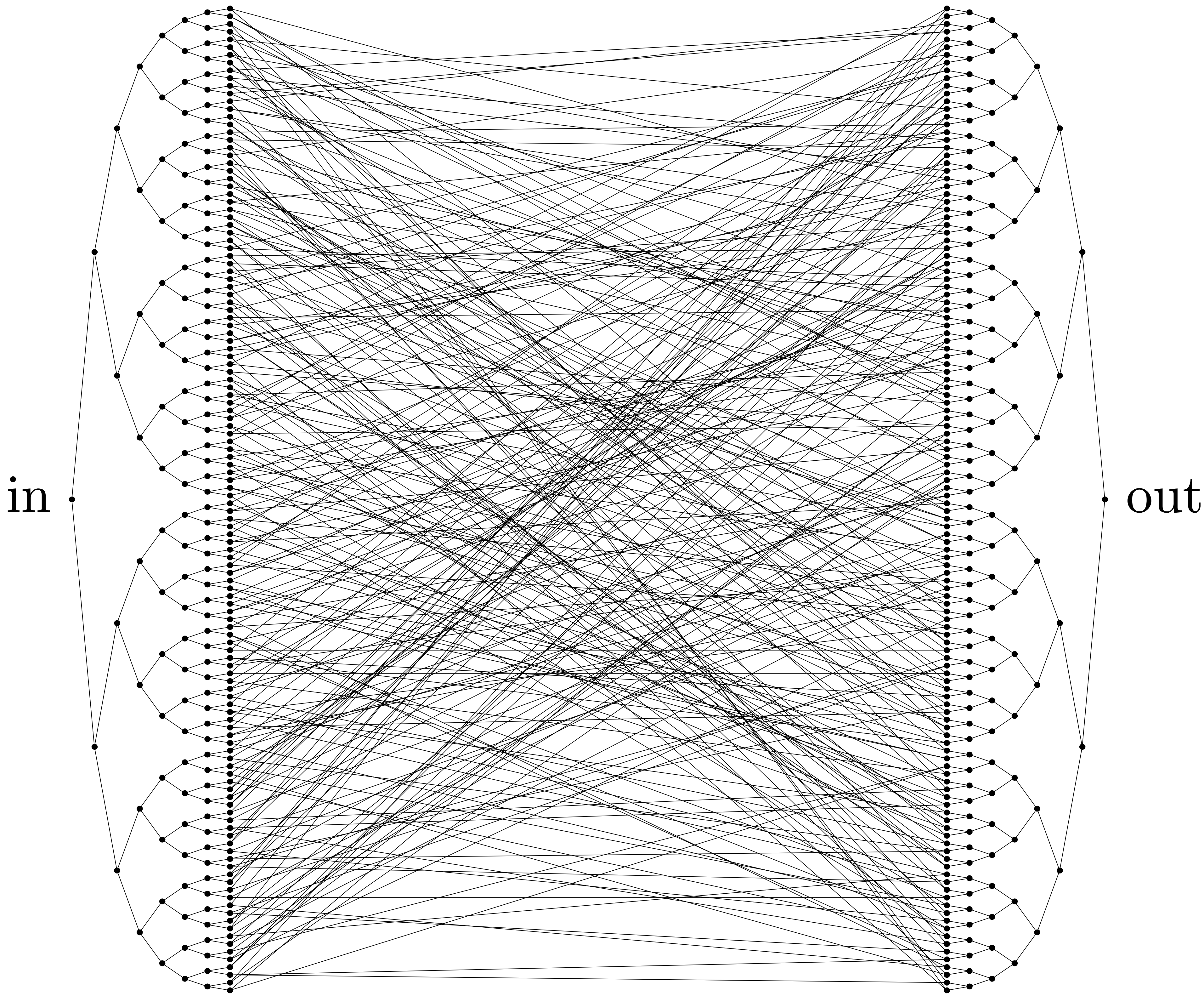
Quantum walk on G

State: Amplitude $q_j(t)$ to be at vertex j at time t

Dynamics: $$i\frac{\mathrm{d}}{\mathrm{d}t}\vec{q} = -\gamma L\vec{q}$$

2. Exponential speedup by quantum walk

- Childs, Farhi, and Gutmann, Quantum Inf. Proc. **1**, 35-43 (2002).
- Childs, Cleve, Deotto, Farhi, Gutmann, and Spielman, in Proc. STOC (2003), pp. 59-68.



in
out

Black box graph traversal problem

Names of vertices are random bit strings (length $2\log|G|$).
Name of in vertex is known.
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Claim: There is a family of graphs G_n (with designated in and out vertices) for which a quantum walk starting at the in vertex finds the out vertex in time $\mathrm{poly}(n)$, but any classical algorithm using $\mathrm{poly}(n)$ queries finds the out vertex with exponentially small probability.

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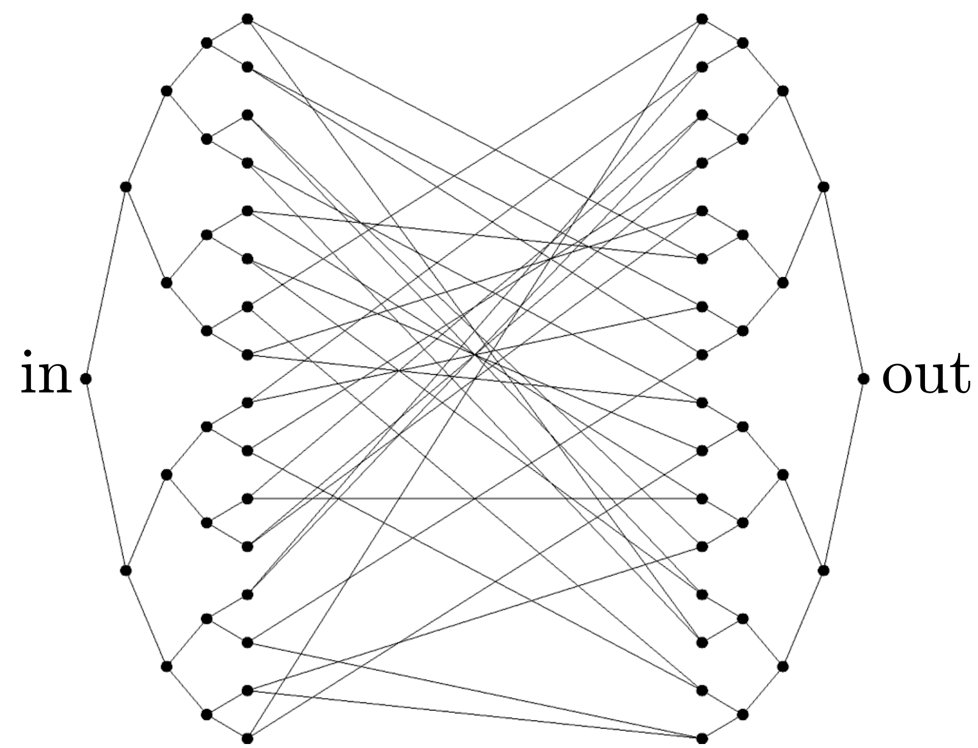
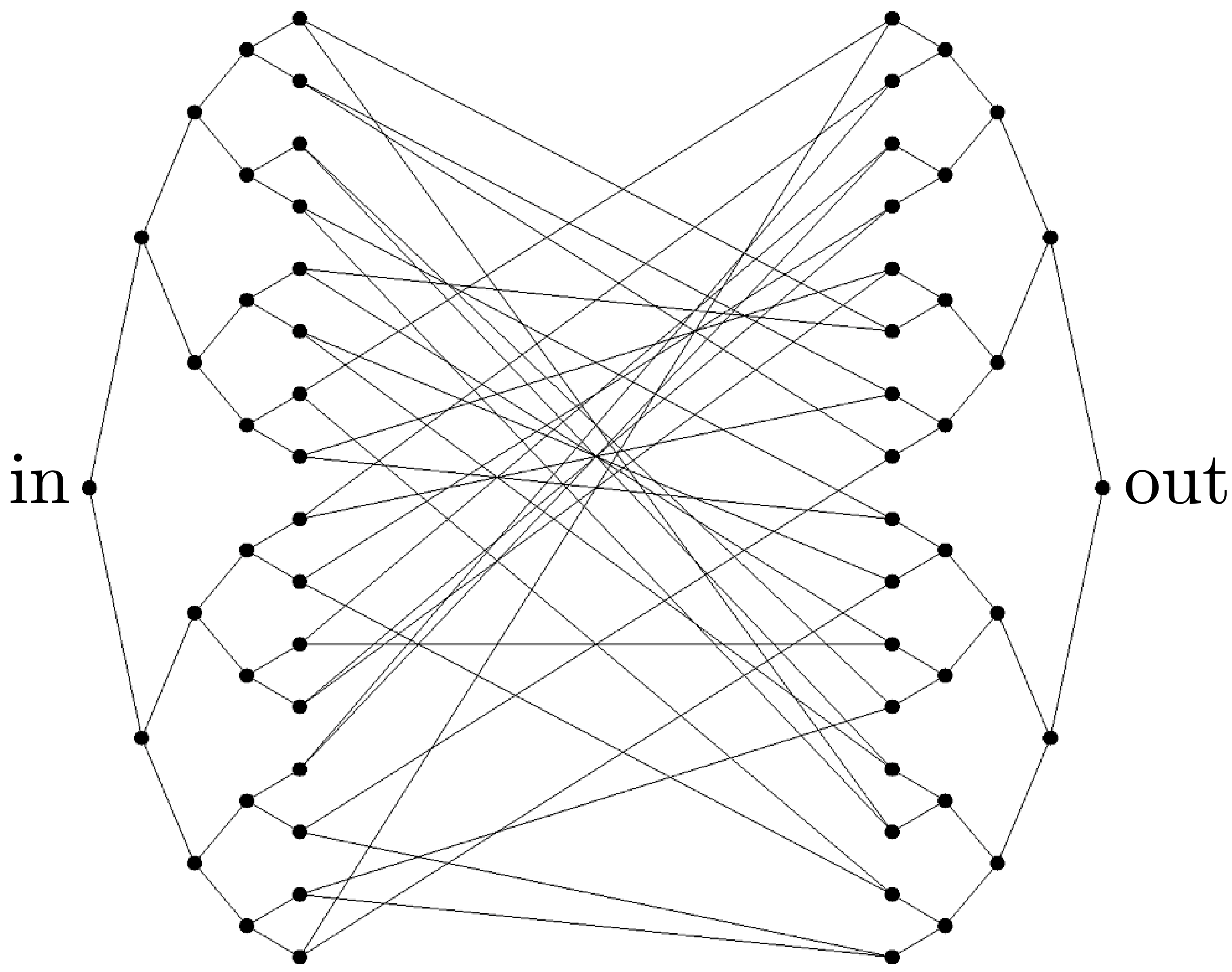
Typical G_4:



Reduction of the quantum walk



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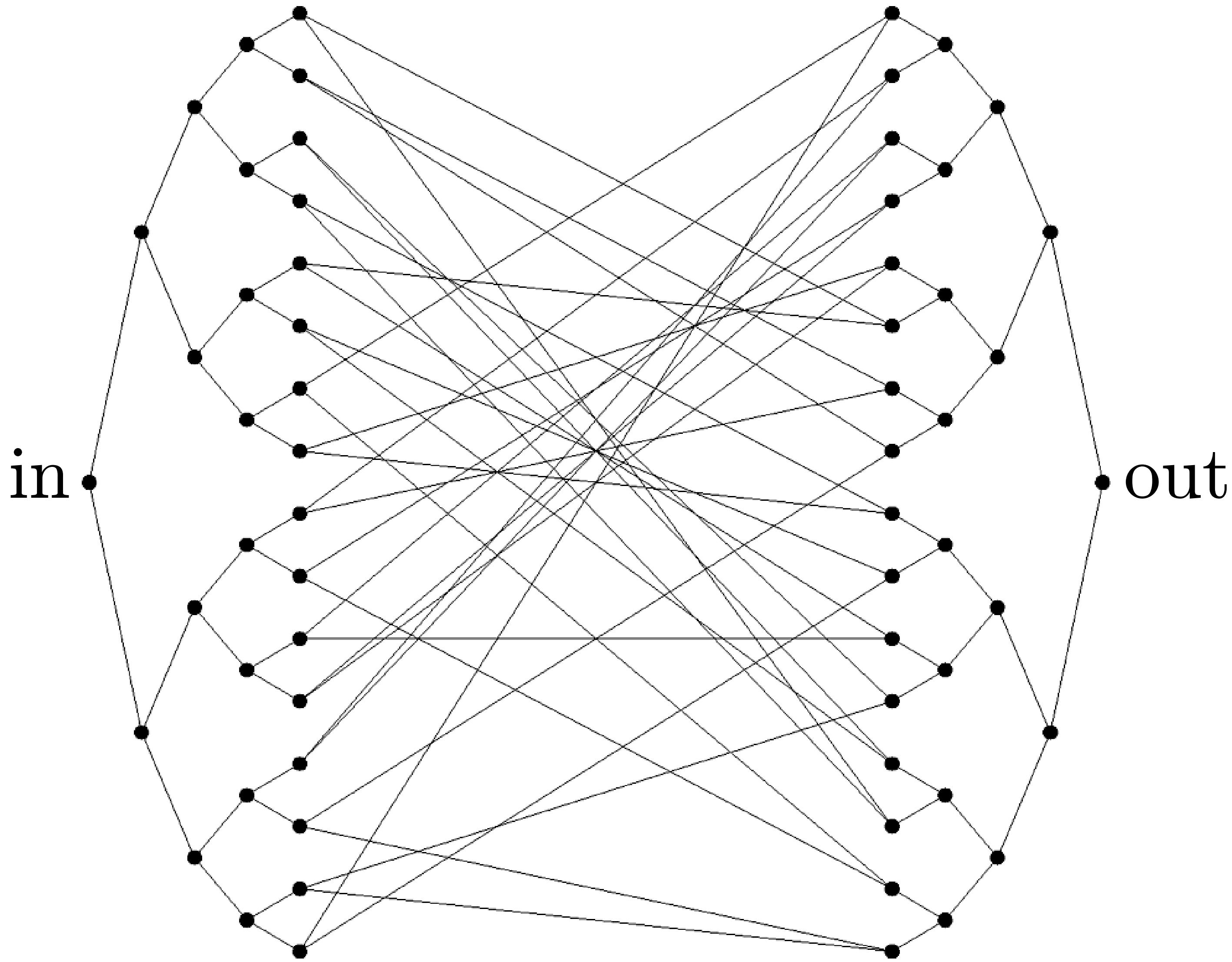

Column subspace

$$|\text{col } j\rangle = \frac{1}{\sqrt{N_j}} \sum_{a \in \text{column } j} |a\rangle$$

where

$$N_j = \begin{cases} 2^j & 0 \le j \le n \\ 2^{2n+1-j} & n+1 \le j \le 2n+1 \end{cases}$$

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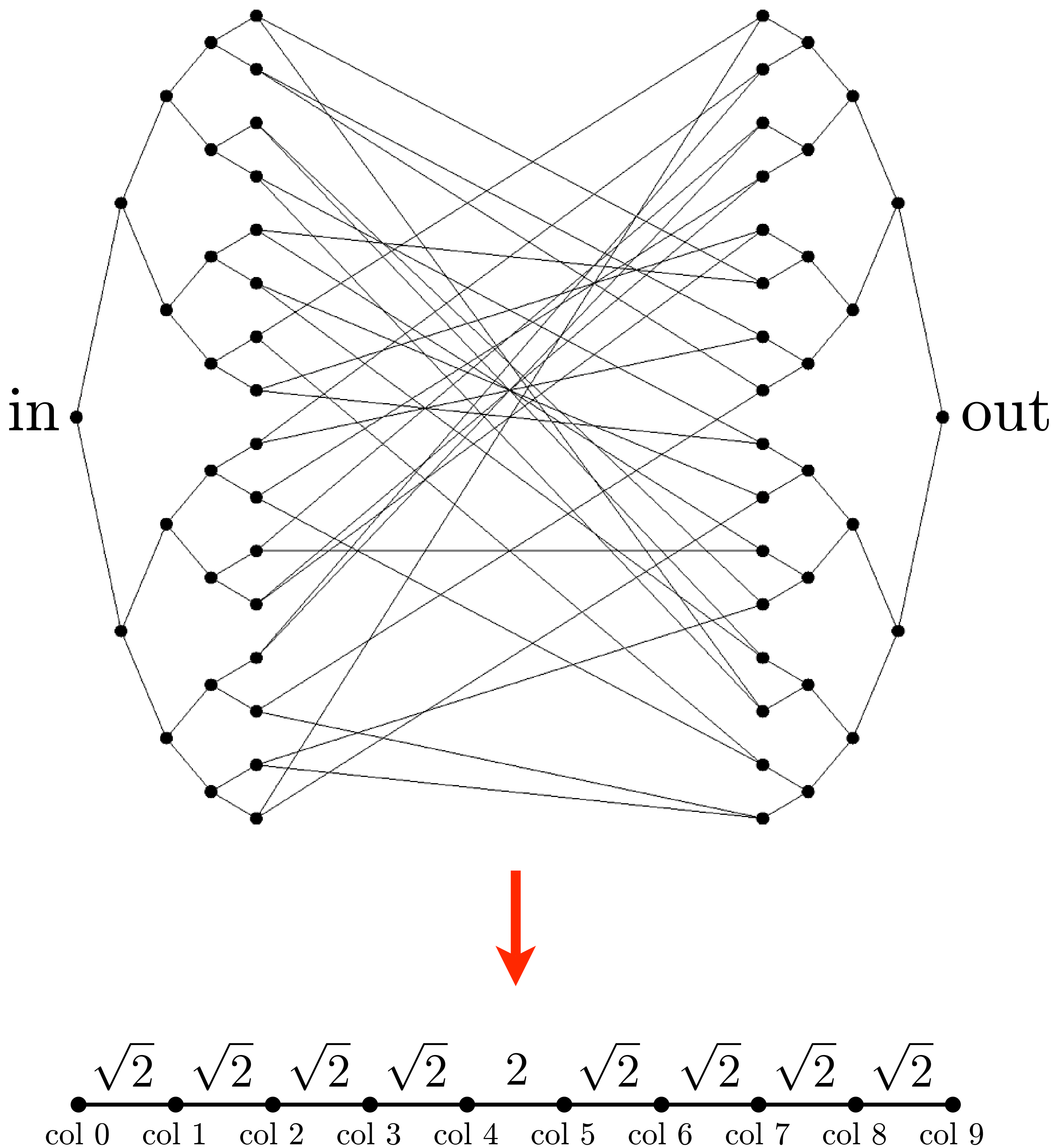

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Reduced adjacency matrix

$$\langle \text{col } j|A|\text{col } j+1\rangle$$

$$= \begin{cases} \sqrt{2} & 0 \le j \le n-1\,, \\ & n+1 \le j \le 2n \\ 2 & j = n \end{cases}$$

Implementing the walk

Problem: Given a black box for G, implement the quantum walk on G, i.e., simulate the unitary time evolution e^{-iHt} where $H{=}L$ (or A).
(Cf. implementing a random walk, which is easy.)

Main idea: Color the graph. Then the simulation breaks into small pieces that are easy to handle.

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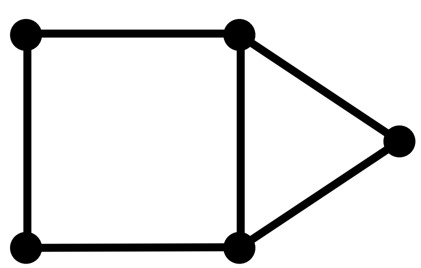

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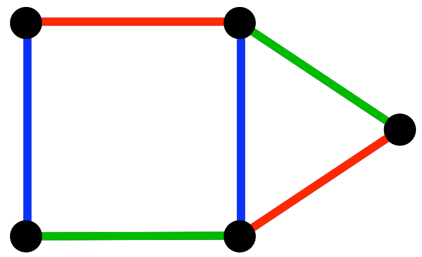

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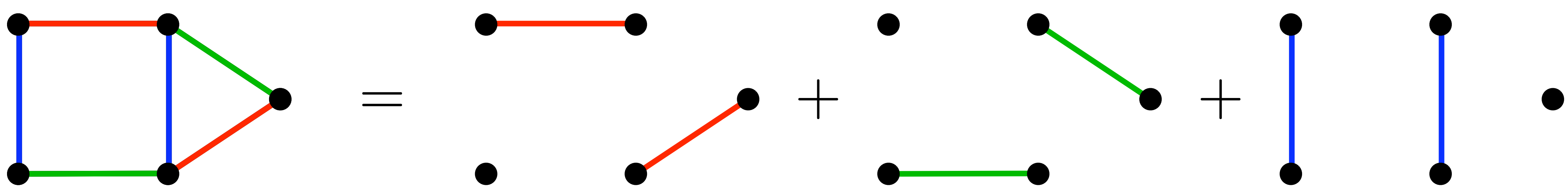

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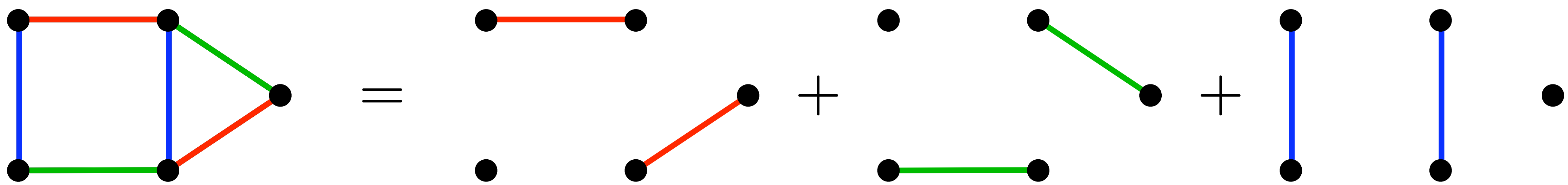

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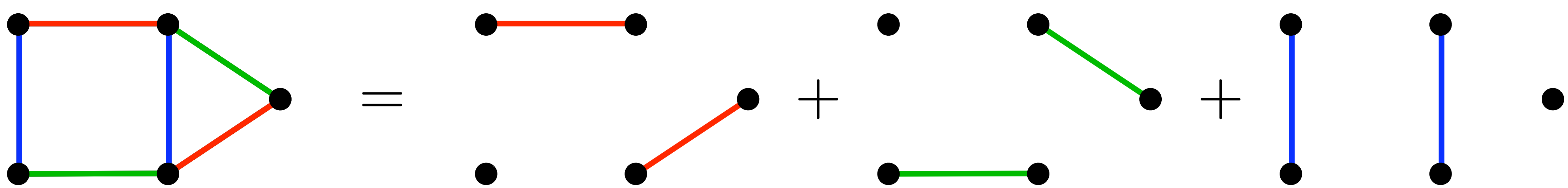

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This idea is useful for simulating Hamiltonians whenever the graph of nonzero matrix elements is sufficiently sparse.

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Theorem: Any classical algorithm that makes at most $2^{n/6}$ queries to the black box finds the out vertex with probability at most $4 \cdot 2^{-n/6}$.

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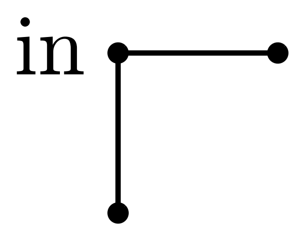

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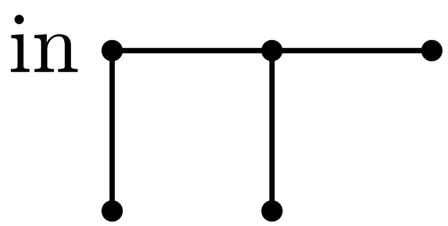

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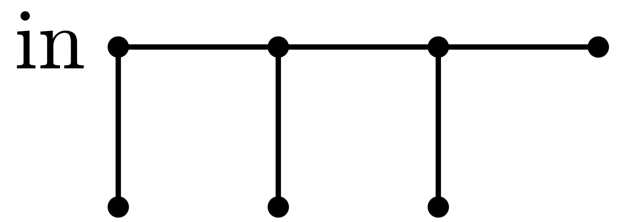

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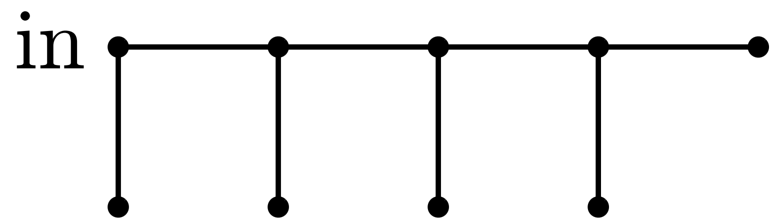

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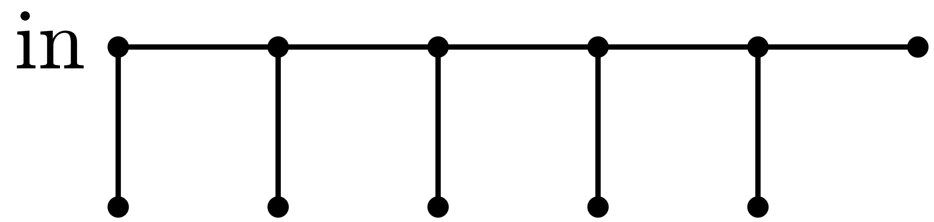

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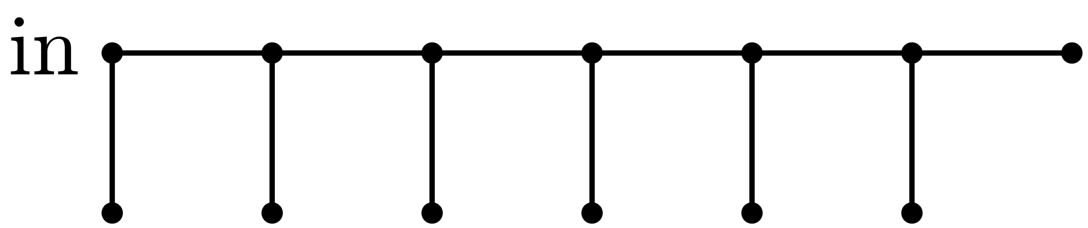

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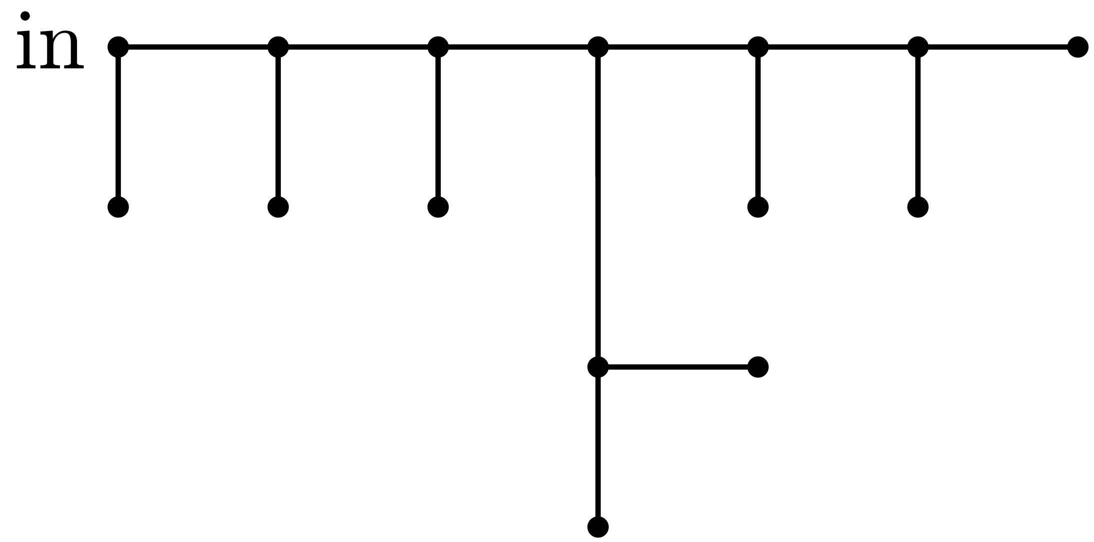

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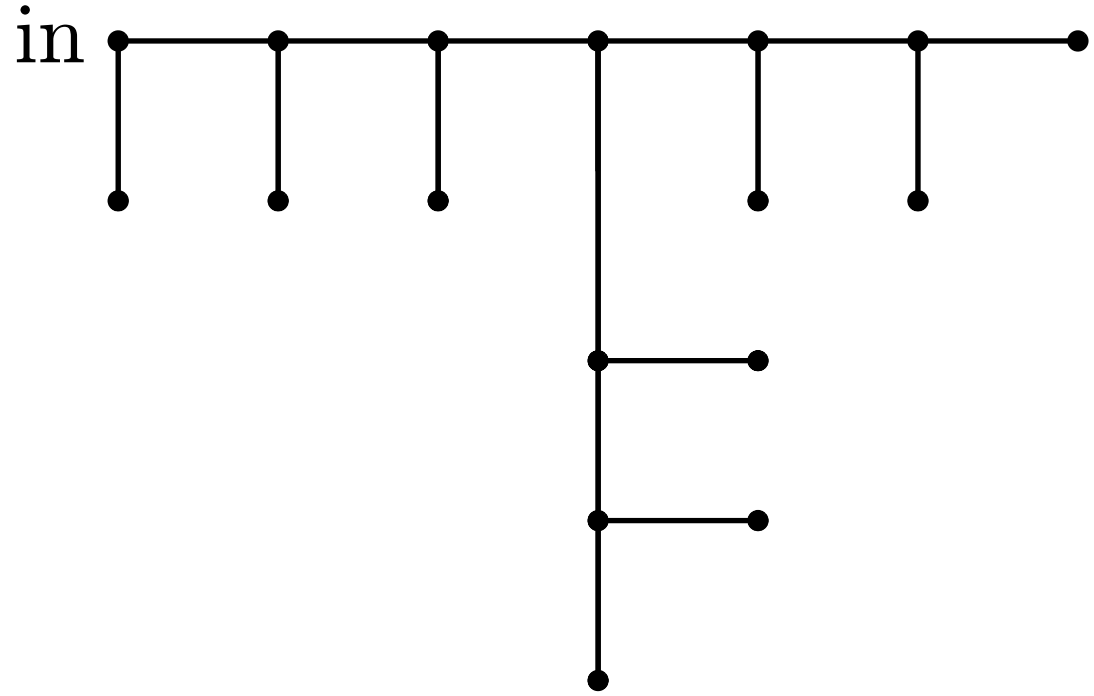

3. Spatial search by quantum walk

- Childs and Goldstone, Phys. Rev. A **70**, 022314 (2004).
- Childs and Goldstone, Phys. Rev. A **70**, 042312 (2004).

Unstructured search

Search space: N items, $x \in \{1, 2, \ldots, N\}$

Goal: Find one *marked item*, w

Query: "is $w = x$?"

i.e., black box function $f(x) = \begin{cases} 0 & x \neq w \\ 1 & x = w \end{cases}$

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Quantum lower bound [BBBV 1996]: $\Omega(\sqrt{N})$

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Quantum lower bound [BBBV 1996]: $\Omega(\sqrt{N})$

Grover searching can be applied to many computational problems.
But can it be used to search a physical database, in which the N items are distributed in space?

Spatial search by quantum walk

Quantum walk approach: $H = -\gamma L - |w\rangle\langle w|$

$$L_{ab} = \begin{cases} 1 & ab \in G \\ -\deg(a) & a = b \\ 0 & \text{otherwise} \end{cases}$$

$$L \approx \nabla^2$$

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$$\gamma \to 0$$

$$H \approx -|w\rangle\langle w|$$

ground state $\approx |w\rangle$

first excited state $\approx |s\rangle$

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ground state $\approx |w\rangle$

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critical γ

ground state $\approx |s\rangle + |w\rangle$

first excited state $\approx |s\rangle - |w\rangle$

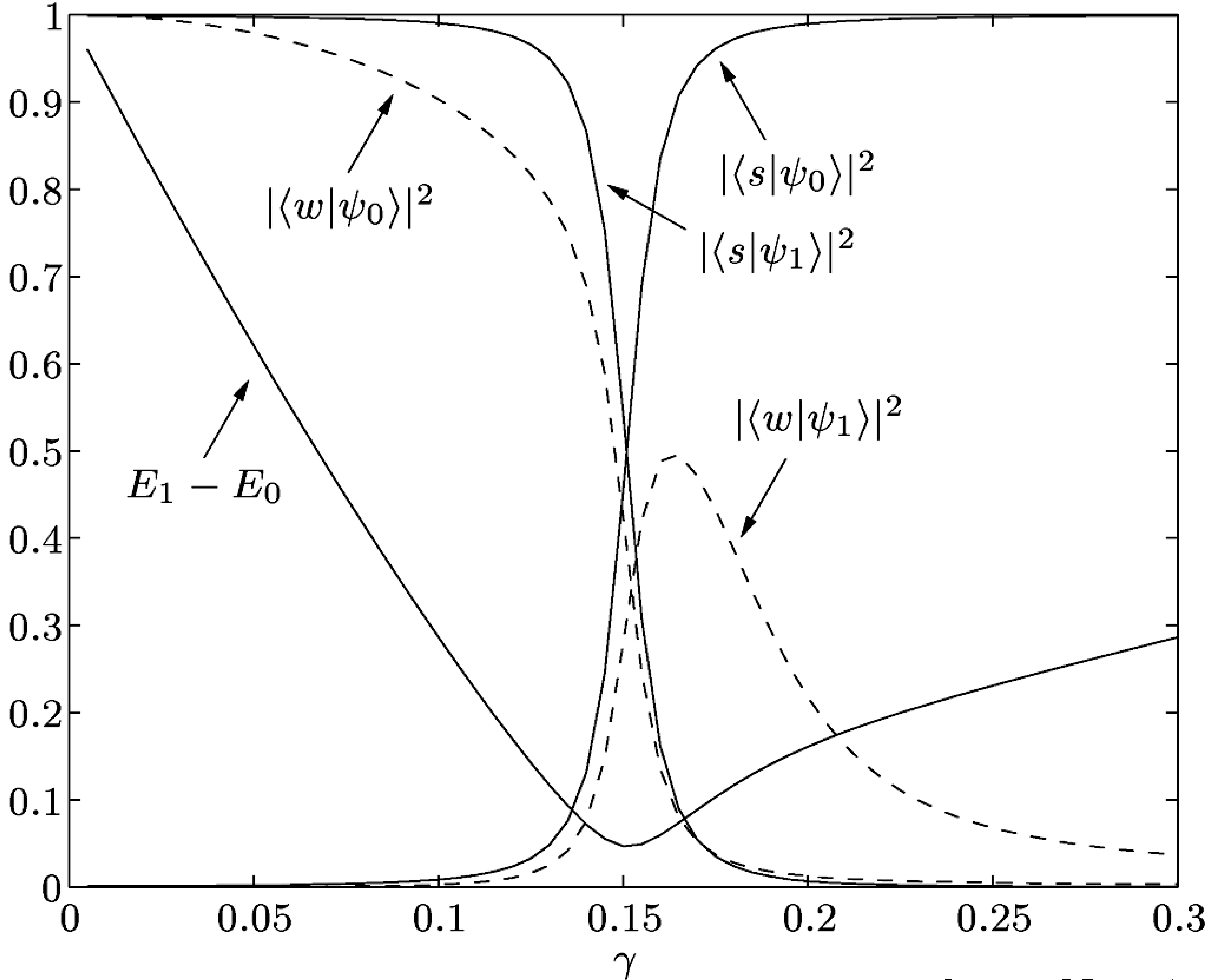
time $\approx \frac{1}{E_1 - E_0}$

$\gamma \to \infty$

$H \approx -\gamma L$

ground state $\approx |s\rangle$

γ



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
γ
$|\langle w|\psi_0\rangle|^2$
$|\langle s|\psi_0\rangle|^2$
$|\langle s|\psi_1\rangle|^2$
$|\langle w|\psi_1\rangle|^2$
$E_1 - E_0$

$d = 4,\ N = 6^4 = 1296$

Results

Graph	Success amplitude	Run time
Complete	$1-o(1)$	$O(N^{1/2})$
Hypercube	$1-o(1)$	$O(N^{1/2})$
Lattice, $d>4$	$O(1)$	$O(N^{1/2})$
Lattice, $d=4$	$O(1/\log^{1/2} N)$	$O((N \log N)^{1/2})$
Lattice, $d=3$	$O(N^{-1/6})$	$O(N^{2/3})$
Lattice, $d=2$	$O((\log N/N)^{1/2})$	$O(N/\log N)$

Behavior for $d>4$

Critical γ: $\gamma_* = I_{1,d}$

Run time: $T = \dfrac{\pi\sqrt{I_{2,d}N}}{2I_{1,d}}$

Success probability: $|\langle w|e^{-iHT}|s\rangle|^2 = \dfrac{I_{1,d}^2}{I_{2,d}}$

where

$$I_{j,d} = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\frac{\mathrm{d}^d\vec{k}}{[\mathcal{E}(\vec{k})]^j}$$

$$\mathcal{E}(\vec{k}) = 2\gamma\big(d - \textstyle\sum_{j=1}^d \cos k_j\big)$$

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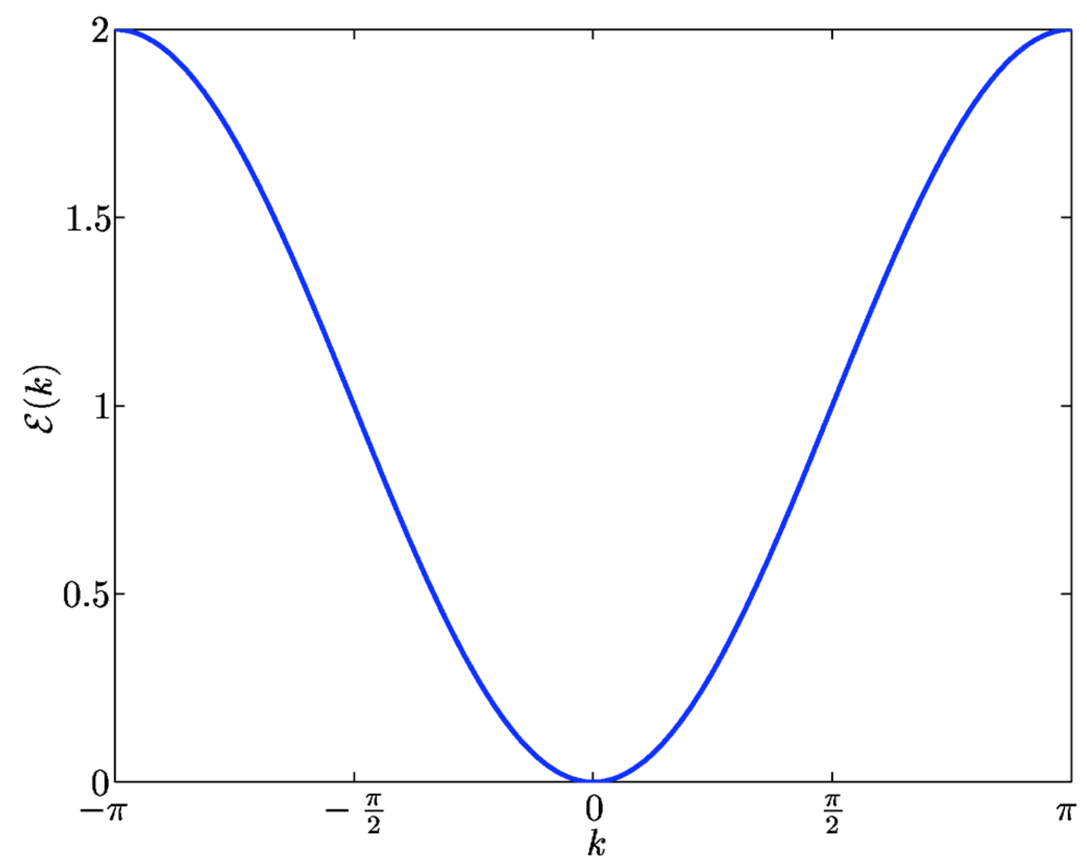

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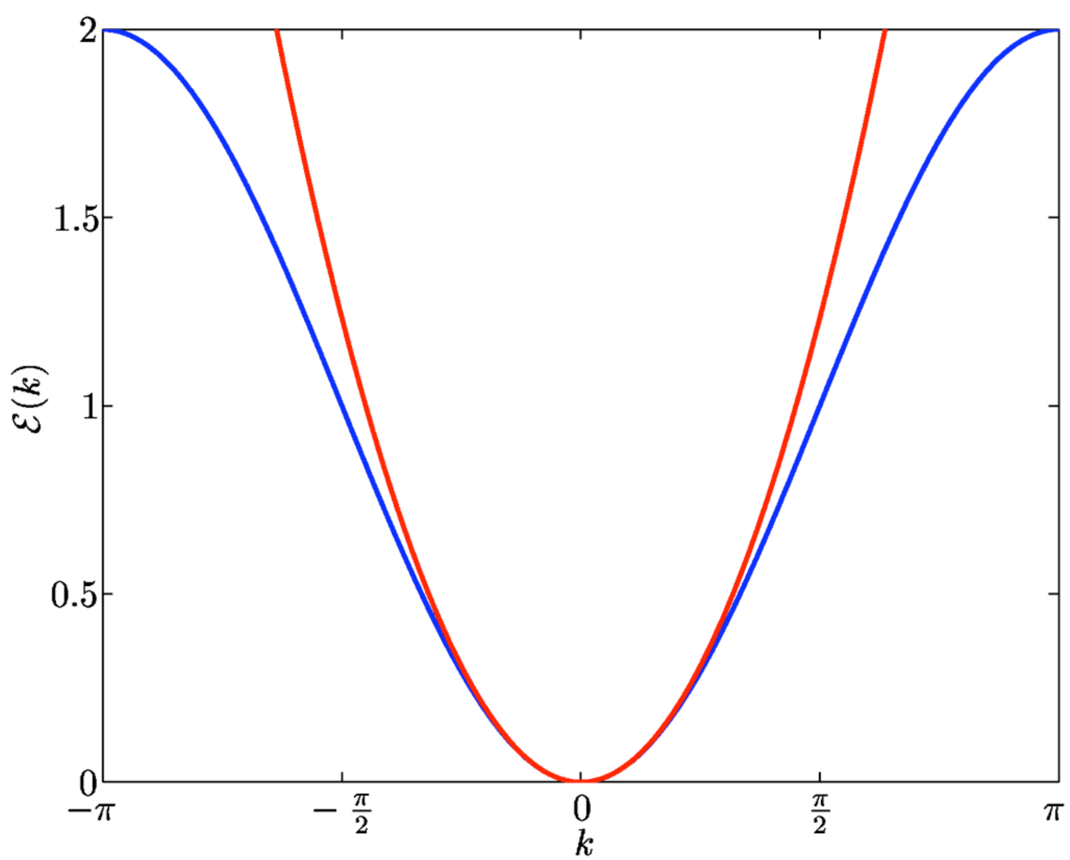

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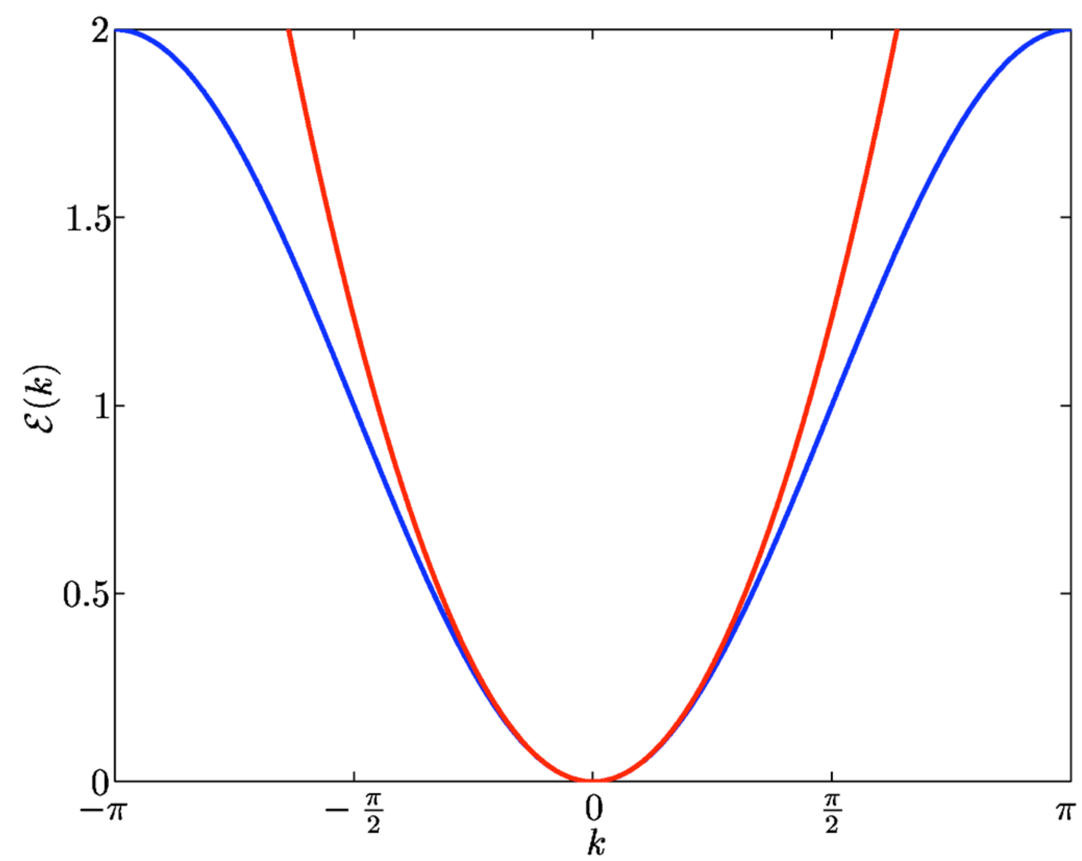
Note: $\displaystyle\int_0 \frac{\mathrm{d}^d\vec{k}}{|\vec{k}|^p} \sim \int_0 \frac{k^{d-1}\mathrm{d}k}{|\vec{k}|^p}$

converges for $d>p$.

dispersion relation



The Dirac equation: Faster search for $d=2,3,4$

Dirac Hamiltonian: $H_{\rm Dirac} = \sum_{j=1}^{d} \alpha_j p_j + \beta m \,, \quad p_j = i\frac{\rm d}{{\rm d}x_j}$

with $\{\alpha_j, \alpha_k\} = 2\delta_{jk} \,, \quad \{\alpha_j, \beta\} = 0 \,, \quad \beta^2 = 1$

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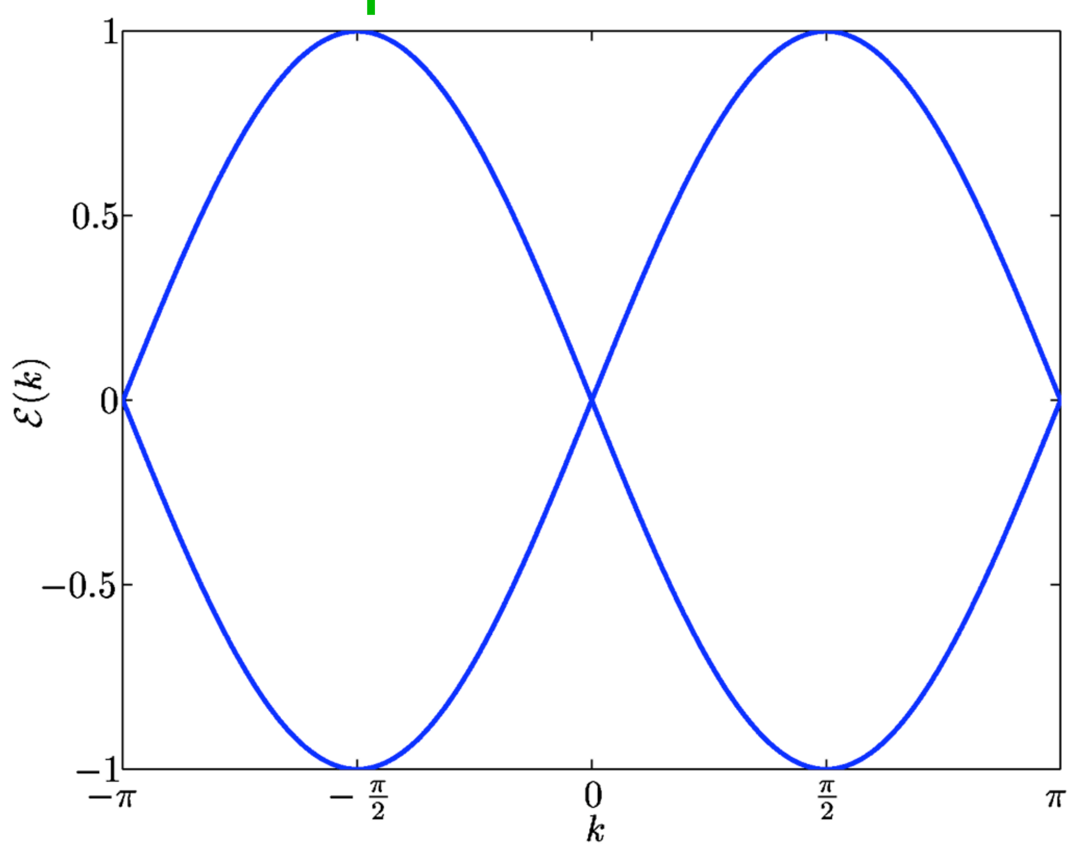

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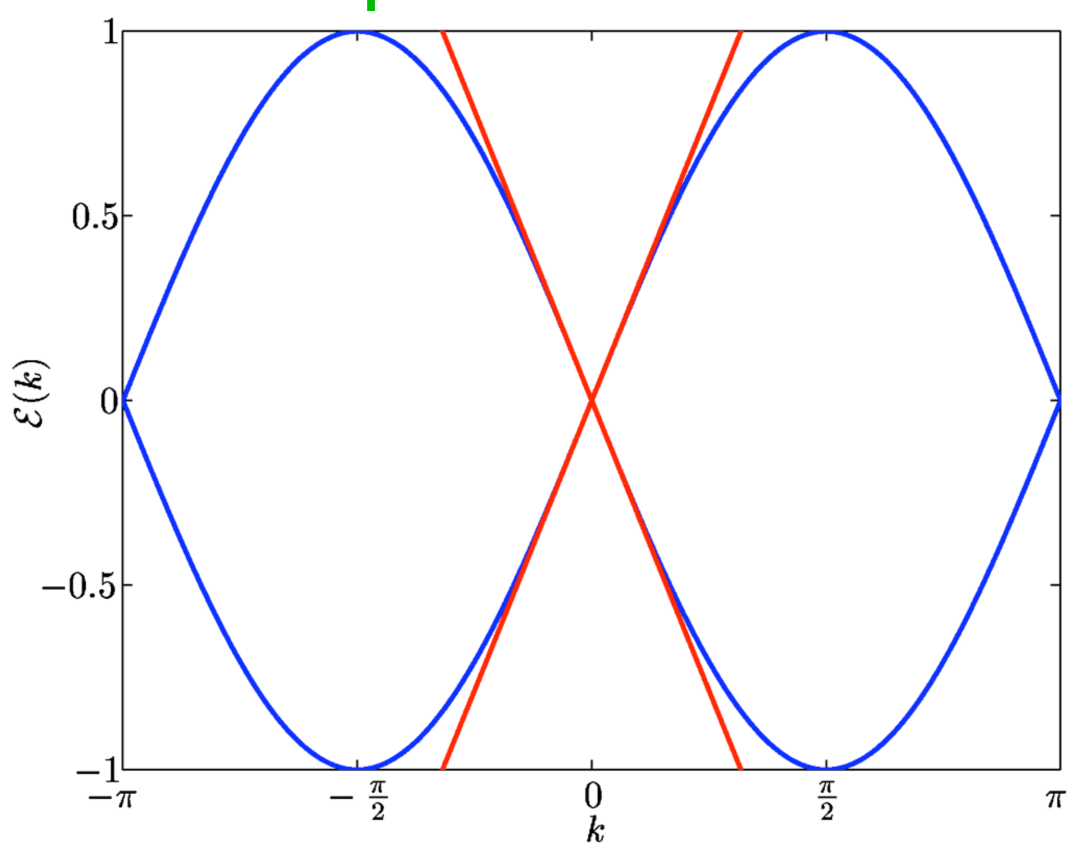

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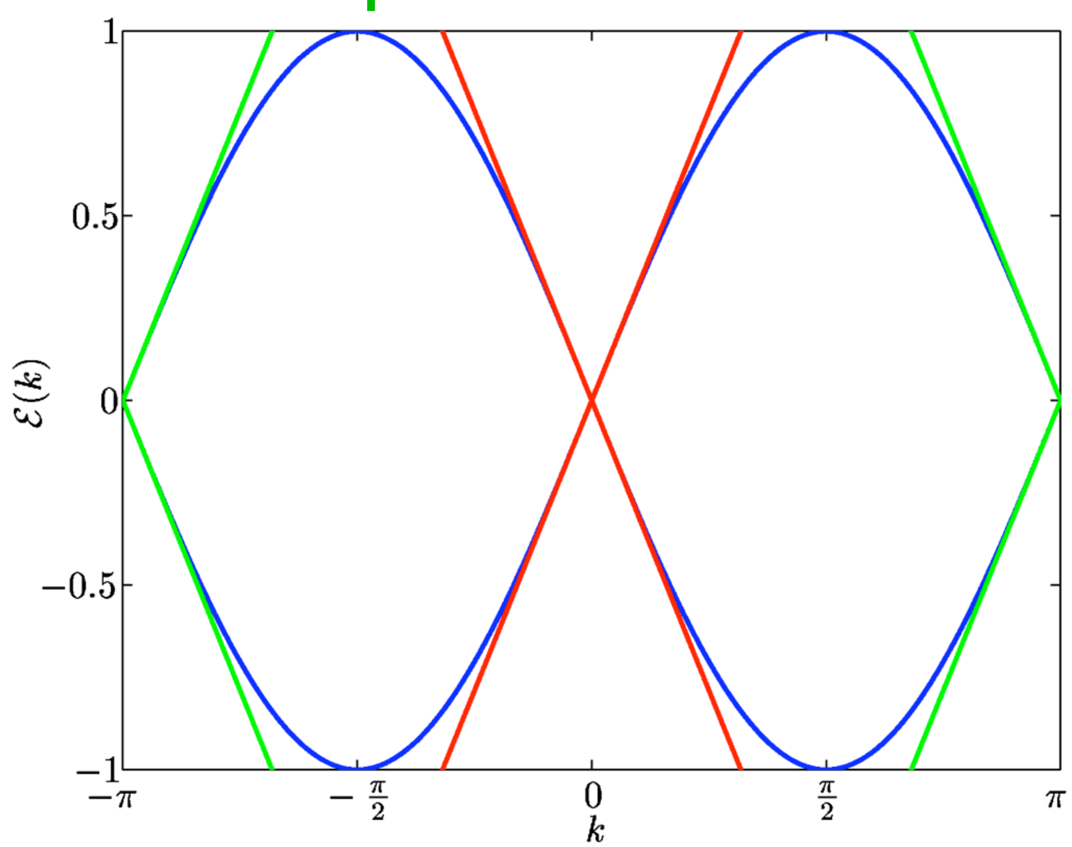

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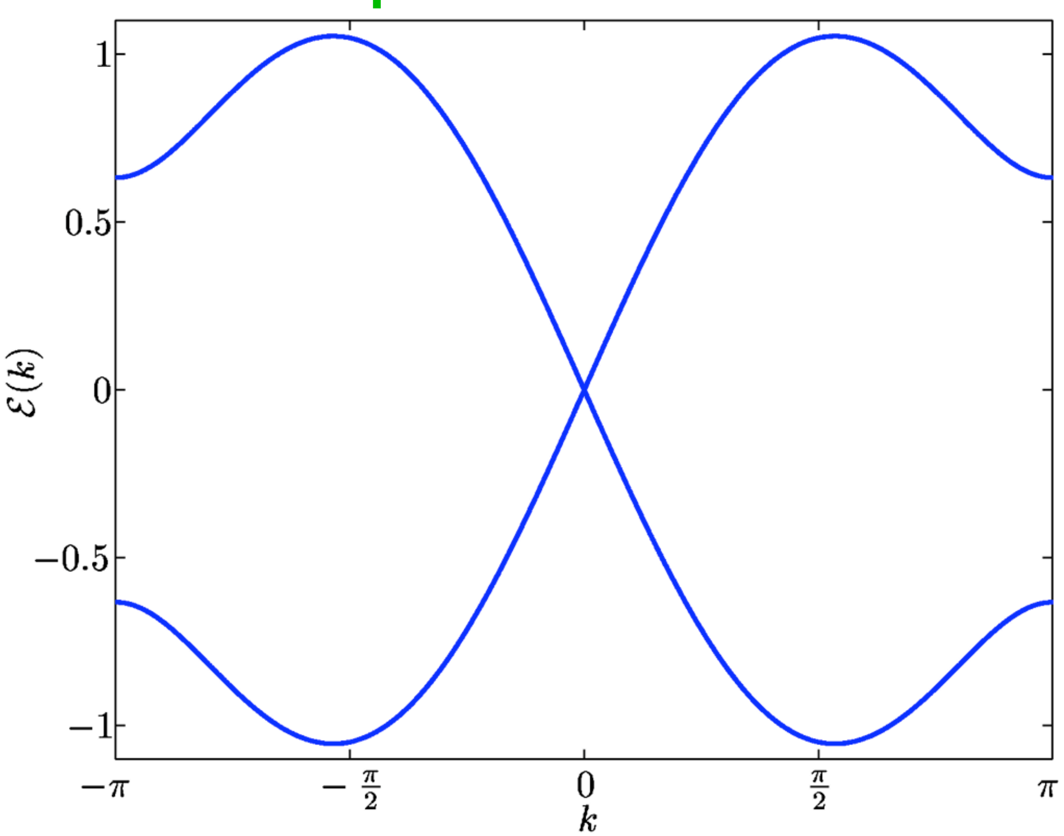

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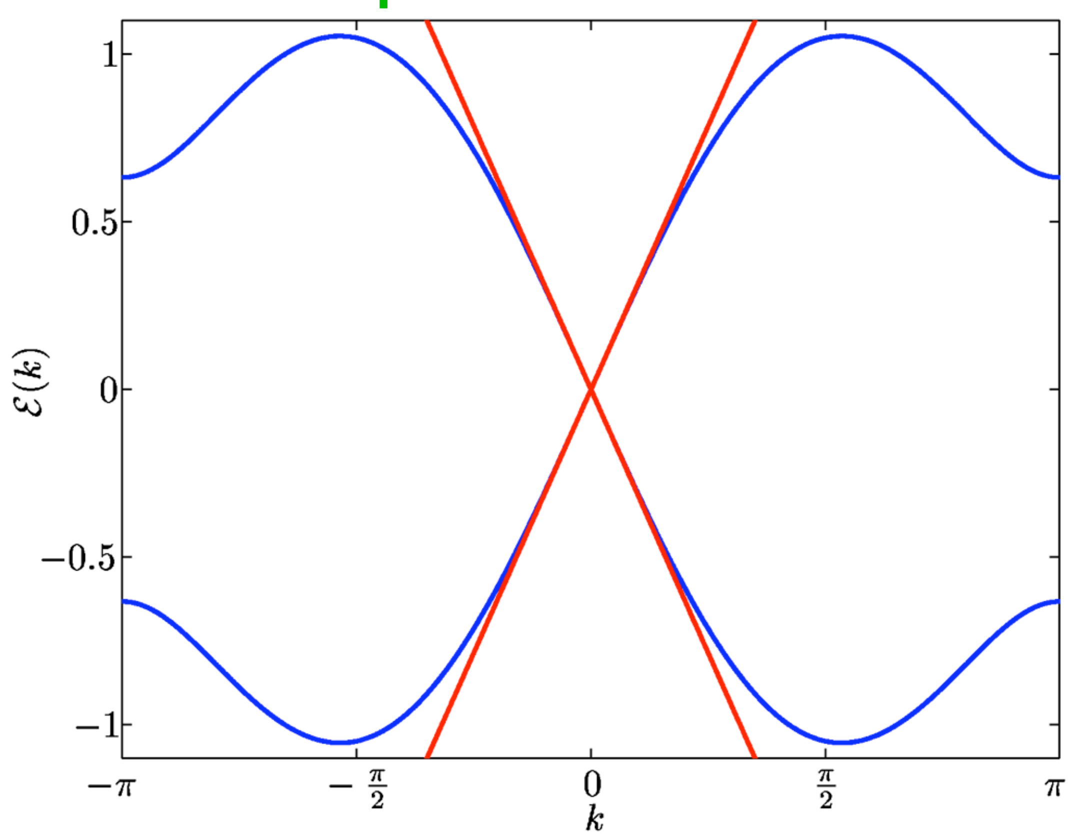

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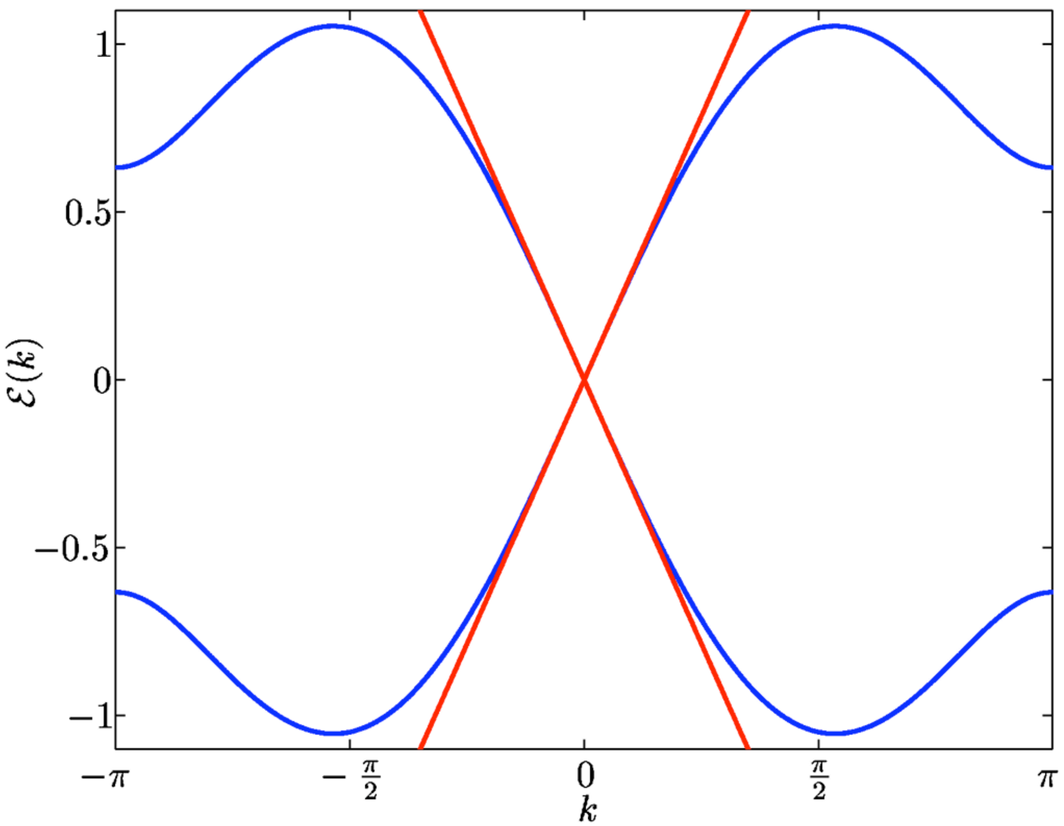

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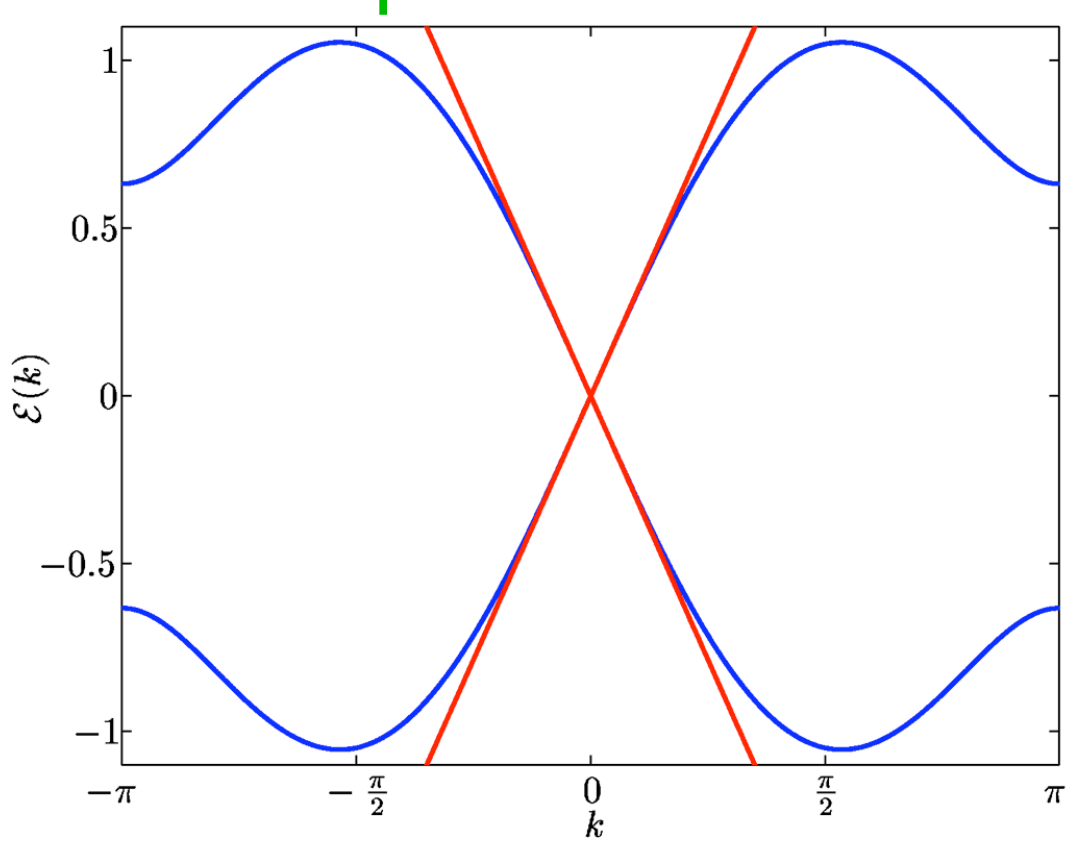

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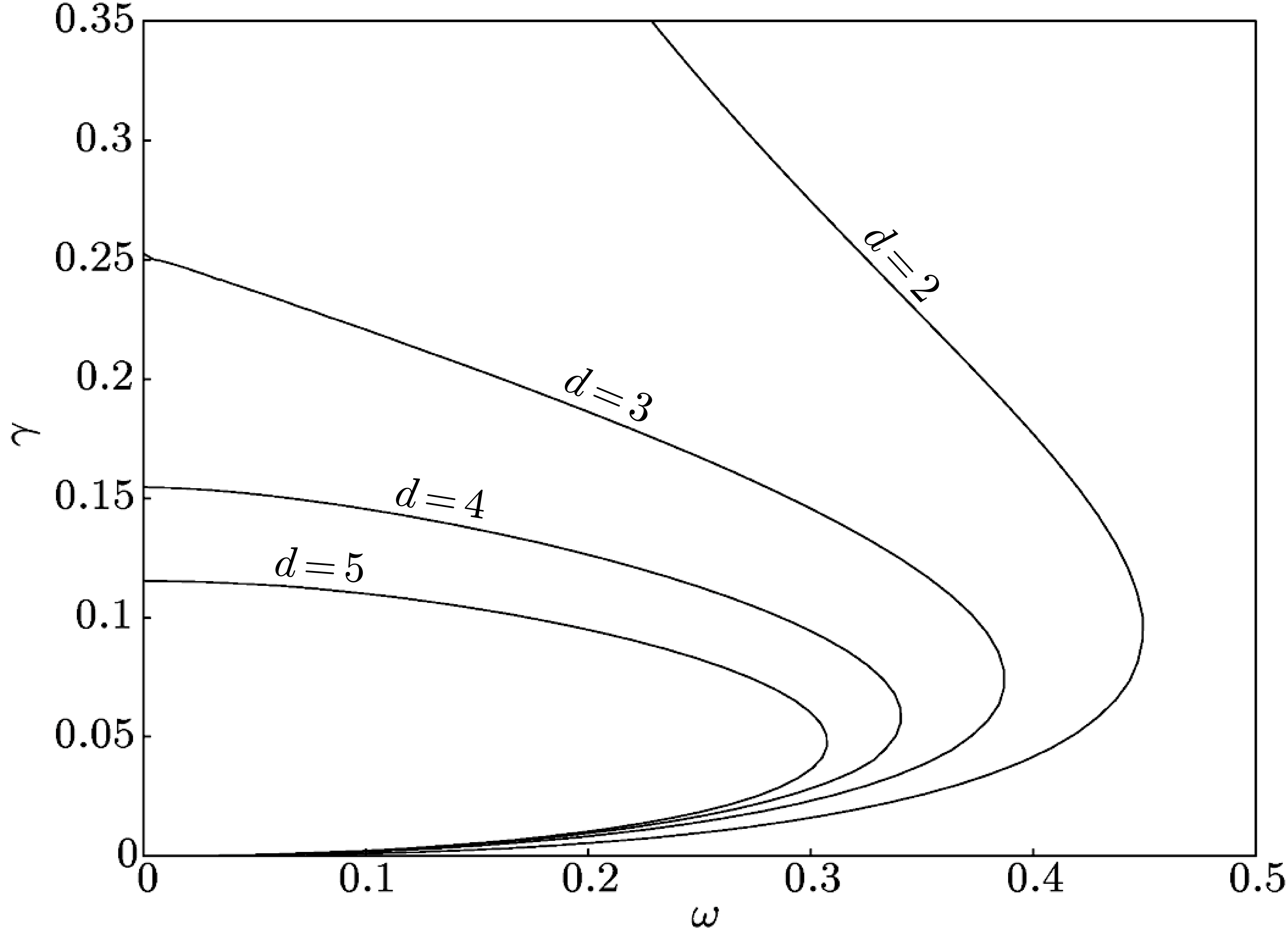
Run time:

$O(\sqrt{N})$ in $d > 2$,
$O(\sqrt{N} \log N)$ in $d = 2$.



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
γ
ω
$d=2$
$d=3$
$d=4$
$d=5$

4. Evaluating Boolean formulas

- Childs, Cleve, Jordan, and Yeung, quant-ph/0702160.
- Childs, Reichardt, Špalek, and Zhang, quant-ph/0703015.

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Fix a Boolean formula on N variables.

Given a black box for determining how the variables are assigned, how many variables must we query to determine the value of the formula?

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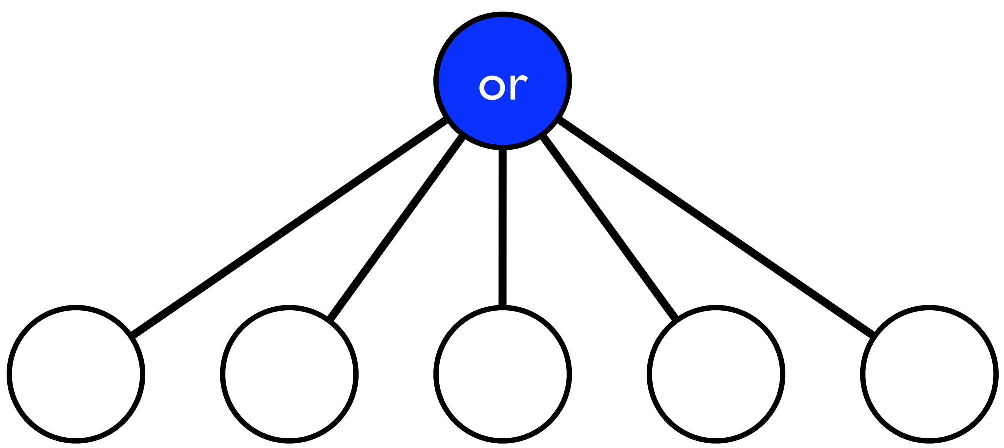

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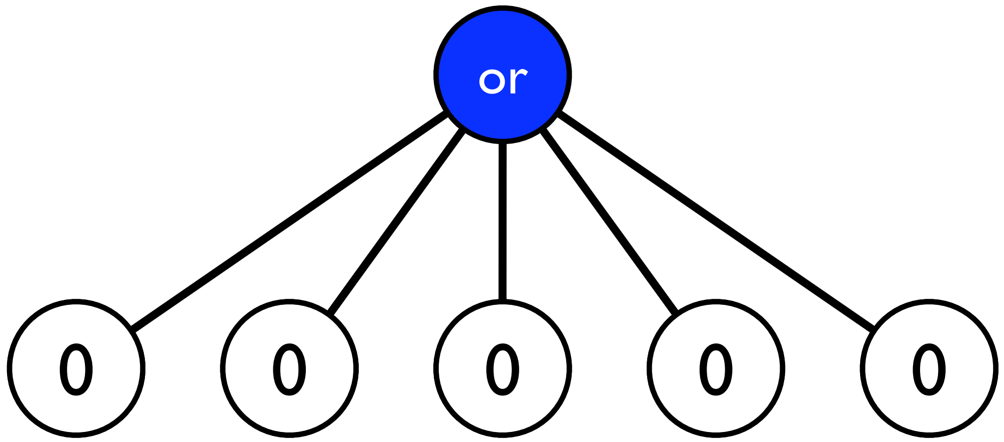

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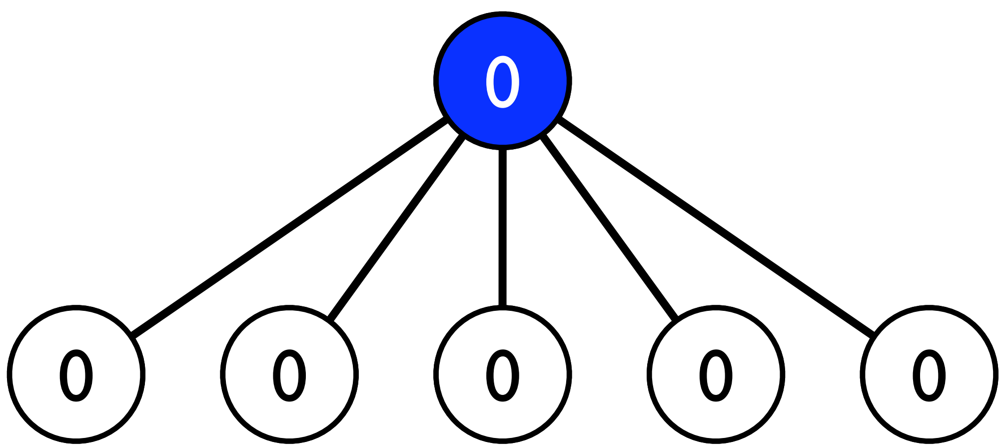

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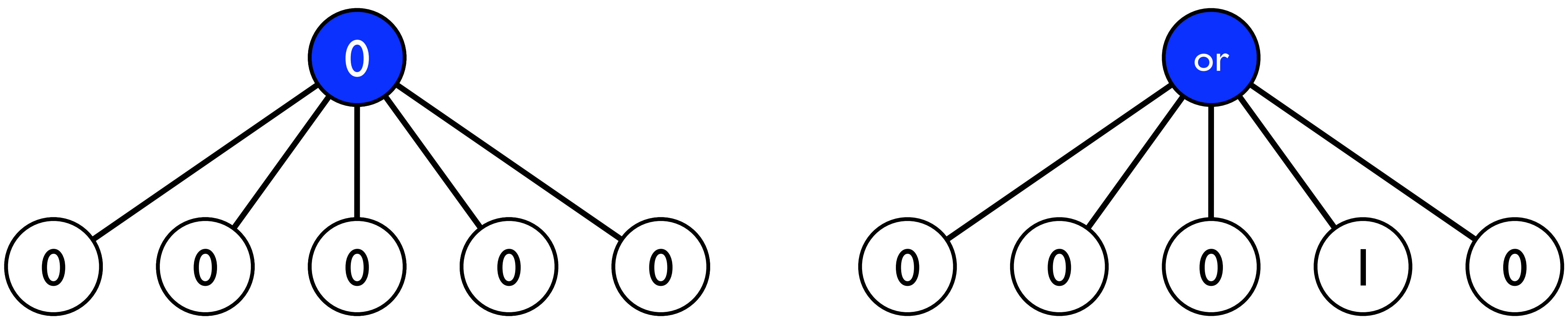

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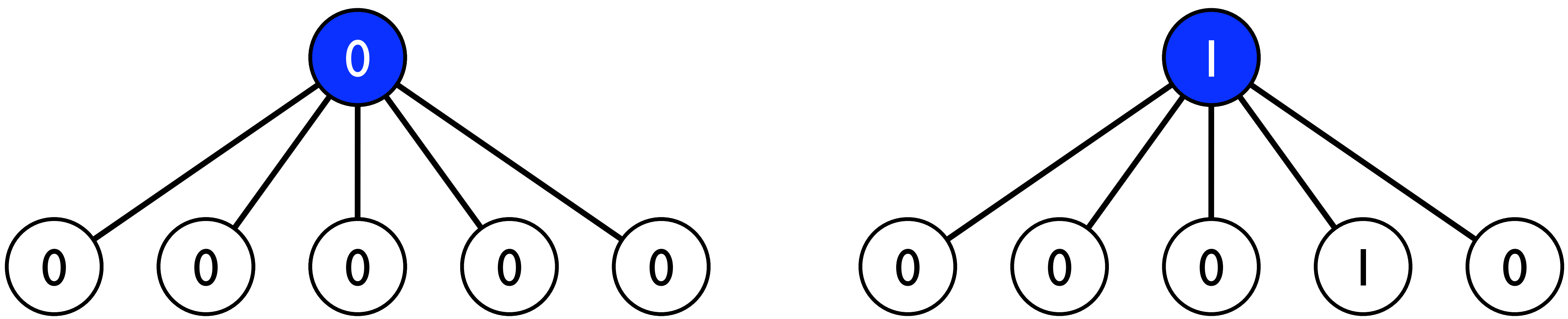

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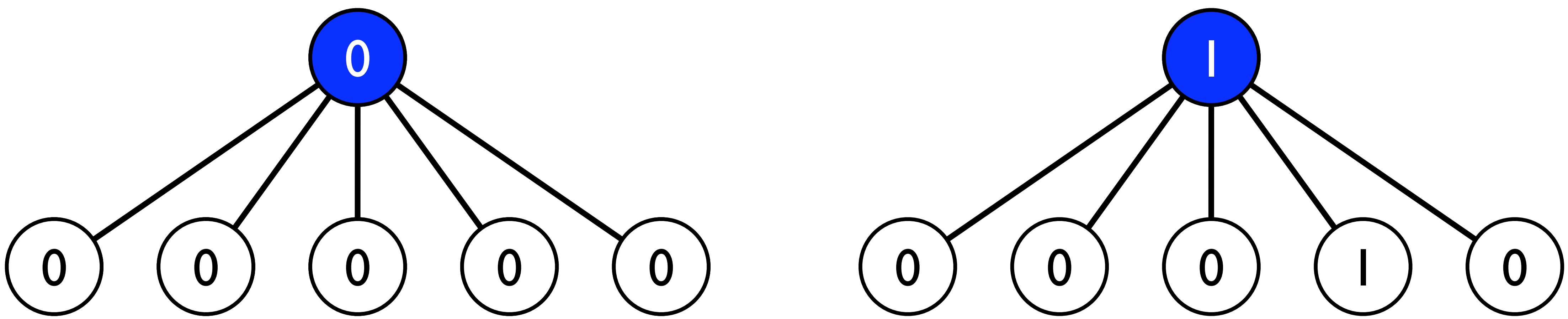
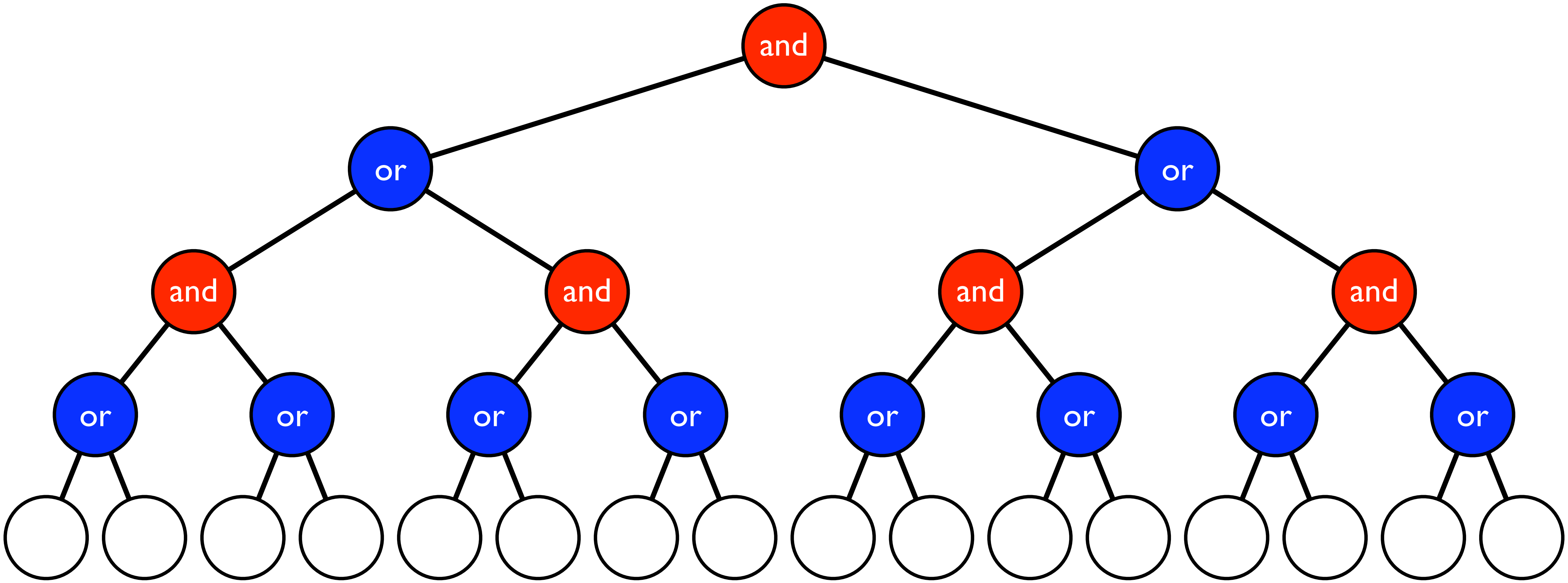


Classical complexity: $\Theta(N)$

Quantum algorithm [Grover 1996]: $O(\sqrt{N})$

Quantum lower bound [BBBV 1996]: $\Omega(\sqrt{N})$

AND-OR trees (aka game trees)



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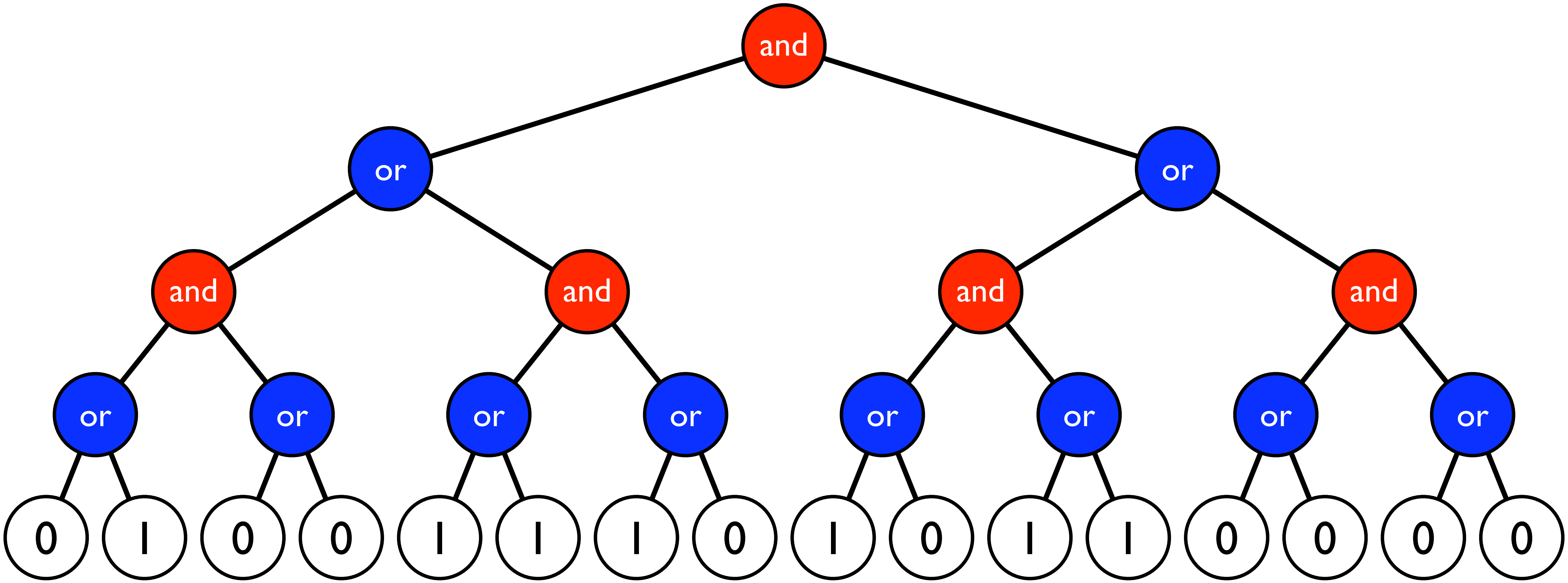

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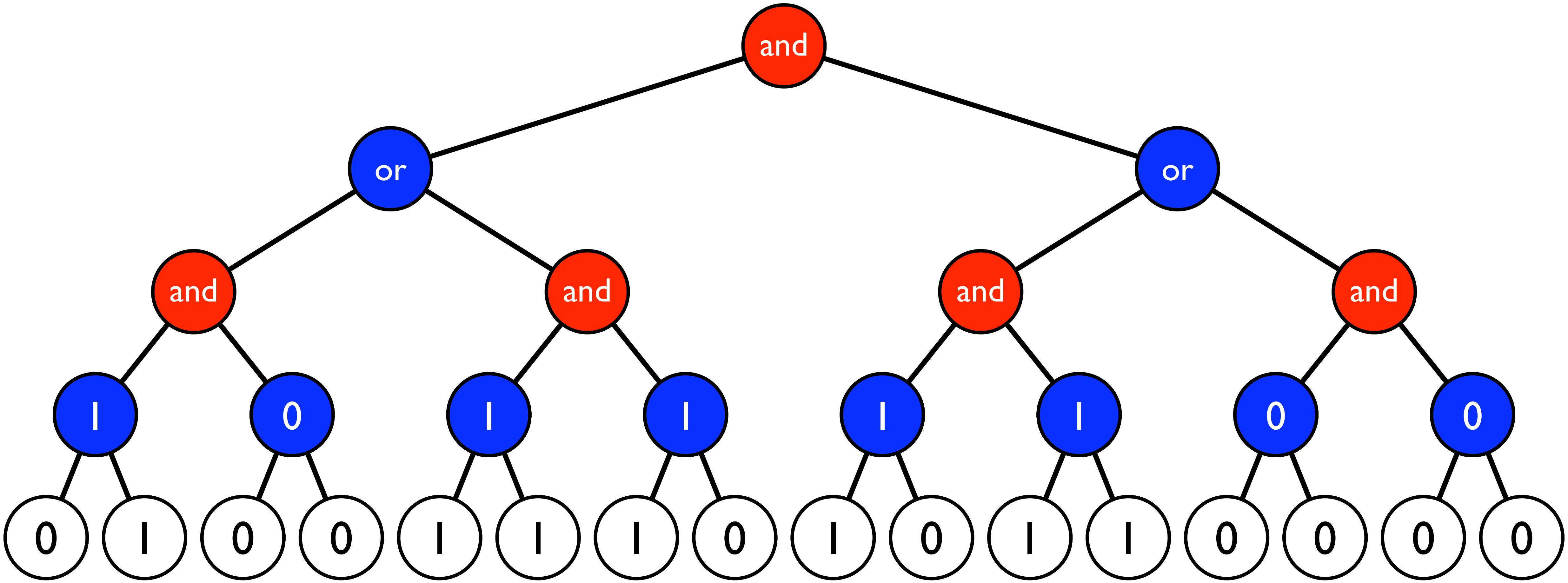

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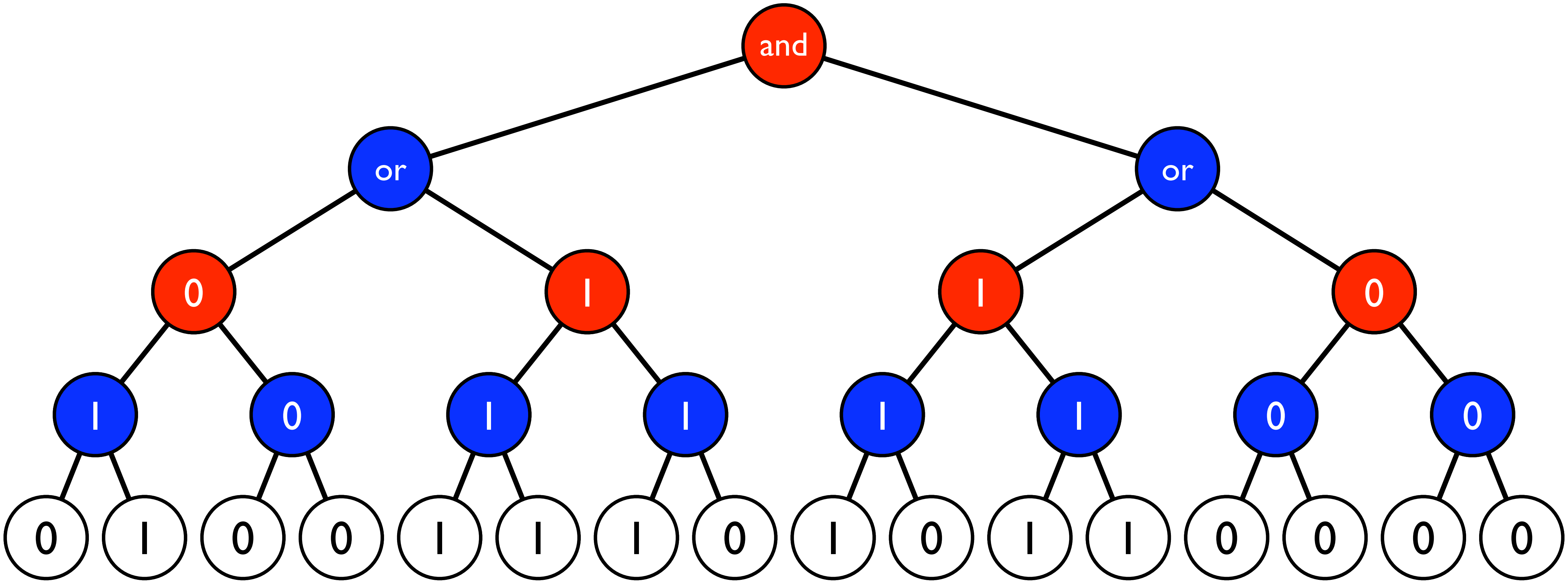

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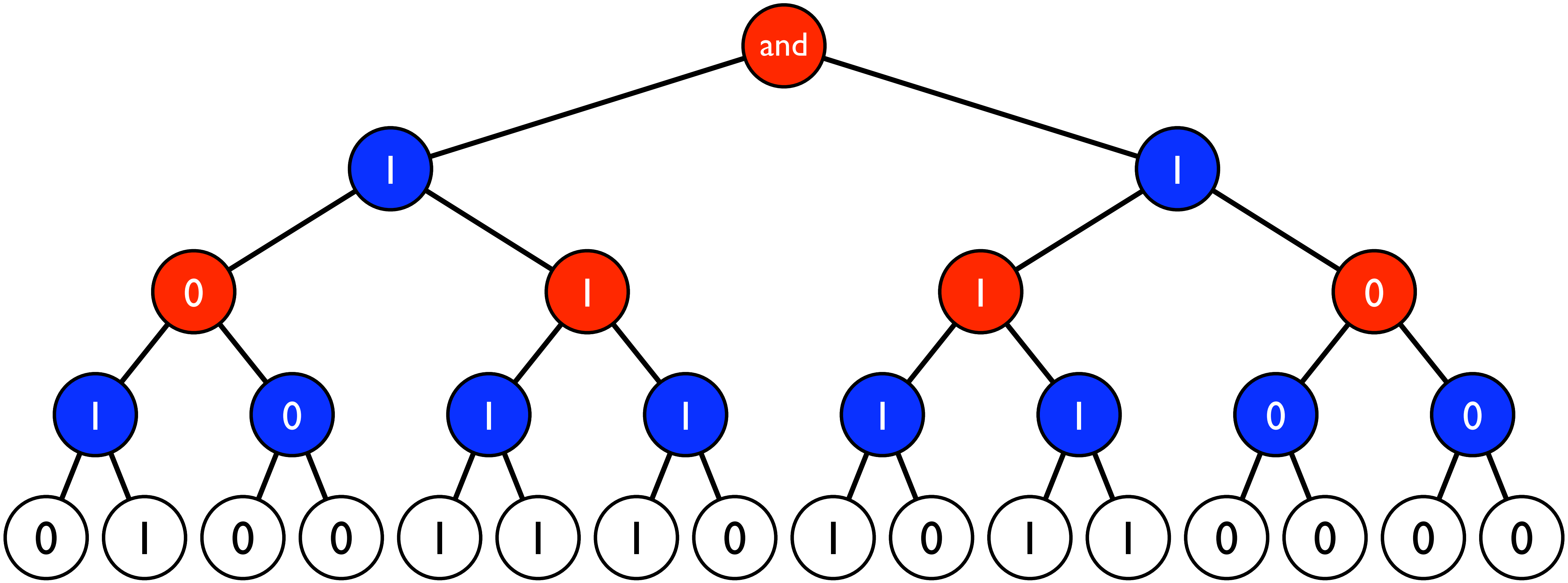

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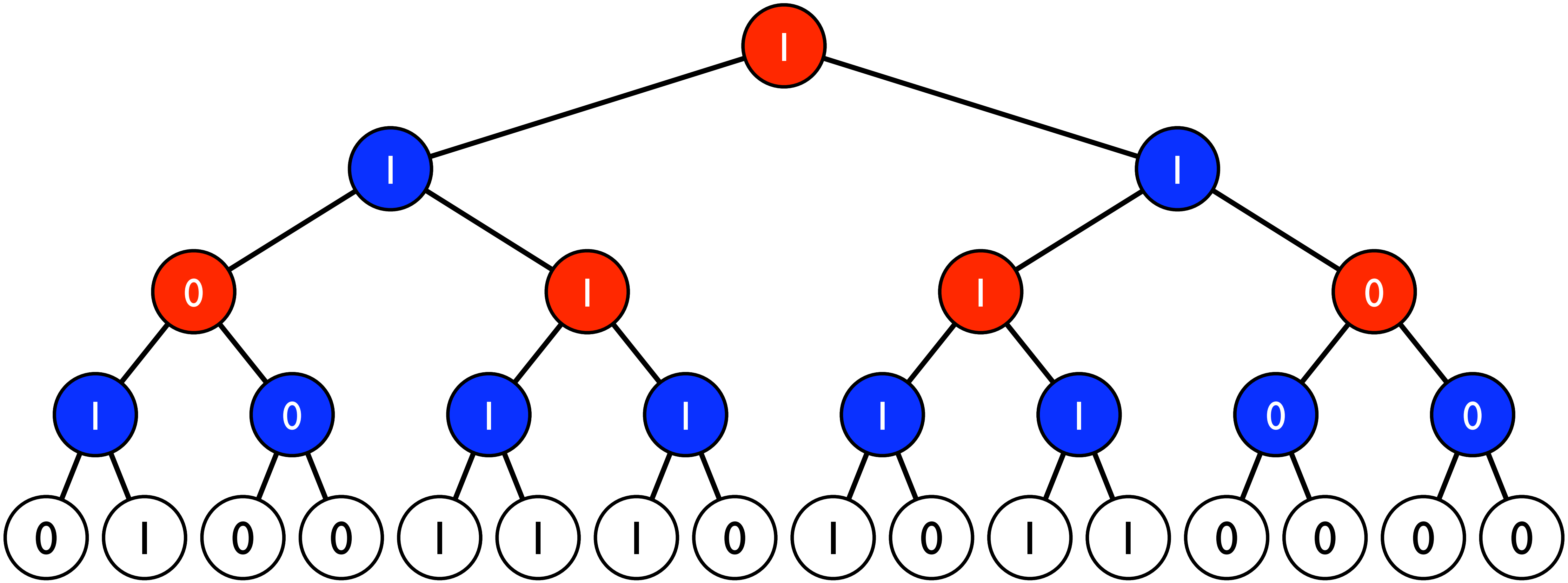

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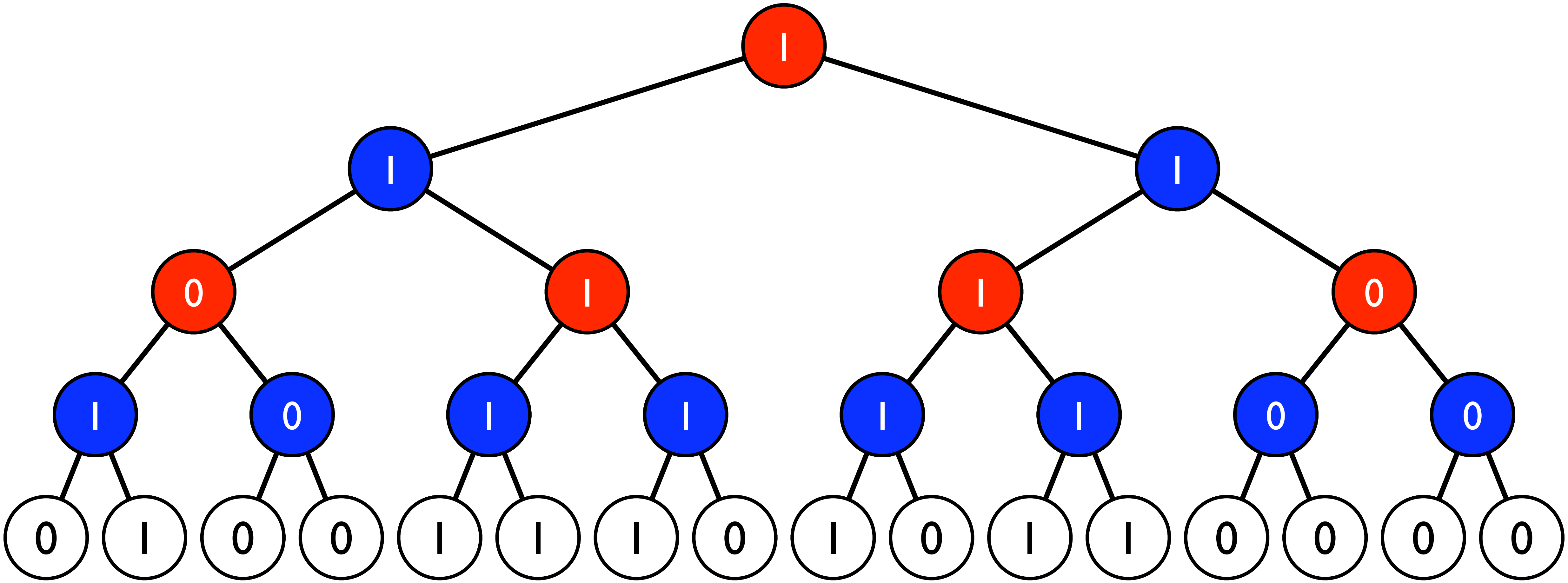

Classical complexity: $\Theta(N^{0.753})$ [Snir 85, Saks-Wigderson 86, Santha 95]

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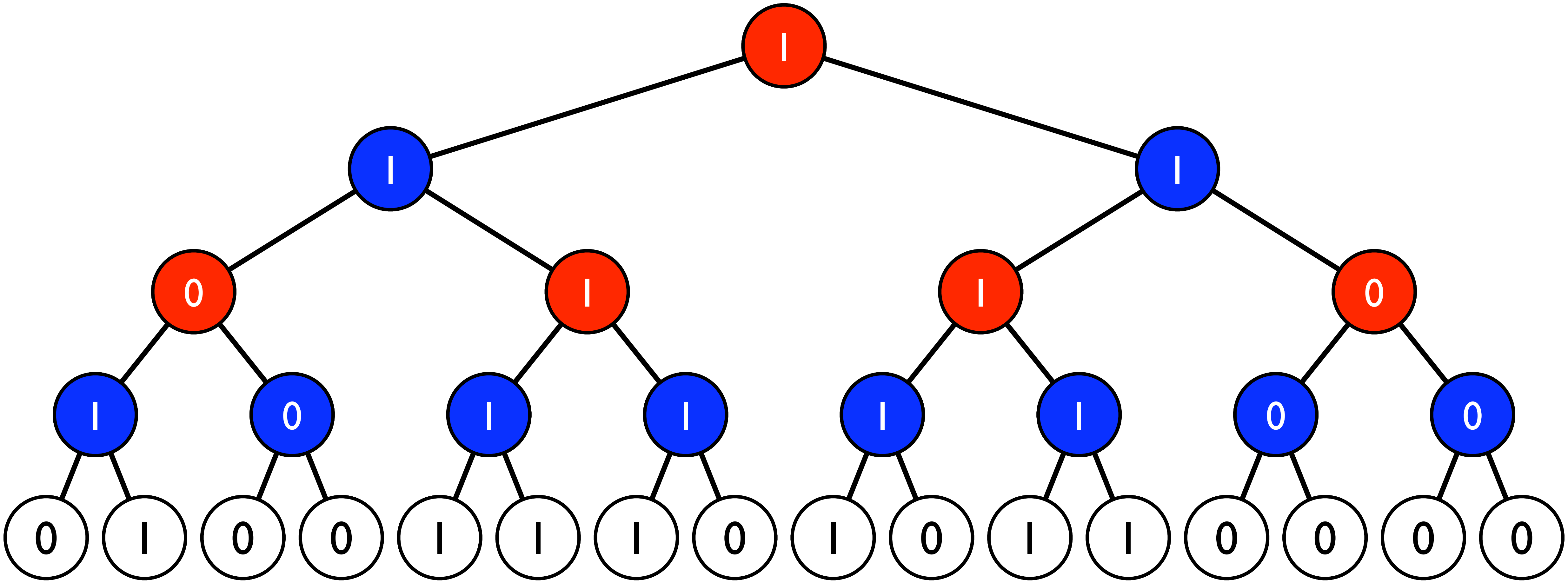

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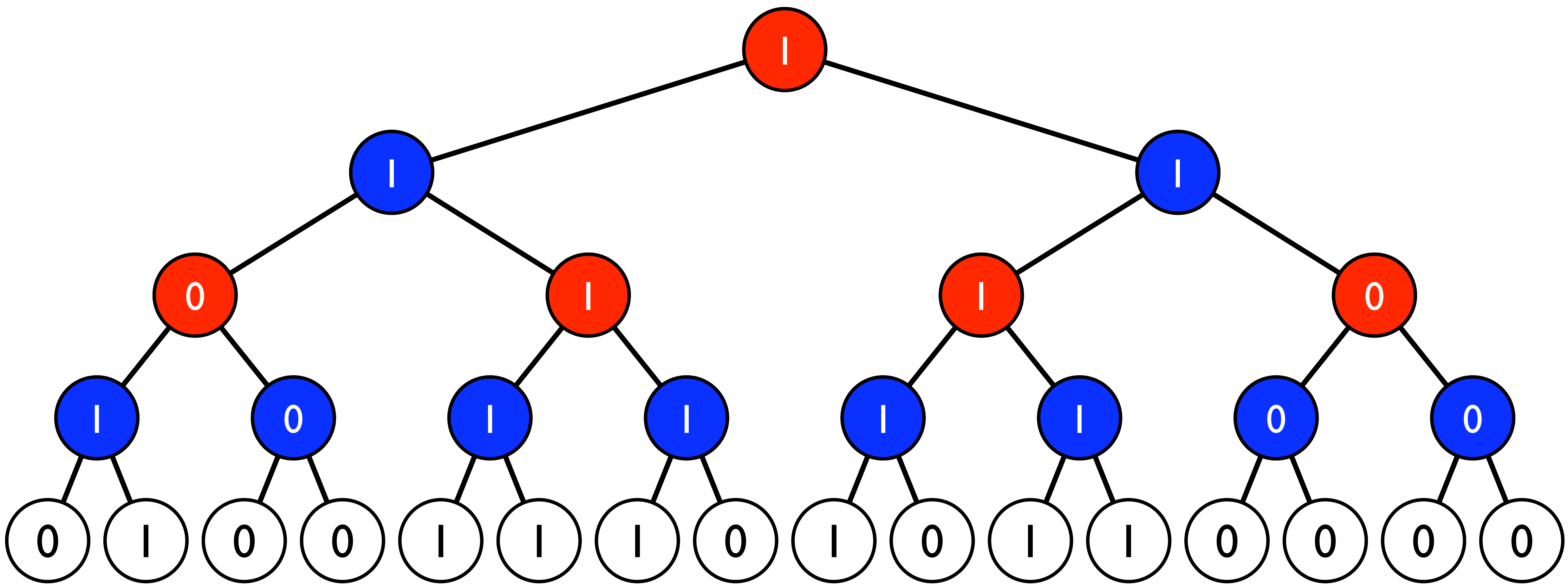

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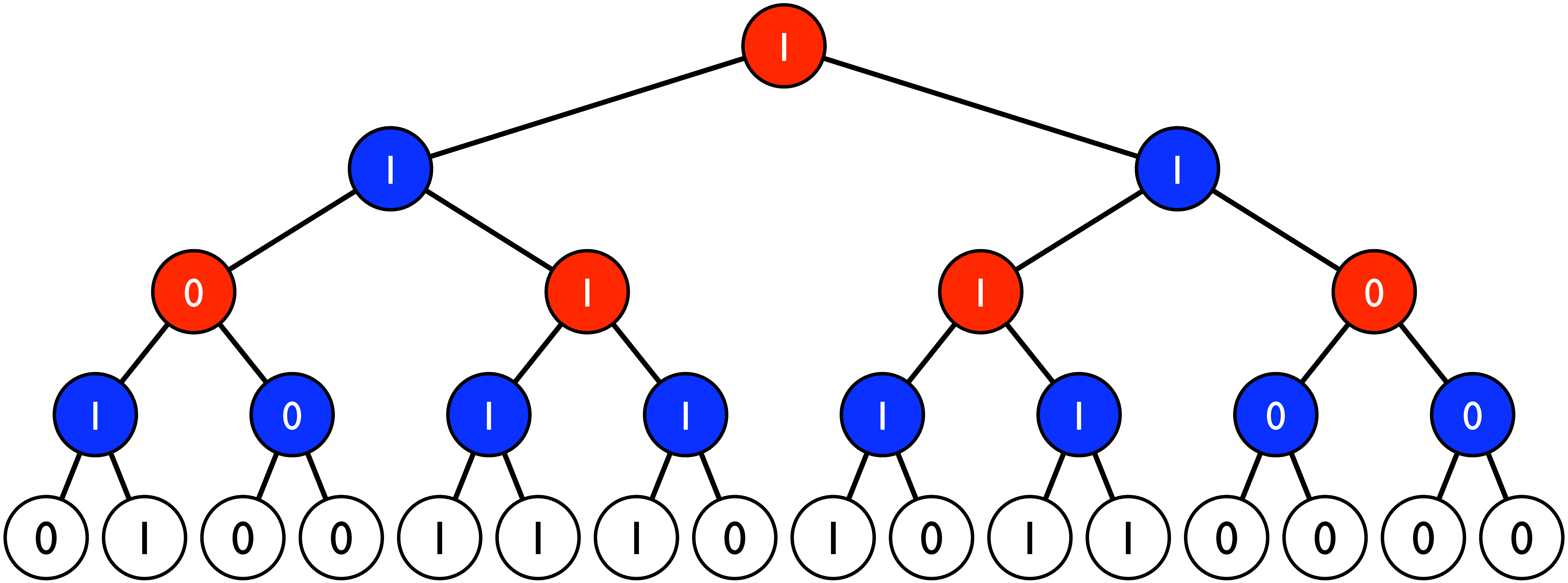

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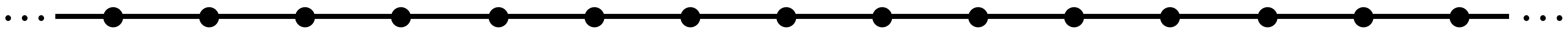
Discretized version [Childs-Cleve-Jordan-Yeung 07]: $O(\sqrt{N^{1+\epsilon}})$

Evaluating AND-OR trees by scattering



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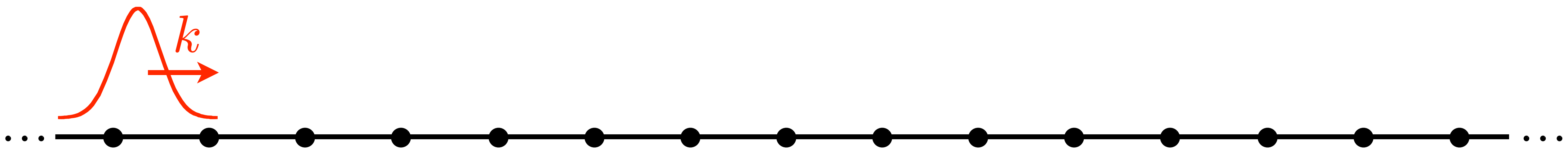

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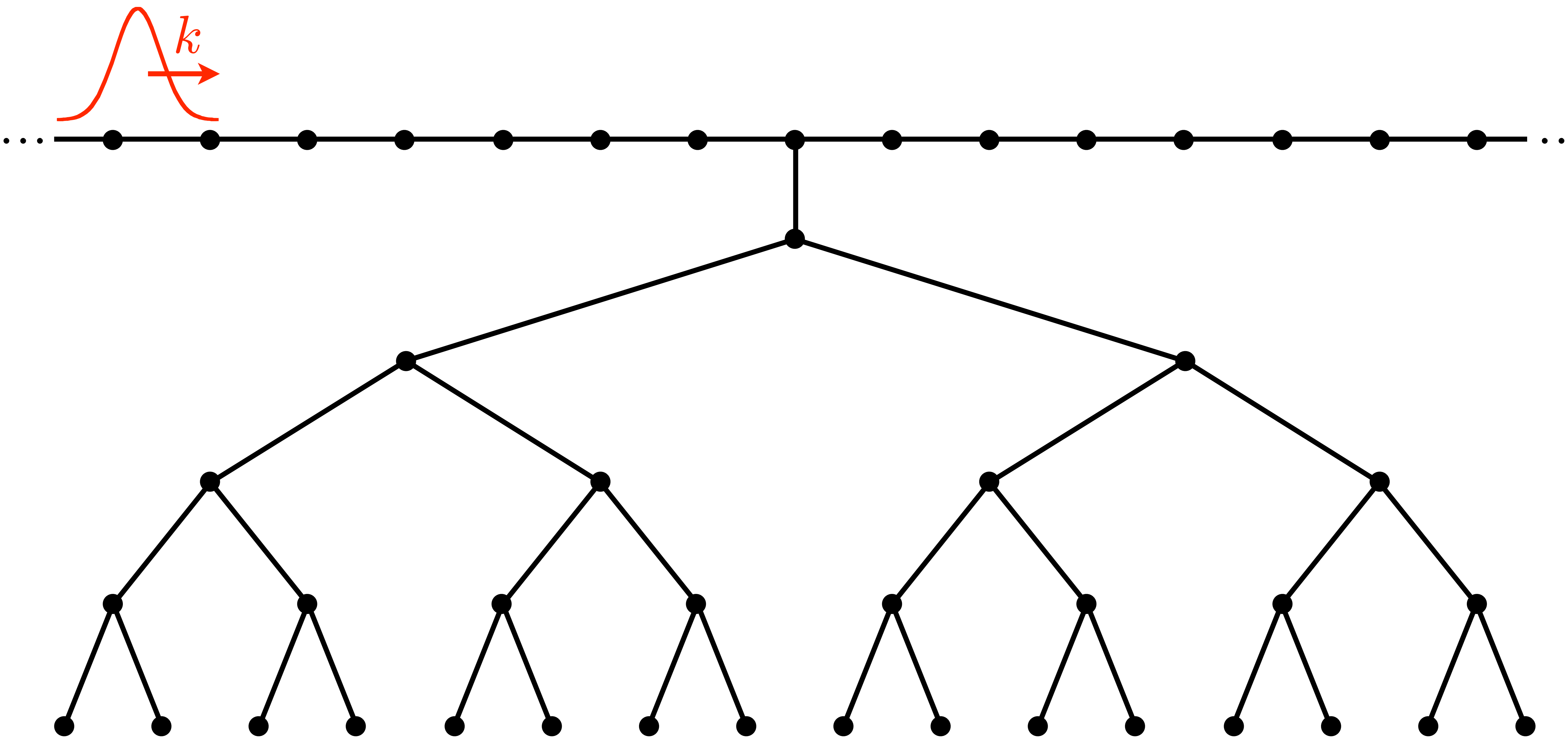

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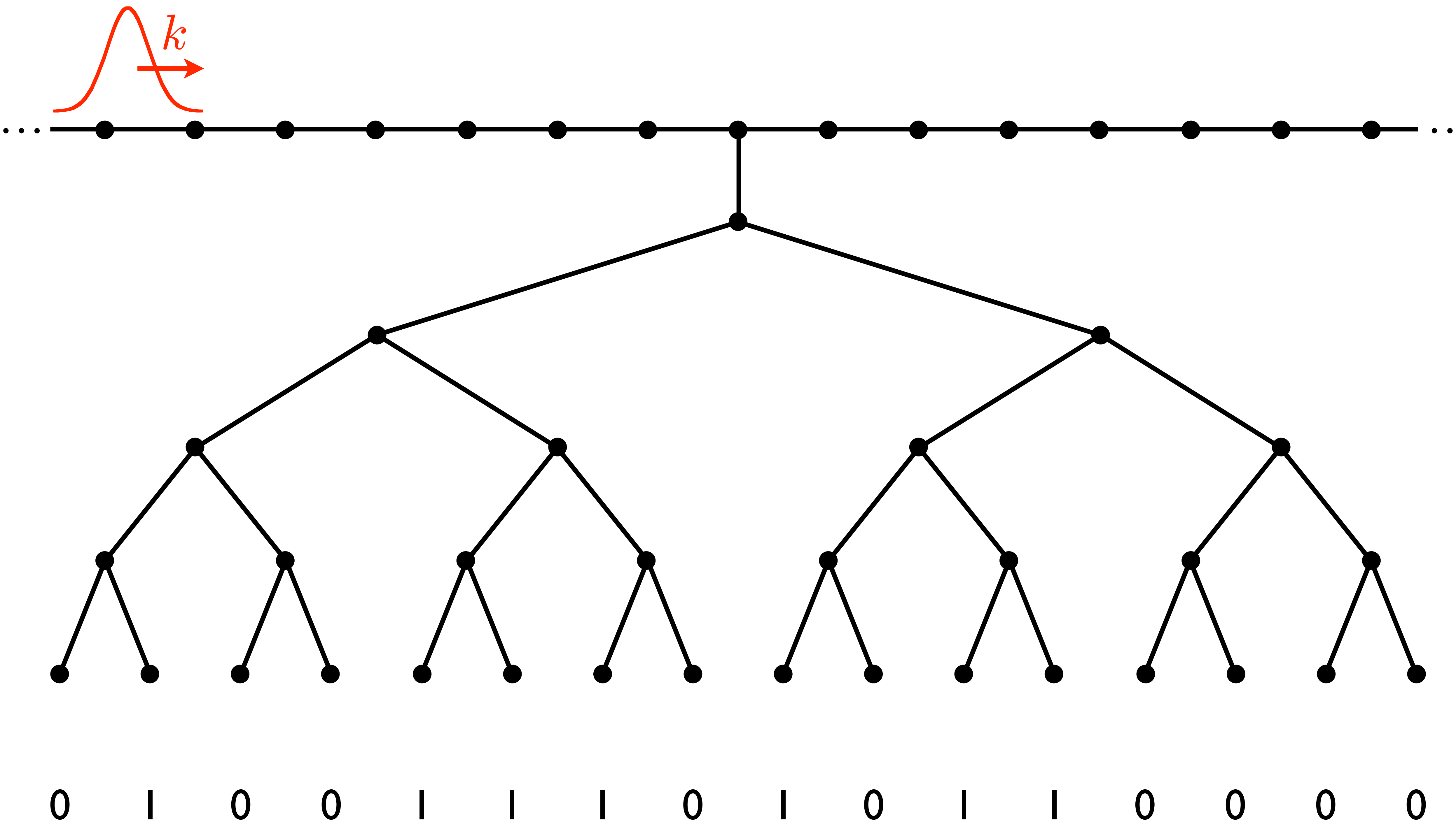

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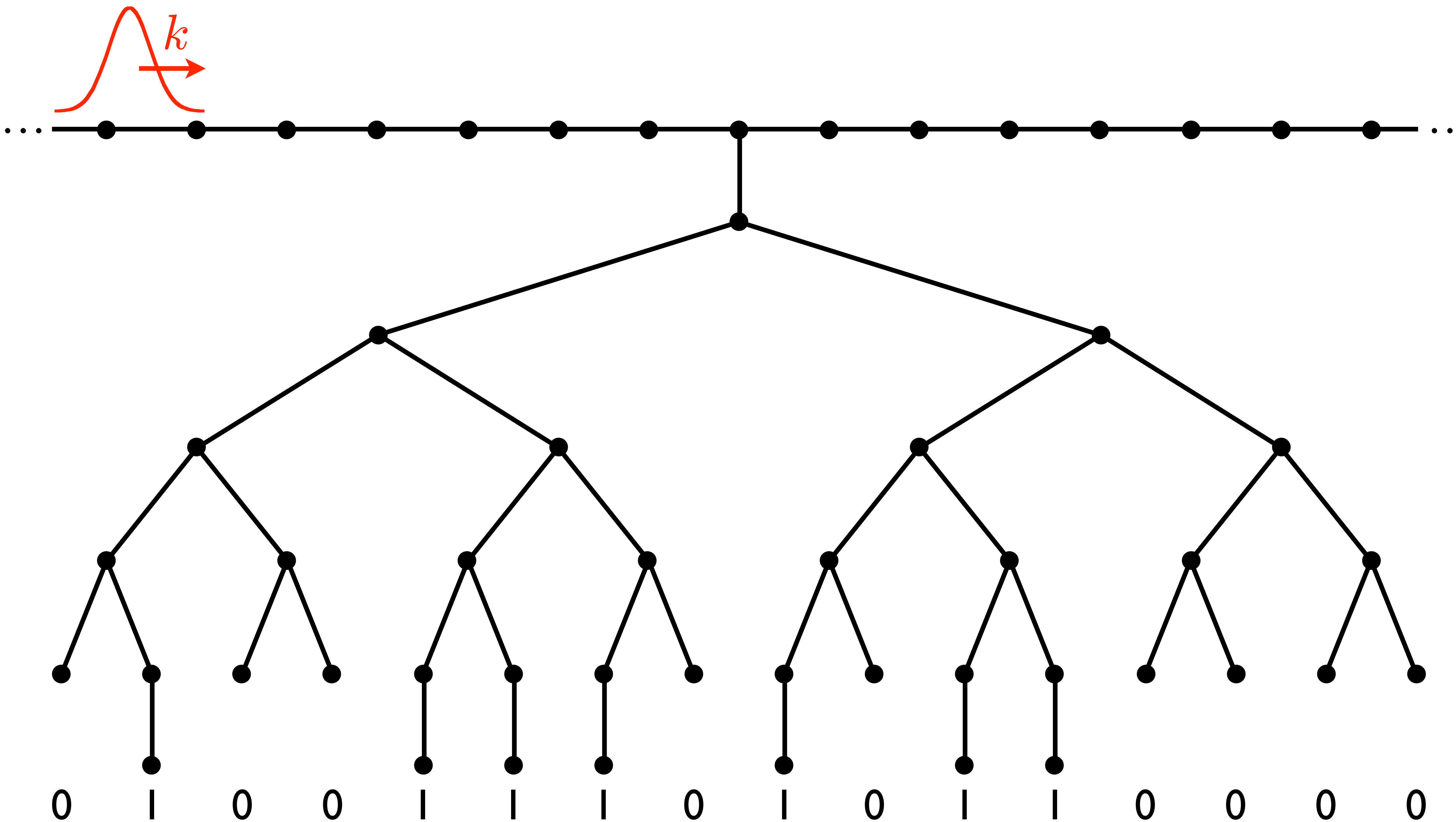

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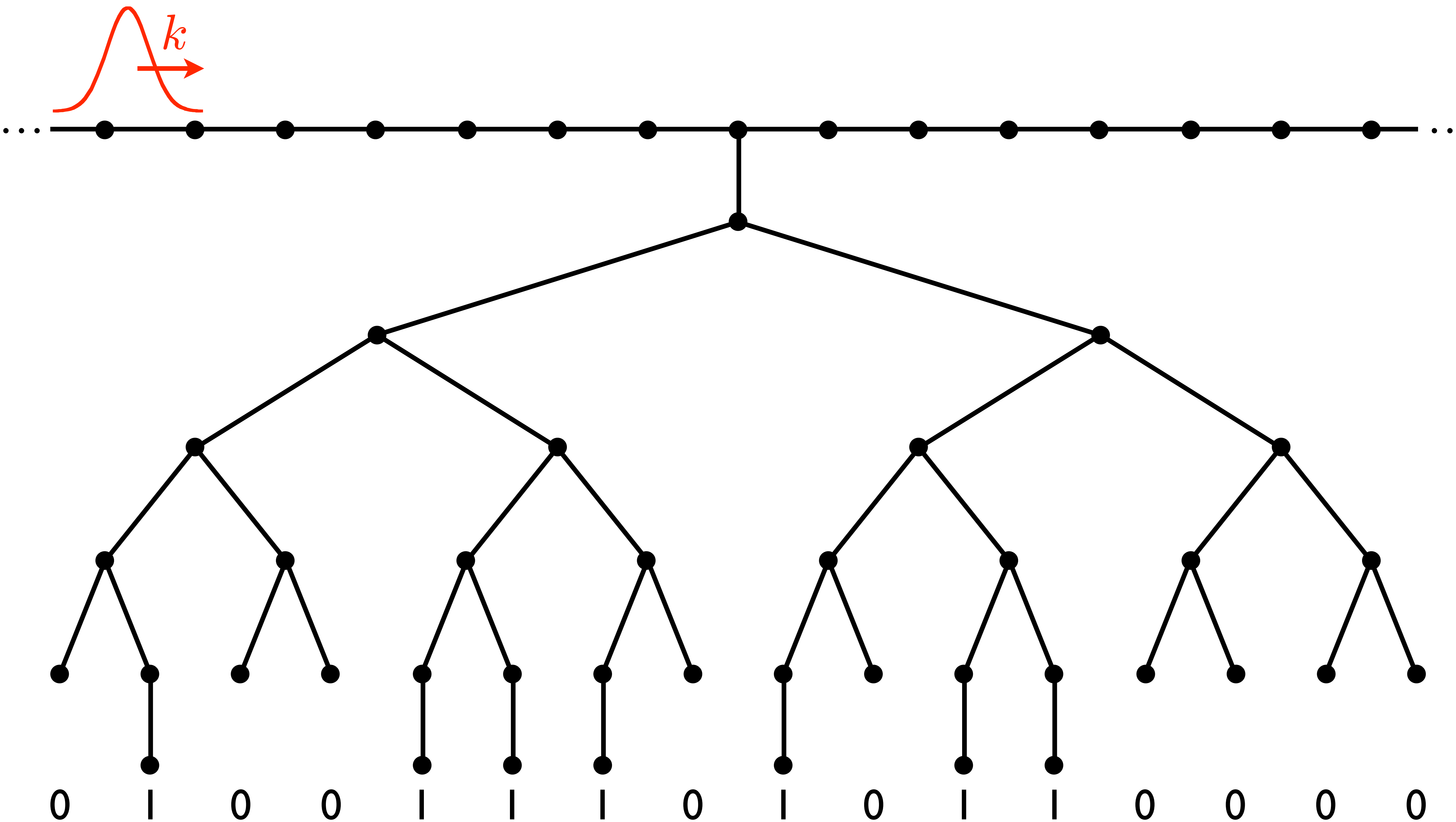
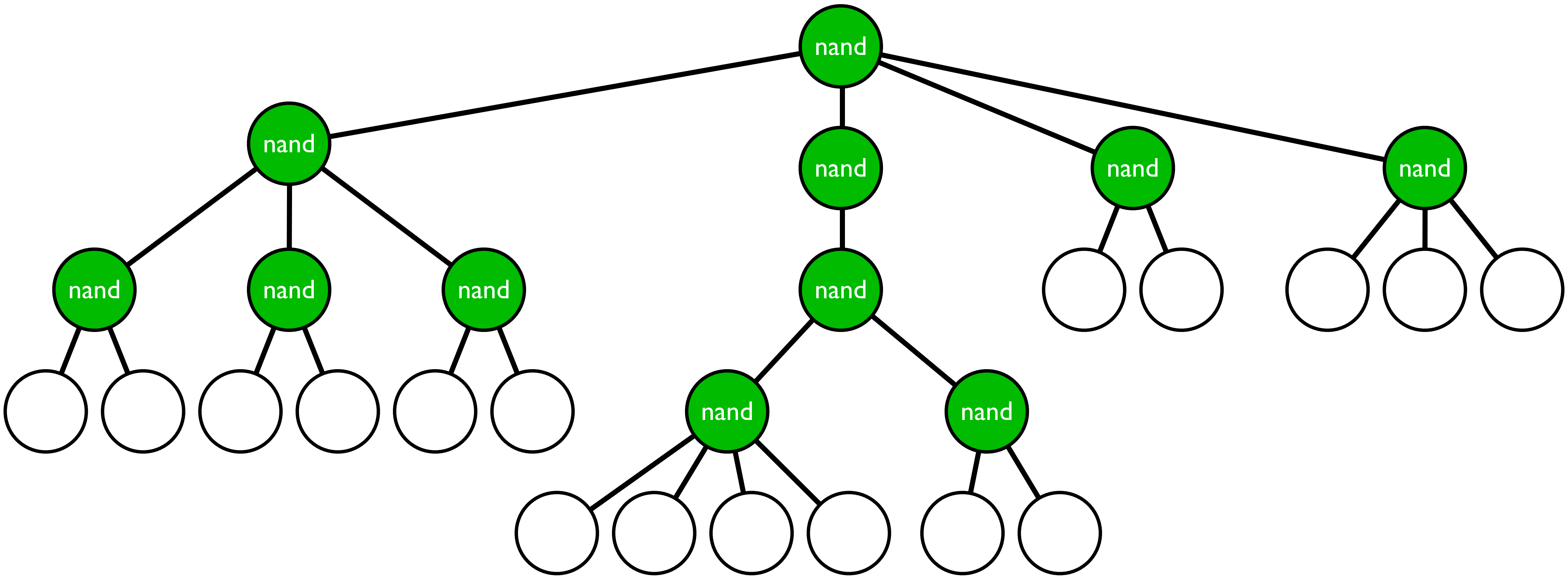


Claim: For small k, the transmission coefficient is large if the formula (translated into NAND gates) evaluates to 0, and small if it evaluates to 1.

General formulas



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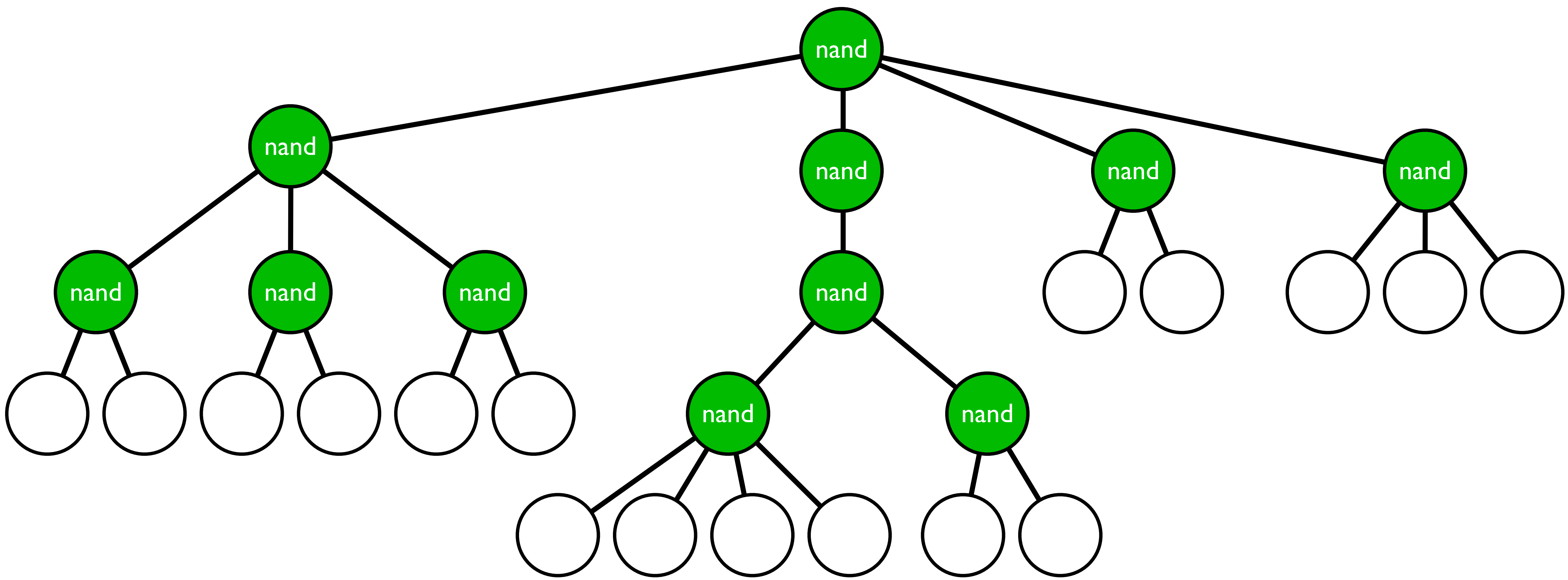

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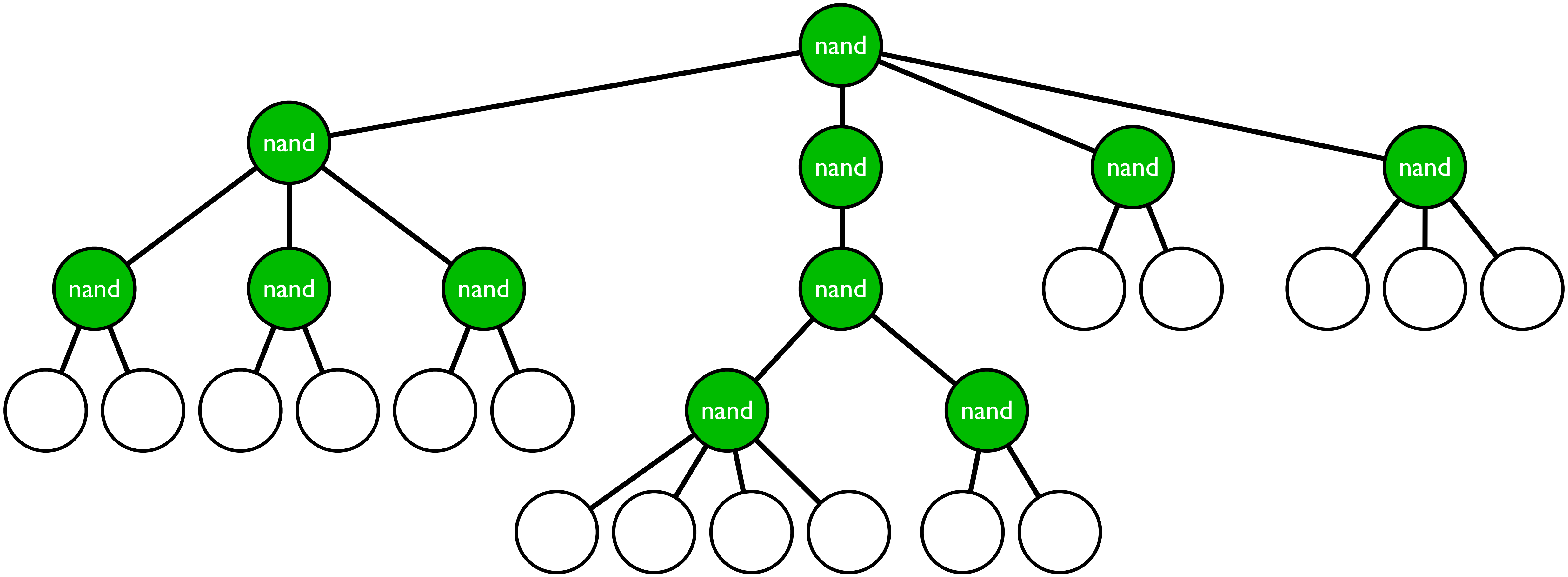

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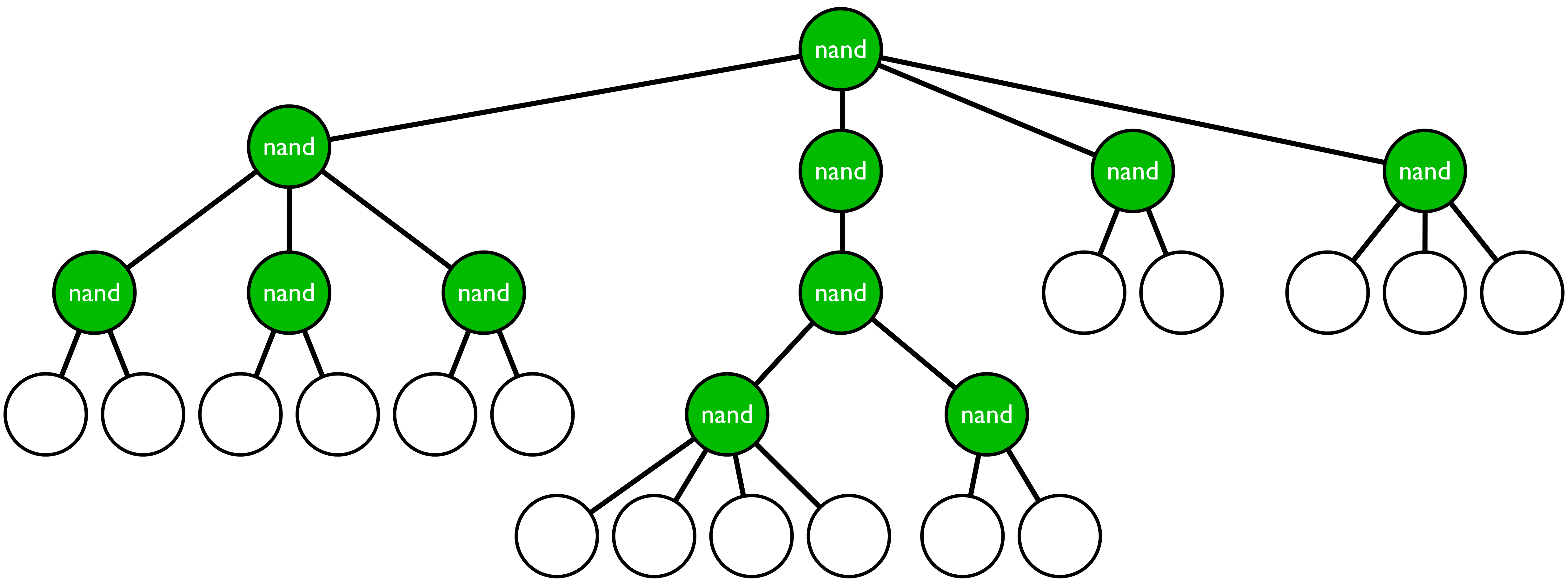

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Main idea: The quantum walk on the (expanded, appropriately weighted) tree has a zero eigenvalue if the formula evaluates to 1, and a smallest eigenvalue $O(\frac{1}{\sqrt{N}})$ if it evaluates to 0. Use phase estimation.

Summary and outlook

- Physics $\leftrightarrow$ Information
- Quantum systems can encode and process information in a fundamentally non-classical way.
- While we have many examples of this phenomenon, we are far from having a general understanding of the information processing power of quantum mechanics.
- In particular, we would like to develop new ways of exploiting quantum interference to perform information processing tasks.