ASSIGNMENT 1

1. Let $V_1, V_2$ be vector spaces. Prove or disprove each of the following.

   (a) If $W_1$ is a subspace of $V_1$ and $W_2$ is a subspace of $V_2$, then $W_1 \oplus W_2$ is a subspace of $V_1 \oplus V_2$.

   (b) If $W_1$ is a subspace of $V_1$ and $W_2$ is a subspace of $V_2$, then $W_1 \otimes W_2$ is a subspace of $V_1 \otimes V_2$.

   (c) If $W$ is a subspace of $V_1 \oplus V_2$, then there exists a subspace $W_1$ of $V_1$ and a subspace $W_2$ of $V_2$ such that $W = W_1 \oplus W_2$.

   (d) If $W$ is a subspace of $V_1 \otimes V_2$, then there exists a subspace $W_1$ of $V_1$ and a subspace $W_2$ of $V_2$ such that $W = W_1 \otimes W_2$.

   (e) If $W_1$ is a subspace of $V_1$ and $W_2$ is a subspace of $V_2$, then $(V_1 \oplus V_2)/(W_1 \oplus W_2) = V_1/W_1 \oplus V_2/W_2$.

   (f) If $W_1$ is a subspace of $V_1$ and $W_2$ is a subspace of $V_2$, then $(V_1 \otimes V_2)/(W_1 \otimes W_2) = V_1/W_1 \otimes V_2/W_2$.

2. Let $V$ be an inner product space with $\dim V \geq 2$. Call $T \in \mathcal{L}(V \otimes V)$ a \textit{unit vector cloner} if there exists a fixed $w \in V$ such that for all $v \in V$ with $\langle v, v \rangle = 1$, $T(v \otimes w) = v \otimes v$. Show that a unit vector cloner does not exist.

3. Define $A := \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $B := \begin{pmatrix} 3 & 3 & 2 \\ 5 & 4 & 3 \\ 0 & 3 & 1 \end{pmatrix}$.

   (a) Compute $\text{rank}(A)$ and $\text{rank}(B)$.

   (b) Compute $A \otimes B$.

   (c) Compute $\text{rank}(A \otimes B)$.

   (d) Prove that for any two matrices $C, D$, $\text{rank}(C \otimes D) = \text{rank}(C) \cdot \text{rank}(D)$.

4. Let $A, B$ be matrices.

   (a) Prove that $A \oplus B$ is invertible if and only if $A, B$ are invertible.

   (b) Is the same true of $A \otimes B$? Why or why not?

5. In this problem, we explore another way to make new vector spaces from old ones. Let $V$ be a vector space with subspaces $U, W$.

   (a) Under what conditions is $U \cap W$ a subspace of $V$?

   (b) Supposing that $V$ is finite-dimensional, give an expression (with proof) for $\dim(U \cap W)$ in terms of $\dim U$, $\dim W$, and $\dim(U + W)$.